Joint distributions Part Two

1 1. Let X and Y be continuous random variables. Prove that X and Y are independent if and only if $f_{xy}(x,y) = f_x(x)\, f_y(y)$. Assume densities are continuous.

(1) Assume X & Y are independent. Then

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) \overset{ind}{=} \frac{\partial^2}{\partial x \partial y} F_X(x) F_Y(y)$$

$$= \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_X(x) F_Y(y) = \frac{\partial}{\partial x} F_X(x) \frac{\partial}{\partial y} F_Y(y)$$

$$= f_X(x)\, f_Y(y) \quad \square$$

(2) Set $f_{XY}(x,y) = f_X(x)\, f_Y(y)$

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(s,t)\, dt\, ds$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} f_X(s) f_Y(t)\, dt\, ds = \int_{-\infty}^{x} f_X(s) \underbrace{\left(\int_{-\infty}^{y} f_Y(t)\, dt\right)}\, ds$$

$$= \int_{-\infty}^{x} f_X(s)\, F_Y(y)\, ds = F_Y(y) \int_{-\infty}^{x} f_X(s)\, ds$$

$$= F_Y(y)\, F_X(x) \quad \square$$

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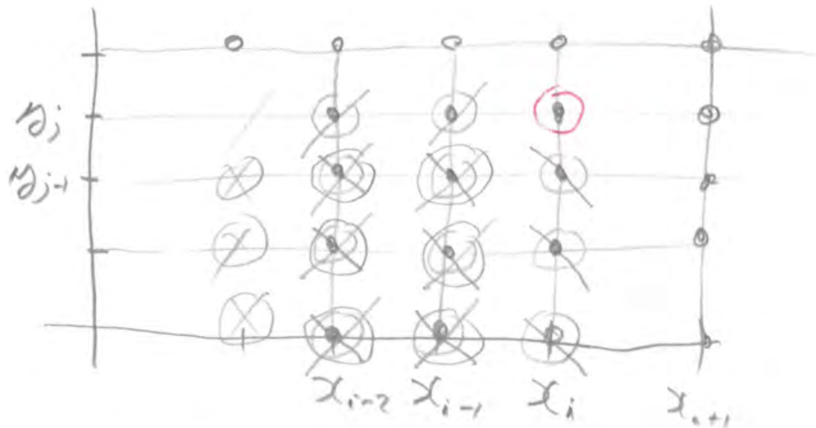
12. Let X and Y be discrete random variables. Prove that if $p_{xy}(x, y) = p_x(x)\, p_y(y)$, then X and Y are independent.

$$F_{XY}(x,y) = \sum_{s \le x} \sum_{t \le y} P_{XY}(s,t)$$

$$= \sum_{s \le x} \sum_{t \le y} P_X(s)\, P_Y(t) = \sum_{s \le x} P_X(s) \sum_{t \le y} P_Y(t)$$

$$= F_X(x)\, F_Y(y) \quad \square$$

13. Let X and Y be discrete random variables. Prove that if X and Y are independent, then $p_{xy}(x, y) = p_x(x)\, p_y(y)$.



$$P_{XY}(x_i, y_j) = F_{XY}(x_i, y_j) - F_{XY}(x_{i-1}, y_j) - F_{XY}(x_i, y_{j-1}) + F_{XY}(x_{i-1}, y_{j-1})$$

$$= F_X(x_i)F_Y(y_j) - F_X(x_{i-1})F_Y(y_j) - F_X(x_i)F_Y(y_{j-1}) + F_X(x_{i-1})F_Y(y_{j-1})$$

$$= \big(F_X(x_i) - F_X(x_{i-1})\big)F_Y(y_j) - \big(F_X(x_i) - F_X(x_{i-1})\big)F_Y(y_{j-1})$$

$$= P_X(x_i)\big(F_Y(y_j) - F_Y(y_{j-1})\big)$$

$$= P_X(x_i)\, P_Y(y_j)$$

4. Let $p_{xy}(x, y) = \frac{|x-2y|}{\cancel{19}\ 20}$ for $x = 1, 2, 3$ and $y = 1, 2, 3$, and zero otherwise.

		x: 1	2	3	
-2	y: 1	$1/20$	$0/20$	$1/20$	$2/20$
-4	2	$3/20$	$2/20$	$1/20$	$6/20$
-6	3	$5/20$	$4/20$	$3/20$	$12/20$
		$9/20$	$6/20$	$5/20$	$20/20$

