Assignment 5

(2) $X \sim Exp(\lambda)$

(a) $F(x) = \int_0^x \lambda e^{-\lambda t} dt$ for $x > 0$ $\quad u = -\lambda t$, $du = -\lambda dt$

t	$u = -\lambda t$
x	$-\lambda x$
0	0

$= -\int_0^{-\lambda x} e^u du = \int_{-\lambda x}^0 e^u du$

$= e^u \Big|_{-\lambda x}^0 = e^0 - e^{-\lambda x} = 1 - e^{-\lambda x}$

so $F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$

(b) 2.4.3 $Z \sim Exp(4)$

(a) $P(Z > 5) = 1 - F(5) = 1 - (1 - e^{-4 \cdot 5}) = e^{-20}$

(b) $P(Z \geq -5) = 1$

(c) $P(Z^2 \geq 9) = P(Z \geq 3) = 1 - F(3)$

$= 1 - (1 - e^{-4 \cdot 3}) = e^{-12}$

NOT 4

(d) $P(Z^2 - 17 \geq 9) = P(Z^2 \geq 25) = P(Z \geq 5)$

$= 1 - (1 - e^{-4 \cdot 5}) = e^{-20}$ NOT e^{-25}

(3) 2.4.5 No. Densities can't be negative

(4) 2.4.7 $$1 = \int_0^M c x^2 \, dx = c \cdot \frac{x^3}{3} \Big|_0^M = \frac{cM^3}{3}$$

$$\Longrightarrow \frac{3}{M^3} = c$$

(5) 2.4.9 For $1 < x < 2$, $f(x) > g(x)$

$$\Longrightarrow \int_1^2 f(x)\, dx > \int_1^2 g(x)\, dx$$

$$\Longrightarrow P(1<X<2) > P(1<Y<2)$$

(6) 2.4.10 No, impossible.

$$f(x) > g(x) \Longrightarrow \int_{-\infty}^{\infty} f(x)\, dx > \int_{-\infty}^{\infty} g(x)\, dx$$

But if f is a density, $\int_{-\infty}^{\infty} f(x)\, dx = 1 > \int_{-\infty}^{\infty} g(x)\, dx$

and g is not a density.

⑦ 2.4.14–19

2.4.14 $P(Y-h \geq y \mid Y \geq h) = P(Y \geq h+y \mid Y \geq h)$

$$= \frac{P(Y \geq h+y \cap Y \geq h)}{P(Y \geq h)} = \frac{P(Y \geq h+y)}{P(Y \geq h)}$$

$$= \frac{1-F(h+y)}{1-F(h)} = \frac{e^{-\lambda(h+y)}}{e^{-\lambda h}} = \frac{\cancel{e^{-\lambda h}}\, e^{-\lambda y}}{\cancel{e^{-\lambda h}}}$$

$$= e^{-\lambda y}$$

2.4.15

(a) $\Gamma(\alpha+1) = \int_0^\infty e^{-t} t^{(\alpha+1)-1} dt$

$\int u\,dv = u \cdot v - \int v\,du$

Set $u = t^\alpha$ $\quad du = \alpha t^{\alpha-1} dt$

$dv = e^{-t} dt \quad v = -e^{-t}$

so $u \cdot v - \int v\,du = t^\alpha(-e^{-t})\Big|_0^\infty - \int_0^\infty (-e^{-t}) \alpha t^{\alpha-1} dt$

$= -\lim\limits_{t\to\infty} \dfrac{t^\alpha}{e^t} + \alpha \int_0^\infty e^{-t} t^{\alpha-1} dt$

$= \underbrace{-\lim\limits_{t\to\infty} \dfrac{t^\alpha}{e^t}} + \alpha\,\Gamma(\alpha)$

This is of the form $\frac{\infty}{\infty}$. Apply l'Hôpital's rule repeatedly until the exponent of t becomes negative. Then the limit is zero

$= \alpha\,\Gamma(\alpha)$

(b) $\Gamma(1)$ is the integral of an exponential density with $\lambda = 1$.

(c) Prove by induction. In part (b), we have seen that for $n = 1$, $\Gamma(n) = (n-1)! = 0! = 1$

Assume for n, show for $n+1$

$\Gamma(n+1) = n\,\Gamma(n)$ by part (a).

