STA 256f19 Assignment Ten[1]

Please read Sections 4.2 and 4.4 in the text. Also, look over your lecture notes. The following homework problems are not to be handed in. They are preparation for the final exam. Use the formula sheet.

1. Let the continuous random variable X_n have density

$$f_X(x) = \begin{cases} \frac{n}{x^{n+1}} & \text{for } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

   Show $T_n \stackrel{p}{\rightarrow} 1$ from the definition.

2. Let $X_1, \ldots, X_n$ be independent Uniform(0,1) random variables with $T_n = \min(X_i)$.
   (a) Find $F_{T_n}(x)$.
   (b) Show $T_n \stackrel{p}{\rightarrow} 0$ from the definition.

3. Prove Markov's inequality for a discrete random variable.

4. Chebyshev's inequality is on the formula sheet, though it may not have been mentioned in lecture.
   (a) Use Markov's inequality to prove Chebyshev's inequality.
   (b) Let $X \sim$ Normal (0,1). Chebyshev's inequality says that $P\{|X| \geq 2\}$ can be no more than ______. The actual probability is ______.

5. Use Markov's inequality to prove the variance rule.

6. Use the variance rule to prove the (weak) Law of Large Numbers.

7. Let X be a random variable with expected value μ and variance σ^2, and let $Y_n = \frac{X}{n}$. Show $Y_n \stackrel{p}{\rightarrow} 0$.

8. Let $X_1, \ldots, X_n$ be a collection of independent gamma random variables with unknown parameter α, and known $\lambda = 6$. Find a random variable $T_n = g(\overline{X}_n)$ that converges in probability to α. The statistic T_n is a good way to estimate α from sample data.

9. Let X_n have a Poisson distribution with parameter $n\lambda$, where $\lambda > 0$. This means $E(X_n) = Var(X_n) = n\lambda$. Let $Y_n = \frac{X_n}{n}$. Does Y_n converge in probability to a constant? Answer Yes or No and "prove" your answer.

10. Let the discrete random variable X_n have probability mass function

$$p_{X_n}(x) = \begin{cases} 1/2 & \text{for } x = \frac{1}{n} \\ 1/2 & \text{for } x = 1 + \frac{1}{n} \\ 0 & \text{Otherwise} \end{cases}$$

   (a) Show that $X_n \stackrel{d}{\rightarrow} X \sim$ Bernoulli($\frac{1}{2}$). Hint: Consider $x < 0$, $0 < x < 1$ and $x > 1$ separately.
   (b) Show that $\lim_{n\rightarrow\infty} M_{X_n}(t) = M_X(t)$, which is another way to prove convergence in distribution.

11. Let $p_{X_n}(x) = \begin{cases} \frac{n+3}{2(n+1)} & \text{for } x = 0 \\ \frac{n-1}{2(n+1)} & \text{for } x = 1 \end{cases}$

   Show that $X_n \stackrel{d}{\rightarrow} X \sim$ Bernoulli($\theta = \frac{1}{2}$).

[1] Copyright information is at the end of the last page.

12. Proposition: If the sequence of discrete random variables $\{X_n\}$ converges in distribution to the discrete random variable X, then $\lim_{n\to\infty} p_{X_n}(x) = p_X(x)$ for all real x. Either prove this statement is true, or prove by a simple counter-example that it is not true in general.

13. Sometimes, a sequence of random variables does not converge in distribution to anything. Let X_n have a continuous uniform distribution on $(0, n)$. Clearly, $\lim_{n\to\infty} F_{X_n}(x) = 0$ for $x \leq 0$. Find $\lim_{n\to\infty} F_{X_n}(x)$ for $x > 0$. Is $\lim_{n\to\infty} F_{X_n}(x)$ continuous? Is it a cumulative distribution function?

14. Consider a *degenerate* random variable X, with $P(X = c) = 1$.

    (a) What is $F_X(x)$, the cumulative distribution function of X? Your answer must apply to all real x.