(a) What is $p_{y|x}(1|2)$? $= \dfrac{P(X=2, Y=1)}{P(X=2)} = \dfrac{0/20}{6/20} = 0$

(b) What is $p_{x|y}(1|2)$? $= \dfrac{P(X=1, Y=2)}{P(Y=2)} = \dfrac{3/20}{6/20} = \dfrac{1}{2}$

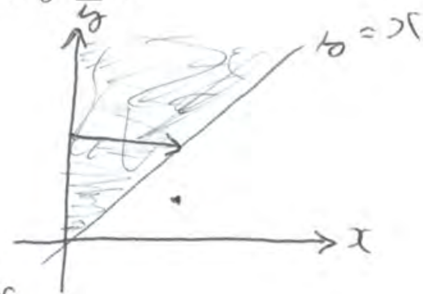
(c) Are x and y independent? Answer Yes or No and "prove" your answer.

NO, not independent, because

$$P_{xy}(2,1) \neq P_x(2)\,P_y(1) = \frac{6}{20} \cdot \frac{2}{20} \quad ✓$$

$$P_{xy}(2,1) = 0$$

5. Let $f_{x,y}(x,y) = \begin{cases} 2e^{-(x+y)} & \text{for } 0 \leq x \leq y \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$



(a) Find $f_{x|y}(x|y)$. $= \dfrac{f_{XY}(x,y)}{f_Y(y)}$

For $y \geq 0$, $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

$$= \int_0^y 2e^{-x}e^{-y}\,dx = 2e^{-y}\int_0^y e^{-x}\,dx \qquad \begin{array}{l} u = -x \\ du = -dx \end{array} \qquad \begin{array}{c|c} x & \\ y & -y \\ 0 & 0 \end{array}$$

$$= -2e^{-y}\int_0^{-y} e^u\,du = -2e^{-y}e^u\Big|_0^{-y}$$

$$= -2e^{-y}(e^{-y}-1) = 2e^{-y}(1-e^{-y}), \text{ so}$$

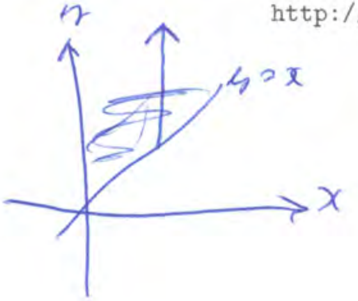
$$f_{x|y}(x|y) = \begin{cases} \dfrac{\cancel{2}e^{-x}\cancel{e^{-y}}}{\cancel{2}\cancel{e^{-y}}(1-e^{-y})} \\ 0 \end{cases} = \begin{cases} \dfrac{e^{-x}}{1-e^{-y}} & \text{for } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

for $y \geq 0$

(b) Are X and Y independent? Answer Yes or No and prove your answer.

No, not independent $f_{XY}(2,1) = 0 = f_X(2)f_Y(1)$

$$= 2e^{-2\cdot 2}\, 2e^{-1}(1-e^{-1}) \neq 0$$



$$f_X(x) = \int_x^\infty 2e^{-x}e^{-y}\,dy$$

$$= 2e^{-x}\int_x^\infty e^{-y}\,dy = 2e^{-2x} \text{ for } x \geq 0$$

6. Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent. Using the convolution formula, find the probability mass function of $Z = X + Y$ and identify it by name.

$$P_z(z) = \sum_x P_x(x) P_y(z-x) =$$

$$= \sum_{x=0}^{z} \frac{e^{-\lambda_1} \lambda_1^x}{x!} \cdot \frac{e^{-\lambda_2} \lambda_2^{z-x}}{(z-x)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{z!} \underbrace{\sum_{x=0}^{z} \frac{z!}{x!(z-x)!} \lambda_1^x \lambda_2^{z-x}}_{(\lambda_1+\lambda_2)^z}$$

$$= \begin{cases} \dfrac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^z}{z!} & \text{for } z = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Poisson $(\lambda_1 + \lambda_2)$

7. Let $X \sim \text{Binomial}(n_1, p)$ and $Y \sim \text{Binomial}(n_2, p)$ be independent. Using the convolution formula, find the probability mass function of $Z = X + Y$ and identify it by name.