$= n\,(n-1)!$ by the induction hypothesis

$= n!$ $\blacksquare$

2.4.16 Because the standard normal is symmetric about zero

$$\int_{-\infty}^{\infty} \phi(x)\,dx = 2\int_0^{\infty} \phi(x)\,dx = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

Let $t = \frac{x^2}{2}$, so $x = \sqrt{2}\,t^{\frac{1}{2}}$, and $dx = \sqrt{2}\cdot\frac{1}{2}t^{-\frac{1}{2}}dt$

x	$t = \frac{x^2}{2}$
∞	∞
0	0

And the integral is $2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t} \sqrt{2}\,\frac{1}{2} t^{-\frac{1}{2}} dt$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-t}\, t^{\frac{1}{2}-1}\,dt$$

$$= \frac{1}{\Gamma(\frac{1}{2})}\cdot\Gamma(\tfrac{1}{2}) = 1$$

2.4.17 $\int_0^{\infty} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}\,dx$ Let $t = \lambda x$

so $x = \frac{1}{\lambda}t$ $dx = \frac{1}{\lambda}dt$

x	t
∞	∞
0	0

$$= \int_0^{\infty} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{-t} \left(\frac{t}{\lambda}\right)^{\alpha-1} \frac{1}{\lambda}\,dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-t} t^{\alpha-1}\,dt = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$$

2.4.18 The density is position and continuous, and

$$\int_{-\infty}^{\infty} \frac{e^{-x}}{(1+e^{-x})^2}\,dx \qquad \text{Let } u = 1+e^{-x},\quad du = -e^{-x}dx \qquad \begin{array}{c|c} x & u \\ \hline \infty & 1 \\ -\infty & \infty \end{array}$$

$$= -\int_{\infty}^{1} \frac{1}{u^2}\,du = \int_{1}^{\infty} u^{-2}\,du = \frac{u^{-1}}{-1}\Big|_{1}^{\infty}$$

$$= (-1)\left(\lim_{u\to\infty} \frac{1}{u} - 1\right) = 0 + 1 = 1$$

2.4.19 The density is non-negative, and has just one discontinuity at zero. Show it integrates to one

$$\int_0^{\infty} \alpha x^{\alpha-1} e^{-x^\alpha}\,dx \qquad \text{Let } u = x^\alpha,\ du = \alpha x^{\alpha-1}dx$$

$$\begin{array}{c|c} x & u \\ \hline \infty & \infty \\ 0 & 0 \end{array}$$

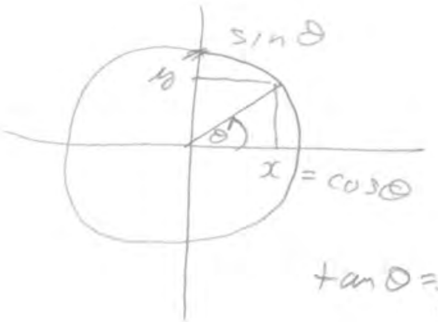
$$= \int_0^{\infty} e^{-u}\,du = 1$$

Because this is an exponential density with $\lambda = 1$

8 (2.4, 21) The density is non-negative & continuous, and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{1}{\pi}\tan^{-1}(x)\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{\pi}\left(\lim_{x\to\infty}\tan^{-1}(x) - \lim_{x\to-\infty}\tan^{-1}(x)\right)$$



As $\theta \to \frac{\pi}{2}$, $\tan\theta \to +\infty$

As $\theta \to -\frac{\pi}{2}$, $\tan\theta \to -\infty$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - -\frac{\pi}{2}\right) = 1$$

9 First, where is the density of Y going to be non-zero? With probability one, $1 \le X < \infty$ $\iff 1 \ge \frac{1}{X} > 0$ and $0 < Y \le 1$. What happens at the end points does not matter, so you could also say $0 \le Y \le 1$ or $0 < Y < 1$.

