Assignment 1: Calculus Review

(1) $$\int_1^3 \frac{1}{t^3}\,dt = \int_1^3 t^{-3}\,dt = \frac{t^{-2}}{-2}\Big|_1^3$$

$$= -\frac{1}{2}\left(\frac{1}{3^2} - \frac{1}{1^2}\right) = -\frac{1}{2}\left(\frac{1}{9} - 1\right) = \frac{1}{2}\cdot\frac{8}{9} = \frac{4}{9}$$

(2) $$\int_0^\infty e^{-\theta x}\,dx \qquad \begin{array}{l} u = -\theta x \\ du = -\theta\,dx \end{array} \qquad \begin{array}{c|c} x & u \\ \hline \infty & -\infty \\ 0 & 0 \end{array}$$

$$= \frac{-1}{\theta}\int_0^\infty e^{-\theta x}(-\theta)\,dx = -\frac{1}{\theta}\int_0^{-\infty} e^u\,du$$

$$= \frac{1}{\theta}\int_{-\infty}^0 e^u\,du = \frac{1}{\theta}e^u\Big|_{-\infty}^0$$

$$= \frac{1}{\theta}\left(e^0 - \lim_{u\to-\infty} e^u\right) = \frac{1}{\theta}(1-0) = \frac{1}{\theta}$$

(3) $$\int_0^\infty x e^{-x}\,dx$$ Integration by parts: $\int u\,dv = u\cdot v - \int v\,du$

$$\begin{array}{ll} u = x & dv = e^{-x}\,dx \quad \text{Integrate both sides} \\ du = dx & v = -e^{-x} \end{array}$$

, so

$$u\cdot v - \int v\,du = x(-e^{-x})\Big|_0^\infty - \int_0^\infty -e^{-x}\,dx$$

$$= \underbrace{(-1)\left(\lim_{x\to\infty}\frac{x}{e^x}\right)}_{=0 \text{ by L'Hôpital}} + \underbrace{\int_0^\infty e^{-x}\,dx}_{=1 \text{ by Problem 2}}$$

$$= 1$$

Assignment 1 Continued

(4) $\frac{d}{dx}\left(x e^x\right) = u'v + v'u$ Product rule

$= e^x + e^x \cdot x = e^x(1+x)$

(5) $\frac{d}{dt}\left(\ln(1+e^x)\right)$ Chain rule

$= \frac{1}{1+e^x} \cdot \frac{d}{dx} e^x = \frac{e^x}{1+e^x}$

(6) $\frac{d}{dx} e^{-\frac{1}{2}(x-\mu)^2} = e^{-\frac{1}{2}(x^2-\mu)^2} \cdot -\frac{1}{2} 2(x-\mu)$

$\overset{set}{=} 0 \Rightarrow x - \mu = 0 \Rightarrow x = \mu$

Is it a minimum or maximum? Could differentiate again, but note that if $x < \mu$, $f' > 0$ increasing & if $x > \mu$, $f' \downarrow$ so it's a max.

(7) This is a special case of #8

Assignment 1 continued

(8) Let $S_j^n = a^j + a^{j+1} + \cdots + a^n$

$$a\, S_j^n = \qquad a^{j+1} + \cdots + a^n + a^{n+1}$$

$$\Rightarrow S_j^n - a S_j^n = a^j - a^{n-1} \Rightarrow (1-a) S_j^n = a^j - a^{n+1}$$

$$\Rightarrow S_j^n = \sum_{k=j}^{n} a^k = \frac{a^j - a^{n+1}}{1-a}, \text{ and}$$

$$\sum_{k=j}^{\infty} a^k = \lim_{n\to\infty} \sum_{k=j}^{n} a^k = \lim_{n\to\infty} \frac{a^j - a^{n+1}}{1-a}$$

$$= \frac{a^j}{1-a}$$

And for problem 7, $j = 0$ & $a = \frac{1}{2}$

$$\frac{\left(\frac{1}{2}\right)^0}{1 - \frac{1}{2}} = 2$$

Assignment 1 continued

(9) The Taylor series expanding $f(x)$ about x_0 is

$$f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots$$

Expanding e^λ about $\lambda = 0$,

$$e^\lambda = 1 + 1\cdot(\lambda - 0) + 1\cdot\frac{(\lambda-0)^2}{2!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}, \text{ so}$$

$$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda}$$

$$= 1$$

Assignment One Continued

(10)

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} \operatorname{Exp}\left\{\ln\left(1+\frac{x}{n}\right)^n\right\}$$

$$= \operatorname{Exp}\left\{\lim_{n\to\infty} n \ln\left(1+\frac{x}{n}\right)\right\}$$

$$= \operatorname{Exp}\left\{\underbrace{\lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{n^{-1}}}_{\text{Form } \frac{0}{0} \text{ L'Hôpital}}\right\}$$

$$= \operatorname{Exp}\left\{\lim_{n\to\infty} \frac{\frac{1}{1+x/n}\cdot x \cdot \cancel{-n^{-2}}}{\cancel{-n^{-2}}}\right\}$$

$$= \operatorname{Exp}\left\{x \lim_{n\to\infty} \frac{1}{1+\frac{x}{n}}\right\} = e^x$$

We were differentiating with respect to n, which only assumes values 1, 2, ..
One could replace n by a continuous variable like t if desired.