Sample Questions: Moment-generating functions

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1. Let X have a moment-generating function $M_x(t)$ and let a be a constant. Show $M_{ax}(t) = M_x(at)$.

$$M_{ax}(t) = E\left(e^{aXt}\right) = E\left(e^{X(at)}\right)$$

$$= M_X(at)$$

2. Let X have a moment-generating function $M_x(t)$ and let a be a constant. Show $M_{a+x}(t) = e^{at} Mx(t)$.

$$M_{a+X}(t) = E\left(e^{(a+X)t}\right)$$

$$= E\left(e^{at+Xt}\right) = E\left(e^{at}e^{Xt}\right)$$

$$= e^{at}E\left(e^{Xt}\right) = e^{at}M_X(t)$$

3. Let X and Y be independent, (continuous) random variables. Show $M_{x+y}(t) = M_x(t)\, M_y(t)$.

$$M_{X+Y}(t) = E\left(e^{(X+Y)t}\right) = E\left(e^{Xt+Yt}\right)$$

$$= E\left(e^{Xt} e^{Yt}\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{xt} e^{yt} f_{XY}(x,y)\, dx\, dy$$

$$\overset{\text{ind}}{\downarrow}$$

$$= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} e^{xt} e^{yt} f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} e^{yt} f_Y(y) \left(\int_{-\infty}^{\infty} e^{xt} f_X(x)\, dx \right) dy$$

$$= M_X(t) \int_{-\infty}^{\infty} e^{yt} f_Y(y)\, dy = M_X(t) M_Y(t)$$

4. Let $X \sim N(0,1)$. Calculate $M_x(t)$.

$$M_X(t) = E(e^{Xt}) = \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x - 2xt + t^2 - t^2)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2xt + t^2)} e^{-\frac{1}{2}(-t^2)} dx$$

$$= e^{\frac{1}{2}t^2} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} dx}_{=1}$$

$$= e^{\frac{1}{2}t^2}$$

5. Let $X \sim N(\mu, \sigma)$. Calculate $M_x(t)$. $Z = \frac{X-\mu}{\sigma}$ $M_Z(t) = e^{\frac{1}{2}t^2}$

$M_X(t) = E(e^{Xt})$ $Z = \frac{X-\mu}{\sigma}$

$X = Z\sigma + \mu$

$= E(e^{(Z\sigma+\mu)t})$

$= E(e^{Z\sigma t} e^{\mu t}) = e^{\mu t} E(e^{Z(\sigma t)})$

$= e^{\mu t} M_Z(\sigma t)$

$= e^{\mu t} e^{\frac{1}{2}\sigma^2 t^2}$

$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$M'(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \cdot (\mu + \frac{1}{2}\sigma^2 2t) \Big|_{t=0}$

$= \mu$

6. Let $X \sim N(\mu, \sigma)$. Show $Y = \frac{X-\mu}{\sigma} \sim N(0,1)$ using moment-generating functions.

$$M_Y(t) = E(e^{Yt}) = E\left(e^{\left(\frac{X-\mu}{\sigma}\right)t}\right)$$

$$= E\left(e^{\left(\frac{Xt}{\sigma} - \frac{\mu}{\sigma}t\right)}\right) = E\left(e^{X\left(\frac{t}{\sigma}\right)} e^{-\frac{\mu t}{\sigma}}\right)$$

$$= e^{-\frac{\mu t}{\sigma}} E\left(e^{X\left(\frac{t}{\sigma}\right)}\right) = e^{-\frac{\mu t}{\sigma}} M_X\left(\frac{t}{\sigma}\right)$$

$$= e^{-\frac{\mu t}{\sigma}} e^{\mu\left(\frac{t}{\sigma}\right) + \frac{1}{2}\sigma^2\left(\frac{t^2}{\sigma^2}\right)}$$

$$= \cancel{e^{-\frac{\mu t}{\sigma}}} \cancel{e^{\frac{\mu t}{\sigma}}} e^{\frac{1}{2}\cancel{\sigma^2}\frac{t^2}{\cancel{\sigma^2}}}$$

$$= e^{\frac{1}{2}t^2}$$ MGF of N(0,1)