Now, for $0 < y < 1$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P\left(\frac{1}{X} \le y\right)$$

$$= \frac{d}{dy} P\left(X \ge \frac{1}{y}\right) = \frac{d}{dy}\left(1 - F_X(y^{-1})\right)$$

$$= -f_X\left(\frac{1}{y}\right)(-1)\, y^{-2}$$

$$= \frac{\alpha\, y^{\alpha+1}}{y^2} = \alpha\, y^{\alpha-1}, \text{ so}$$

$$f_Y(y) = \begin{cases} \alpha\, y^{\alpha-1} & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Notice that the beta density with $\beta = 1$ & general α is

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)}\, y^{\alpha-1}(1-y)^{1-1} = \frac{\alpha\, \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}\, y^{\alpha-1}$$

$$= \alpha\, y^{\alpha-1}$$

10 For $0 \le x \le 1$ $f(x) = F'(x) = 3x^2$, so

$$f(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Another beta with $\beta = 1$

11 (a) $f(\mu + x) - f(\mu - x)$

$$= \frac{1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{1}{2\sigma^2}(\mu+x-\mu)^2} - e^{-\frac{1}{2\sigma^2}(\mu-x-\mu)^2}\right)$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\left(e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2\sigma^2}}\right) = 0$$

b $-\infty < X < \infty \iff -\infty < \frac{X-\mu}{\sigma} < \infty$, so the support of Z is the whole real line.

$$f_Z(z) = \frac{d}{dz}P(Z \le z) = \frac{d}{dz}P\left(\frac{X-\mu}{\sigma} \le z\right)$$

$$= \frac{d}{dz}P(X \le \sigma z + \mu) = \frac{d}{dz}F_X(\sigma z + \mu) = f_X(\sigma z + \mu)\cdot\sigma$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \exp -\frac{1}{2\sigma^2}(\sigma z + \cancel{\mu} - \cancel{\mu})^2 \cdot \cancel{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$ standard normal

10c Density of Y is > 0 for all real y
There are 2 cases.

$a < 0$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \leq y) = \frac{d}{dy} P(aX+b \leq y)$$

$$= \frac{d}{dy} P(aX \leq y-b) = \frac{d}{dy} P\left(X \geq \frac{y-b}{a}\right)$$

$$= \frac{d}{dy}\left(1 - F_X\left(\frac{y-b}{a}\right)\right) = -f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

This is positive, since $a<0$

$$= \frac{1}{(-a\sigma)\sqrt{2\pi}} \operatorname{Exp} - \frac{1}{2a^2\sigma^2}\left(y - (\mu+b)\right)^2$$

$N(\mu+b, a^2\sigma^2)$

$a > 0$

$$f_Y(y) = \text{as before} = \frac{d}{dy} P(aX \leq y-b) = \frac{d}{dy} P\left(X \leq \frac{y-b}{a}\right)$$

$$= \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

As before

$$\downarrow$$

$$= \frac{1}{a\sigma\sqrt{2\pi}} \operatorname{Exp} - \frac{1}{2a^2\sigma^2}\left(y - (\mu+b)\right)^2$$

$N(\mu+b, a^2\sigma^2)$

10d) $F_Z(-x) = \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$ Let $y = -x$, $dy = -dx$

x	y
$-x$	x
$-\infty$	∞

$$= -\int_{\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{(-y)^2}{2}}\, dy$$

$$= \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy = 1 - F_Z(x)$$

e) $X \sim N(50, 100)$ so $\sigma = 10$

(i) $P(X < 60) = P\left(\frac{X-\mu}{\sigma} < \frac{60-50}{10}\right) = P(Z < 1)$

$= 1 - F_Z(-1) = 1 - 0.1587$
$= 0.8413$

(ii) $P(X > 30) = P\left(\frac{X-\mu}{\sigma} > \frac{30-50}{10}\right) = P(Z > -2)$

$= 1 - F_Z(-2) = 1 - 0.0228$
$= 0.9772$

(iii) $P(30 < X < 55) = P\left(\frac{30-50}{10} < \frac{X-\mu}{\sigma} < \frac{55-50}{10}\right)$
$= P(-2 < Z < \frac{1}{2})$

$= 1 - F_Z(-\frac{1}{2}) - F_Z(-2) = 1 - 0.3085 - 0.0228$
$= 0.6687$

(12)

