NAME (PRINT): Jerry
Last/Surname First /Given Name

STUDENT #: ______________ SIGNATURE: ______________

UNIVERSITY OF TORONTO MISSISSAUGA
DECEMBER 2018 FINAL EXAMINATION
STA256H5F
Probability and Statistics I
Jerry Brunner
Duration - 3 hours
Aids: Calculator Model(s): Any calculator with no wifi connection; Statistical Table and Formula Sheet supplied

The University of Toronto Mississauga and you, as a student, share a commitment to academic integrity. You are reminded that you may be charged with an academic offence for possessing any unauthorized aids during the writing of an exam. Clear, sealable, plastic bags have been provided for all electronic devices with storage, including but not limited to: cell phones, SMART devices, tablets, laptops, calculators, and MP3 players. Please turn off all devices, seal them in the bag provided, and place the bag under your desk for the duration of the examination. You will not be able to touch the bag or its contents until the exam is over.

If, during an exam, any of these items are found on your person or in the area of your desk other than in the clear, sealable, plastic bag, you may be charged with an academic offence. A typical penalty for an academic offence may cause you to fail the course.

Please note, once this exam has begun, you ***CANNOT*** *re-write it.*

Question	Value	Points		Question	Value	Points
1	4			7	18	
2	4			8	12	
3	10			9	10	
4	8			10	8	
5	8			11	8	
6	10			Total Marks = 100 Points		

4 points

1. Ten students are standing in line. If they lined up completely at random, what is the probability that Romeo is standing next to Juliet? The answer is a number. Circle your answer.

$$\frac{9! * 2}{10!} = \frac{2}{10} = \frac{1}{5}$$

4 points

2. A box of 25 Valentine's Day chocolates has 10 that are cream filled and 15 that are not cream filled. If you eat 7 chocolates at random, what is the probability that you get exactly 2 cream filled? Just write down the answer. There is no need to simplify.

$$\frac{\binom{10}{2}\binom{15}{5}}{\binom{25}{7}}$$

Continued on page 3

10 points

3. In this question, you will show that if the events A and B are independent, then A^c and B^c are independent.

   (a) In symbols, *not words, but symbols*, what are you trying to show? If you don't have this right, it is hard to imagine how you could get any marks.

   $$P(A^c \cap B^c) = P(A^c)P(B^c)$$

   (b) Now do the proof. You do *not* have to use the tabular format from the first part of the course. You will be using several facts from the formula sheet, but you don't have to specifically cite them.

$$P(A^c \cap B^c) = P\{(A \cup B)^c\} = 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A)P(B)$$ By independence, but they don't have to say it.

Now $P(A^c)P(B^c) = (1 - P(A))(1 - P(B))$

$$= 1 - P(A) - P(B) + P(A)P(B)$$

same.

Continued on page 4

Continue the answer to Question 3 if necessary.

Continued on page 5

8 points

4. Let X have a Uniform(0,1) distribution, and let $Y = -\ln X$. Find the density $f_Y(y)$. Show your work. In your final answer, do not forget to specify where the density of Y is non-zero.

$$f_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(-\ln X \le y)$$ 2 points to here

$$= \frac{d}{dy} P(\ln X \ge -y) = \frac{d}{dy} P(X \ge e^{-y})$$

$$= \frac{d}{dy}\left(1 - F_X(e^{-y})\right) = -f_X(e^{-y}) \cdot e^{-y} \cdot -1$$

$$= f_X(e^{-y}) e^{-y}$$

Nonzero iff $0 < e^{-y} < 1 \iff -\infty < -y < 0$

$\iff \infty > y > 0$ (They don't have to show this work as long as the answer below is right.)