    (b) Give a formula for $M_X(t)$, the moment-generating function of X.

15. For $n = 1, 2, \ldots$, let X_n have a beta distribution with $\alpha = n$ and $\beta = 1$.

    (a) What is $\lim_{n\to\infty} F_{X_n}(x)$ for $x < 1$?

    (b) What is $\lim_{n\to\infty} F_{X_n}(x)$ for $x > 1$?

    (c) What do you conclude?

16. Show that if $T_n \stackrel{p}{\to} c$, then $T_n \stackrel{d}{\to} c$.

17. Show that if $T_n \stackrel{d}{\to} c$, then $T_n \stackrel{p}{\to} c$. To avoid unpleasant technicalities, you may assume that T_n is continuous.

18. Let X_n be a binomial (n, θ_n) random variable with $\theta_n = \frac{\lambda}{n}$, so that $n \to \infty$ and $\theta_n \to 0$ in such a way that the value of $n\,\theta_n = \lambda$ remains fixed. Using moment-generating functions, find the limiting distribution of X_n.

19. Use moment-generating functions to prove the Law of Large Numbers. It starts like this. Let $X_1, X_2, \ldots$ be a sequence of independent random variables with common moment-generating function $M(t)$, and let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.
$$\lim_{n\to\infty} M_{\overline{X}_n}(t) = \lim_{n\to\infty} \exp \ln M_{\overline{X}_n}(t) = \ldots$$

20. Let $X_1, \ldots, X_{64}$ be independent Poisson random variables with parameter $\lambda = 2$. Using the Central Limit Theorem, find the approximate probability that $P\left(\sum_{i=1}^{64} X_i > 100\right)$. Answer: 0.9925

21. Let $X_1, \ldots, X_n$ be independent random variables from an unknown distribution with expected value 5.1 and standard deviation 4.8. Find the approximate probability that the sample mean will be greater than 6 for $n = 25$. Answer: 0.1736

22. The "normal approximation to the binomial" says that if $X \sim$ Binomial(n, θ), then for large n,
$$Z = \frac{X - n\theta}{\sqrt{n\theta(1-\theta)}}$$
may be treated as standard normal to obtain approximate probabilities. Where does this formula come from? Hint: What is the distribution of a sum of independent Bernoulli random variables?

23. Suppose that $X_1, \ldots, X_{200}$ are independent continuous random variables with common density function

$$f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} .$$

   (a) Find $E(X_i)$ and $Var(X_i)$.

   (b) Define $S = X_1 + \ldots + X_{200}$. Using the central limit theorem, find $P(S > 70)$.

   (c) Now define

   $$Y_i = \begin{cases} 1 & \text{if } X_i > 0.9 \\ 0 & \text{if } X_i \leq 0.9 \end{cases} .$$

   and set $T = Y_1 + \ldots + Y_{200}$. Compute $P(T \geq 3)$. **Hint:** Y_i is a Bernoulli random of variable and T is a binomial random variable.

24. Suppose that $X_1, \ldots, X_{100}$ are independent continuous random variables from Uniform$(-1, 1)$. Find $P\left(\sum_{i=1}^{100} X_i^2 \leq 40\right)$.

25. Suppose that $X_1, \ldots, X_n$ are independent continuous random variables from Uniform$(0, \theta)$. Let $Y_n = \max(X_1, \ldots, X_n)$.

   (a) Prove that $\sqrt{Y_n} \stackrel{p}{\to} \sqrt{\theta}$.

   (b) Prove that $Z_n = n(\theta - Y_n) \stackrel{d}{\to} Z$, where $Z \sim$Exponential$(\lambda = 1/\theta)$.

26. Suppose that $X_1, \ldots, X_n$ are independent continuous random variables with cdf $F(x)$ and pdf $f(x)$. Let $Y_n = \max(X_1, \ldots, X_n)$. Prove that $Z_n = n(1 - F(Y_n)) \stackrel{d}{\to} Z$, where $Z \sim$Exponential$(\lambda = 1)$.