10f By symmetry, $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$

(Set $t = \frac{z^2}{2} \Leftrightarrow z = \sqrt{2t}$ since z is positive

$z = \sqrt{2}\, t^{\frac{1}{2}} \Rightarrow dz = \sqrt{2}\, \frac{1}{2} t^{-\frac{1}{2}} dt$

z	$t = \frac{z^2}{2}$
∞	∞
0	0

)

$$= \frac{\cancel{2}}{\cancel{\sqrt{2}}\sqrt{\pi}} \int_0^{\infty} e^{-t} \cancel{\sqrt{2}}\, \cancel{\tfrac{1}{2}}\, t^{-\frac{1}{2}} dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt = \frac{1}{\sqrt{\pi}} \Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

g) $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$ Set $z = \frac{x-\mu}{\sigma}$ $dz = \frac{1}{\sigma} dx$

x	t
∞	∞
$-\infty$	∞

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1 \text{ by part (f)}$$

10 h) $Z \sim N(0,1)$ & $Y = Z^2$

(a) $-\infty < Z < \infty \Rightarrow 0 \le Y < \infty$ but endpoints don't matter

(b) $f_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(Z^2 \le y)$

$= \frac{d}{dy} P(|Z| \le y^{\frac{1}{2}}) = \frac{d}{dy} P(-y^{\frac{1}{2}} \le Z \le y^{\frac{1}{2}})$

$= \frac{d}{dy}\left(F_Z(y^{\frac{1}{2}}) - F_Z(-y^{\frac{1}{2}})\right)$

$= f_Z(y^{\frac{1}{2}}) \cdot \frac{1}{2} y^{-\frac{1}{2}} - f_Z(-y^{\frac{1}{2}}) \cdot (-1) \frac{1}{2} y^{-\frac{1}{2}}$

$= \frac{1}{2} y^{-\frac{1}{2}} \left(f_Z(y^{\frac{1}{2}}) + f_Z(-y^{\frac{1}{2}}) \right)$

symmetry of $N(0,1)$ density

$= \cancel{\frac{1}{2}} y^{-\frac{1}{2}} \cancel{2} f_Z(y^{\frac{1}{2}}) = y^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y}$

$= \frac{(1/2)^{1/2}}{\Gamma(\frac{1}{2})} e^{-\frac{1}{2} y} y^{\frac{1}{2} - 1}$ for $y \ge 0$, and zero otherwise

This is a Gamma($\alpha = \frac{1}{2}$, $\lambda = \frac{1}{2}$)

$\frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda y} y^{\alpha - 1}$

12 If $\lambda > 0$ & $X > 0$, then $Y = \lambda X > 0$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \leq y) = \frac{d}{dy} P(\lambda X \leq y)$$

$$= \frac{d}{dy} P(X \leq y/\lambda) = \frac{d}{dy} F_X(y/\lambda)$$

$$= f_X\left(\frac{y}{\lambda}\right) \cdot \frac{1}{\lambda} = \lambda e^{-\lambda \frac{y}{\lambda}} \cdot \frac{1}{\lambda}$$

$= e^{-y}$ for $y > 0$, and zero otherwise

13 $Y = F_X(X)$

(a) $-\infty < X < \infty \Rightarrow 0 < F_X(X) < 1 \Rightarrow 0 < Y < 1$

So for $y \in (0,1)$,

(b) $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(F_X(X) \le y)$

$= \frac{d}{dy} P(X \le F_X^{-1}(y)) = \frac{d}{dy} F_X(F_X^{-1}(y))$

$= \frac{d}{dy} y = 1$, and $f_Y(y) = \begin{cases} 1 & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

This is the continuous $U(0,1)$ density.

14 First, recall that $F_U(u) = u$ for $0 \le u \le 1$*. Then,

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(F_X^{-1}(U) \le y)$

$= \frac{d}{dy} P(F_X(F_X^{-1}(U)) \le F_X(y)) = \frac{d}{dy} P(U \le F_X(y))$

$\overset{*}{=} \frac{d}{dy} F_X(y) = f_X(y)$, the density of X!

So $Y = F_X^{-1}(U)$ has both the cdf and density of the RV X.