So

$$f_Y(y) = \begin{cases} e^{-y} & \text{for } y \geq 0 \\ 0 & \text{for } y < 0 \end{cases}$$

5 marks

3 marks

Continued on page 6

8 points

5. There is a continuous version of Bayes' Theorem, which says $f_{y|x}(y|x) = \frac{f_{x|y}(x|y)f_y(y)}{\int_{-\infty}^{\infty} f_{x|y}(x|t)f_y(t)\,dt}$. Prove it. It's helpful to start with the right-hand side.

$$\frac{f_{X|Y}(x|y)\,f_Y(y)}{\int_{-\infty}^{\infty} f_{X|Y}(x|t)\,f_Y(t)\,dt} = \frac{\frac{f_{XY}(x,y)}{\cancel{f_Y(y)}}\cancel{f_Y(y)}}{\int_{-\infty}^{\infty}\frac{f_{XY}(x,t)}{\cancel{f_Y(t)}}\cancel{f_Y(t)}\,dt}$$

2 pts

3 pts

$$= \frac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{XY}(x,t)\,dt} = \frac{f_{XY}(x,y)}{f_X(x)} = f_{Y|X}(y|x)$$

3 pts

Continued on page 7

10 points 6. Let X have a beta distribution with parameters α and β. Calculate $E(X)$. Simplify! For full marks, no gamma functions should appear in your answer. Remember that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. **Circle your answer**.

$$E(X) = \int_0^1 x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$ one pt

2 more

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}} \frac{\Gamma(\alpha+1)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 x^{(\alpha+1)-1}(1-x)^{\beta-1}\,dx$$

3: Integral = 1

$= 1$

$$= \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)} = \frac{\alpha\cancel{\Gamma(\alpha)}\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}}$$

4 pts for simplifying gammas

$$= \frac{\alpha}{\alpha+\beta}$$

Continued on page 8

18 points

7. The random variables X_1 and X_2 are independent. X_1 has a gamma distribution with parameters $\alpha = a$ and $\lambda = 1$, and X_2 has a gamma distribution with parameters $\alpha = b$ and $\lambda = 1$. Let $Y_1 = \frac{X_1}{X_1+X_2}$ and $Y_2 = X_1 + X_2$.

   (a) Give the joint density of Y_1 and Y_2. Factor, separating y_1 and y_2 as much as possible. In your final statement of the answer to this part, *specify where the joint density is non-zero.*

$y_1 = \dfrac{x_1}{x_1+x_2}$ $\quad x_1 = y_1 y_2 \quad \dfrac{\partial x_1}{\partial y_1} = y_2 \quad \dfrac{\partial x_1}{\partial y_2} = y_1$

$y_2 = x_1 + x_2$ $\quad x_2 = y_2 - x_1 = y_2 - y_1 y_2 \quad \dfrac{\partial x_2}{\partial y_1} = -y_2 \quad \dfrac{\partial x_2}{\partial y_2} = 1 - y_1$

$\det = y_2 - y_1 y_2 - - y_1 y_2 = y_2$

So $f_{Y_1,Y_2}(y_1, y_2)$

$$= \frac{1}{\Gamma(a)} e^{-y_1 y_2} (y_1 y_2)^{a-1} \cdot \frac{1}{\Gamma(b)} e^{-(y_2 - y_1 y_2)} (y_2 - y_1 y_2)^{b-1} \cdot y_2$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} e^{-y_1 y_2} e^{-y_2} e^{y_1 y_2} \cdot y_1^{a-1} y_2^{a-1} y_2^{b-1} (1-y_1)^{b-1} y_2$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} y_1^{a-1} (1-y_1)^{b-1} e^{-y_2} y_2^{(a+b)-1}$$

for $0 \le y_1 \le 1$ and $y_2 \ge 0$

zero otherwise

Continued on page 9

Continue the answer to Question 7a if necessary.

Continued on page 10

(b) Find $f_{y_1}(y_1)$, the marginal density of Y_1. Again, do not forget to specify where the density is non-zero.

$$f_{Y_1}(y_1) = \int_0^\infty f_{Y_1,Y_2}(y_1, y_2)\, dy_2$$

$$= \int_0^\infty \frac{1}{\Gamma(a)\,\Gamma(b)} y_1^{a-1}(1-y_1)^{b-1} e^{-y_2} y_2^{(a+b)-1}\, dy_2$$

$$= \frac{1}{\Gamma(a)\,\Gamma(b)} y_1^{a-1}(1-y_1)^{b-1}\, \Gamma(a+b)$$

$$* \underbrace{\int_0^\infty \frac{1}{\Gamma(a+b)} e^{-y_2} y_2^{(a+b)-1}\, dy_2}_{=1}$$

$$= \begin{cases} \dfrac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} y_1^{a-1}(1-y_1)^{b-1} & \text{for } 0 \le y_1 \le 1 \\ 0 & \text{otherwiso} \end{cases}$$

(c) Identify the distribution of Y_1 by name; it is on the formula sheet.