$$P_Z(z) = \sum_x P_X(x) P_Y(z-x) -$$

$$= \sum_{x=0}^{z} \binom{n_1}{x} p^x (1-p)^{n_1-x} \binom{n_2}{z-x} p^{z-x} (1-p)^{n_2-(z-x)}$$

$$= \sum_{x=0}^{z} \binom{n_1}{x}\binom{n_2}{z-x} \cancel{p^x} (1-p)^{n_1} \cancel{(1-p)^{-x}} p^z \cancel{p^{-x}} (1-p)^{n_2-z} \cancel{(1-p)^{x}}$$

$$= \binom{n_1+n_2}{z} p^z (1-p)^{n_1+n_2-z} \underbrace{\sum_{x=0}^{z} \frac{\binom{n_1}{x}\binom{n_2}{z-x}}{\binom{n_1+n_2}{z}}}_{} \Big\} = 1$$

jar with n_1 red, n_2 blue
choose z w/o replacement
Get x red
sum $= 1$

$$= \begin{cases} \binom{n_1+n_2}{z} p^z (1-p)^{n_1+n_2-z} & z = 0, 1, \ldots n_1+n_2 \\ 0 & \text{otherwise} \end{cases}$$

Binomial (n_1+n_2, p)

8. Let X and Y be independent exponential random variables with parameter λ. Using the convolution formula, find the probability density function of $Z = X + Y$ and identify it by name. Gamma (α, λ) : $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)}\, dx$$

$z - x \geq 0 \quad x \leq z$

$$= \lambda^2 \int_0^z \cancel{e^{-\lambda x}} e^{-\lambda z} \cancel{e^{\lambda x}}\, dx$$

$$= \lambda^2 e^{-\lambda z} \int_0^z dx = \frac{\lambda^2}{\Gamma(2)} e^{-\lambda z} z^{2-1}$$

$$= \begin{cases} \frac{\lambda^2}{\Gamma(2)} e^{-\lambda z} z^{2-1} & \text{for } z > 0 \\ 0 & \text{otherwise} \end{cases}$$

Gamma $(\alpha = 2, \lambda)$

9. Let X_1 and X_2 be independent standard normal random variables. Find the probability density function of $Y_1 = X_1/X_2$.

$$y_1 = x_1/x_2 \quad \Rightarrow \quad x_1 = y_1 y_2$$
$$y_2 = x_2 \qquad\qquad x_2 = y_2$$

$$\text{abs} \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \text{abs} \begin{vmatrix} \frac{\partial x_1}{\partial y_1} = y_2 & \frac{\partial x_1}{\partial y_2} = y_1 \\ \frac{\partial x_2}{\partial y_1} = 0 & \frac{\partial x_2}{\partial y_2} = 1 \end{vmatrix}$$

$$= |y_2|$$

$$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(x_1(y_1, y_2), x_2(y_1, y_2)) \cdot \text{abs} \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y_1 y_2)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y_2^2} |y_2|$$

$$f_{Y_1}(y_1) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1+y_1^2) y_2^2} |y_2| \, dy_2$$

SYMMETRY

$$\stackrel{\downarrow}{=} 2 \frac{1}{2\pi} \int_0^{\infty} e^{-\frac{1}{2}(1+y_1^2) y_2^2} y_2 \, dy_2$$

$$u = \frac{1}{2} y_2^2 , \quad du = \cancel{\tfrac{1}{2}}\,\cancel{2}\, y_2 \, dy_2 \qquad \begin{array}{c|c} y_2 & u \\ \hline \infty & \infty \\ 0 & 0 \end{array}$$

$$= \frac{1}{\pi} \int_0^\infty e^{-\overset{\lambda}{(1+y_1^2)}u} \, du$$

$$= \frac{1}{\pi} \frac{1}{(1+y_1^2)} \underbrace{\int_0^\infty (1+y_1^2) e^{-(1+y_1^2)u} \, du}_{\substack{\text{Integral of exponential} \\ \lambda = (1+y_1^2)}} = 1$$

So

$$f_{Y_1}(y_1) = \frac{1}{\pi (1+y_1^2)}$$

Cauchy density

10. Use the Jacobian method to prove the convolution formula for continuous random variables.

X_1 & X_2 have joint density $f_{X_1 X_2}(x_1, x_2)$

$$= f_{X_1}(x_1) f_{X_2}(x_2)$$

$$Y_1 = X_1 + X_2 \implies X_1 = Y_1 - Y_2$$

$$Y_2 = X_2 \qquad X_2 = Y_2$$

$$\left| \begin{matrix} \frac{\partial x_1}{\partial y_1} = 1 & \frac{\partial x_1}{\partial y_2} = -1 \\ \frac{\partial x_2}{\partial y_1} = 0 & \frac{\partial x_2}{\partial y_2} = 1 \end{matrix} \right| = 1$$