This handout was prepared by Jerry Brunner, Department of Mathematical and Computational Sciences, University of Toronto. The LaTeX source code is available from the course website:

http://www.utstat.toronto.edu/~brunner/oldclass/256f18

$$M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

7. Let $X \sim N(\mu, \sigma)$. Find the distribution of $Y = a + bX$ using moment-generating functions.

$$M_Y(t) = E(e^{Yt}) = E(e^{(a+bX)t})$$

$$= E(e^{at} e^{X(bt)}) = e^{at} E(e^{X(bt)})$$

$$= e^{at} M_X(bt) = e^{at} e^{\mu(bt) + \frac{1}{2}\sigma^2 (bt)^2}$$

$$= e^{(a+b\mu)t + \frac{1}{2}(b^2\sigma^2)t^2}$$

MGF of $N(a+b\mu, b\sigma)$ ✓

$$e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

8. Let $X_1 \sim N(\mu_1, \sigma_1)$ and $X_2 \sim N(\mu_2, \sigma_2)$ be independent. Find the distribution of $Y = aX_1 + bX_2 + c$.

$$M_Y(t) = E(e^{Yt}) = E\left(e^{(aX_1 + bX_2 + c)t}\right)$$

$$= E\left(e^{aX_1 t} e^{bX_2 t} e^{ct}\right)$$

$$= e^{ct} \underbrace{E\left(e^{aX_1 t}\right) E\left(e^{bX_2 t}\right)}_{\text{independence of } X_1 \,\&\, X_2 \,\&\, \text{hence of } aX_1 t \,\&\, bX_2 t}$$

$$= e^{ct} M_{aX_1}(t) M_{bX_2}(t)$$

$$= e^{ct} M_{X_1}(at) M_{X_2}(bt) \qquad \left(e^{\mu t + \frac{1}{2}\sigma^2 t^2}\right)$$

$$= e^{ct} e^{\mu_1 a t + \frac{1}{2}\sigma_1^2 a^2 t^2} e^{\mu_2 b t + \frac{1}{2}\sigma_2^2 (bt)^2}$$

$$e^{(a\mu_1 + b\mu_2 + c)t + \frac{1}{2}(a^2\sigma_1^2 + b^2\sigma_2^2)t^2}$$

MGF of

$$N\left(a\mu_1 + b\mu_2 + c, \sqrt{a^2\sigma_1^2 + b^2\sigma_2^2}\right)$$

9. Let $Z \sim N(0,1)$ and let $Y = Z^2$. Find the distribution of Y. Recall that the MGF of a chi-squared random variable is $M(t) = (1-2t)^{-\frac{\nu}{2}}$.

$$M_Y(t) = E(e^{Yt}) = E(e^{Z^2 t})$$

$$= \int_{-\infty}^{\infty} e^{z^2 t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} \, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} z^2 (1-2t)} \, dz$$

Integral of normal $= 1$

$$(1-2t)^{-\frac{1}{2}} \frac{1}{(1-2t)^{-\frac{1}{2}} \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{z^2}{(1-2t)^{-1}}} \, dz$$

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

$$= (1-2t)^{-\frac{1}{2}}$$ like $(1-2t)^{-\frac{\nu}{2}}$

MGF of $\chi^2(1)$

10. Independently for $i = 1, \ldots, n$, let $Y_i \sim \chi^2(\nu_i)$. Find the distribution of $W = \sum_{i=1}^n Y_i$.

$$M_W(t) = M_{\Sigma Y_i}(t) \overset{\text{ind}}{=} \prod_{i=1}^{n} M_{Y_i}(t)$$

$$= \prod_{i=1}^{n} (1-2t)^{-\frac{\nu_i}{2}} = (1-2t)^{-\frac{\sum_{i=1}^{n} \nu_i}{2}}$$

MGF of $\chi^2\left(\sum_{i=1}^{n} \nu_i\right)$

11. Independently for $i = 1, \ldots, n$, let $X_i \sim N(\mu_i, \sigma_i)$. What is the distribution of $Y = \sum_{i=1}^n \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2$? Justify your answer.

$\chi^2(n)$ because

$\frac{X_i - \mu_i}{\sigma_i} \sim N(0,1)$

square of standard normal is $\chi^2(1)$

sum of independent χ^2s is

$\chi^2(\sum \nu_i)$

so $\chi^2(n)$