Beta (a, b)

12 points 8. Let X have a binomial distribution with parameters n and p.

8 (a) Derive the moment-generating function of X. Show your work.

$$M_X(t) = E(e^{Xt}) = \sum_{x=0}^{n} e^{xt} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

Binomial Formula

$$\stackrel{\downarrow}{=} (pe^t + 1 - p)^n$$

4 (b) Use the moment-generating function (which is also on the formula sheet) to obtain $E(X)$. Show some work and **circle your answer.**

$$M'(t) = n(pe^t + 1 - p)^{n-1} \cdot pe^t$$ set $t = 0$ and

$$E(X) = n \cdot 1 \cdot p \cdot 1 = np$$

Continued on page 12

10 points

9. Let $X_1, \dots, X_n$ be independent exponential random variables with parameter $\lambda > 0$. The sample mean is $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the distribution of $\overline{X}$. Show your work. Identify the distribution by name; it is on the formula sheet. *What are the parameters?* Note that this question is asking for the *exact* distribution of $\overline{X}$. It is not a Central Limit Theorem problem.

$$M_{\bar{X}}(t) = M_{\frac{1}{n}\Sigma X_i}(t) = M_{\Sigma X_i}\left(\frac{t}{n}\right)$$

No need to mention independence

$$= \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{n}\right) = \prod_{i=1}^{n}\left(1 - \frac{t}{n\lambda}\right)^{-1}$$

$$= \left(1 - \frac{t}{n\lambda}\right)^{-n} \quad \text{MGF of Gamma}$$

So $\bar{X} \sim \text{Gamma}(\alpha = n, \lambda = \lambda n)$

Continued on page 13

8 points

10. Question 9 was not a Central Limit Theorem problem, but this one is. In Question 9, the exponential distribution of X_i means that $E(X_i) = \frac{1}{\lambda}$ and $Var(X_i) = \frac{1}{\lambda^2}$. If $\lambda = \frac{1}{2}$ and $n = 49$, find the approximate probability that $\overline{X} > 2.5$. The answer is a number. Circle your answer.

$$P(\bar{X} > 2.5) = P\left(\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma} > \frac{\sqrt{49}(2.5-2)}{2} \right)$$

$$= P(Z_n > 1.75) \approx 1 - \Phi(1.75)$$

$$= 1 - 0.9599 = 0.0491$$

Continued on page 14

8 points 11. Let $X_1, \ldots, X_n$ be independent random variables from a distribution with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$. Prove the Law of Large Numbers, which says that for all $\epsilon > 0$, $\lim_{n\to\infty} P(|\overline{X}_n - \mu| \geq \epsilon) = 0$.

You may use the facts that $E(\overline{X}_n) = \mu$ and $Var(\overline{X}_n) = \frac{\sigma^2}{n}$.

Because $E(\bar{X}) = \mu$ and $SD(\bar{X}) = \sqrt{\sigma^2/n} = \frac{\sigma}{\sqrt{n}}$,

Chebyshev says

$$P\left(|\bar{X}_n - \mu| \geq k\frac{\sigma}{\sqrt{n}}\right) \leq \frac{1}{k^2}$$

Setting $k\frac{\sigma}{\sqrt{n}} = \varepsilon$ so that $k = \frac{\varepsilon\sqrt{n}}{\sigma}$

A different k for each n but that's okay – They don't have to say this

$\frac{1}{k^2} = \frac{\sigma^2}{\varepsilon^2 n}$, and

$$0 \leq P\{|\bar{X}_n - \mu| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2 n} \downarrow 0$$

Squeeze