And

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1 Y_2}(y_1, y_2)\, dy_2$$

$$= \int_{-\infty}^{\infty} f_{X_1}(y_1 - y_2)\, f_{X_2}(y_2) \cdot 1 \, dy_2$$

convolution formula

$$X_1 = Y_1 - Y_2 \,, \quad X_2 = Y_2$$

$$\frac{\partial x_1}{\partial y_1} = \frac{\partial}{\partial y_1}(y_1 - y_2) = \frac{\partial y_1}{\partial y_1} - \frac{\partial y_2}{\partial y_1} = 1 - 0$$

$$\frac{\partial x_1}{\partial y_2} = \frac{\partial}{\partial y_2}(y_1 - y_2) = 0 - 1$$

$$\frac{\partial x_2}{\partial y_1} = \frac{\partial}{\partial y_1} y_2 = 0$$

11. Show that the normal probability density function integrates to one.

$$\text{Let } I = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(t-\mu)^2}\, dt$$

$$\text{Let } x = \frac{t-\mu}{\sigma} \qquad dx = \frac{1}{\sigma}\, dt$$

x	t
∞	∞
$-\infty$	$-\infty$

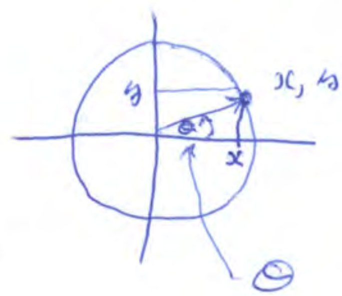
$$\text{So } I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx$$

Trick

$$I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}\, dx\, dy$$

"Change to polar co-ordinates"



$x_1 = x = r\, \cancel{\sin\theta} \cos\theta$

x_2 $y = r\sin\theta$ $\qquad x^2+y^2=r^2$

$\theta = y_1$, $r = y_2$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}r^2} \text{abs} \underbrace{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}}_{= r} dr\, d\theta$$

$$\begin{vmatrix} \dfrac{\partial x}{\partial r} = \dfrac{\partial}{\partial r} r\cos\theta & \dfrac{\partial x}{\partial \theta} = \dfrac{\partial}{\partial \theta} r\cos\theta \\[2ex] \dfrac{\partial y}{\partial y} = \dfrac{\partial}{\partial r} r\sin\theta & \dfrac{\partial y}{\partial \theta} = \dfrac{\partial}{\partial \theta} r\sin\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & r(-\sin\theta) \\[2ex] \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - r(-\sin^2\theta)$$

$$= r(\underbrace{\sin^2\theta + \cos^2\theta}_{=1}) = r$$

$x = r\cos\theta$

$y = r\sin\theta$

$$\begin{vmatrix} \dfrac{\partial x}{\partial r} = \cos\theta & \dfrac{\partial x}{\partial \theta} = -r\sin\theta \\[2em] \dfrac{\partial y}{\partial r} = \sin\theta & \dfrac{\partial y}{\partial \theta} = r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta - -r\sin^2\theta = r(\sin^2\theta + \cos^2\theta)$

$= r$, so

$$I^2 = \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}r^2} \, r\,dr\,d\theta$$

$u = \frac{1}{2}r^2 \quad du = r\,dr$

$$\begin{array}{c|c} r & u \\ \hline \infty & \infty \\ 0 & 0 \end{array}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\int_0^{\infty} e^{-u}\,du}_{=1} \, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta = \frac{2\pi}{2\pi} = 1 = I^2, \text{ so}$$

$I = 1$ (Because it can't be negative θ)

11. Show that the normal probability density function integrates to one.

$$I = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx$$

$$z = \frac{x-\mu}{\sigma} \qquad dz = \frac{1}{\sigma} dx$$

$$x = \sigma z + \mu$$

x	$z = \frac{x-\mu}{\sigma}$
∞	∞
$-\infty$	$-\infty$

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(z\sigma + \cancel{\mu - \mu})^2} dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{z^2\cancel{\sigma^2}}{\cancel{\sigma^2}}} dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

TRICK

$$I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dx\, dy$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$
$$dx\, dy = r\, dr\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}r^2}\, r\, dr\, d\theta$$

$u = \frac{1}{2} r^2 \quad du = \frac{1}{2} 2r\, dr = r\, dr$

r	$u = \frac{1}{2} r^2$
∞	∞
0	0

$$= \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\int_0^{\infty} e^{-u}\, du}_{\text{Exponatial, integral} = 1}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} 1\, d\theta \quad \frac{1}{2\pi}(2\pi - 0) = 1$$

$= I^2$ so $I = 1$ since $I > 0$

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, dt$$

12. Prove $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

$$x = t^{\frac{1}{2}}$$

$$t = x^2 \quad dt = 2x\,dx$$

t	x
∞	∞
0	0

$$\Gamma(\alpha) = \int_0^\infty e^{-x^2} (x^2)^{\alpha-1} \, 2x \, dx$$

$$= 2\int_0^\infty e^{-x^2} x^{2\alpha-2+1} \, dx = 2\int_0^\infty e^{-x^2} x^{2\alpha-1} \, dx$$

WITH $\alpha = \frac{1}{2}$ $= 2\int_0^\infty e^{-x^2} x^{1-1} \, dx = 2\int_0^\infty e^{-x^2} \, dx$

$$\Gamma\left(\tfrac{1}{2}\right)^2 = 4\int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$$

$$= 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$$

$$= 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2} \, r \, dr \, d\theta$$

This handout was prepared by Jerry Brunner, Department of Mathematical and Computational Sciences, University of Toronto. The LaTeX source code is available from the course website:

http://www.utstat.toronto.edu/~brunner/oldclass/256f18

$$= 2\int_0^{\pi/2}\underbrace{\int_0^{\infty} e^{-r^2}\, 2r\,dr}\, d\theta$$

$$u = r^2$$

$$du = 2r\,dr$$

r	u
∞	∞
0	0

$$= 2\int_0^{\pi/2}\underbrace{\int_0^{\infty} e^{-u}\,du}_{=1}\, d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\,\frac{\pi}{2} = \pi = \Gamma\left(\tfrac{1}{2}\right)^2$$

$$\Longrightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

13. Let $X_1, \ldots, X_n$ be independent random variables with probability density function $f(x)$ and cumulative distribution function $F(x)$. Let $Y = \max(X_1, \ldots, X_n)$. Find the density $f_y(y)$.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(\max_i X_i \le y)$$

all less than y

$$= \frac{d}{dy} P\left(\bigcap_{i=1}^{n} \{X_i \le y\}\right) \overset{\text{ind}}{=} \frac{d}{dy} \prod_{i=1}^{n} P(X_i \le y)$$

$$= \frac{d}{dy} \prod_{i=1}^{n} F(y) = \frac{d}{dy} F(y)^n =$$

$$= n F(y)^{n-1} f(y)$$

14. Let $X_1, \ldots, X_n$ be independent random variables with probability density function $f(x) = e^{-x}$ for $x \geq 0$. Let $Y = \max(X_1, \ldots, X_n)$. Find the density $f_y(y)$.

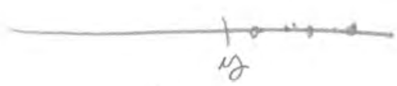
From #13, for $y \geq 0$

$$f_Y(y) = n\, F_X(y)^{n-1} f(y) \qquad F(x) = 1 - e^{-x} \text{ for } x \geq 0$$

$$= \begin{cases} n(1-e^{-y})^{n-1} e^{-y} & \text{for } y \geq 0 \\ 0 & \text{for } y < 0 \end{cases}$$

15. Let $X_1, \ldots, X_n$ be independent random variables with probability density function $f(x)$ and cumulative distribution function $F(x)$. Let $Y = \min(X_1, \ldots, X_n)$. Find the density $f_y(y)$.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \leq y) = \frac{d}{dy} \underline{P\left[\min_i X_i \leq y\right]}$$



$$= \frac{d}{dy}\left(1 - P(\min_i X_i > y)\right)$$

$$= -\frac{d}{dy} P\left(\bigcap_{i=1}^{n} \{X_i > y\}\right) \quad \text{All bigger}$$

$$\stackrel{\text{ind}}{=} -\frac{d}{dy} \prod_{i=1}^{n} P(X_i > y) = -\frac{d}{dy} \prod_{i=1}^{n} (1 - F(y))$$

$$= -\frac{d}{dy}(1 - F(y))^n = -n(1 - F(y))^{n-1}(-1) f(y)$$

$$= n(1 - F(y))^{n-1} f(y) \checkmark$$
