Sample Questions: Conditional Probability[1]

1. The table below shows percentages of passengers on the Titanic.

	Died	Lived	
1st Class	.09	.15	.24
2nd Class	.13	.09	.22
3d Class	.40	.14	.54
	.62	.38	1.00

For a randomly chosen passenger, what is

(a) The probability of living?

.38

(b) The probability of living

i. Given 1st class?

$$P(L \mid 1) = \frac{P(L \cap 1)}{P(1)} = \frac{.15}{.24} = .625$$

ii. Given 2nd class?

$$P(L/2) = \frac{P(L \cap 2)}{P(2)} = \frac{.09}{.22} = .409$$

iii. Given 3d class?

$$P(L \mid 3) = \frac{P(L \cap 3)}{P(3)} = \frac{.14}{.54} = .259$$

(c) The probability of being in first class given that the person died?

$$P(1 \mid D) = \frac{P(1 \cap D)}{P(D)} = \frac{.09}{.62} = .145$$

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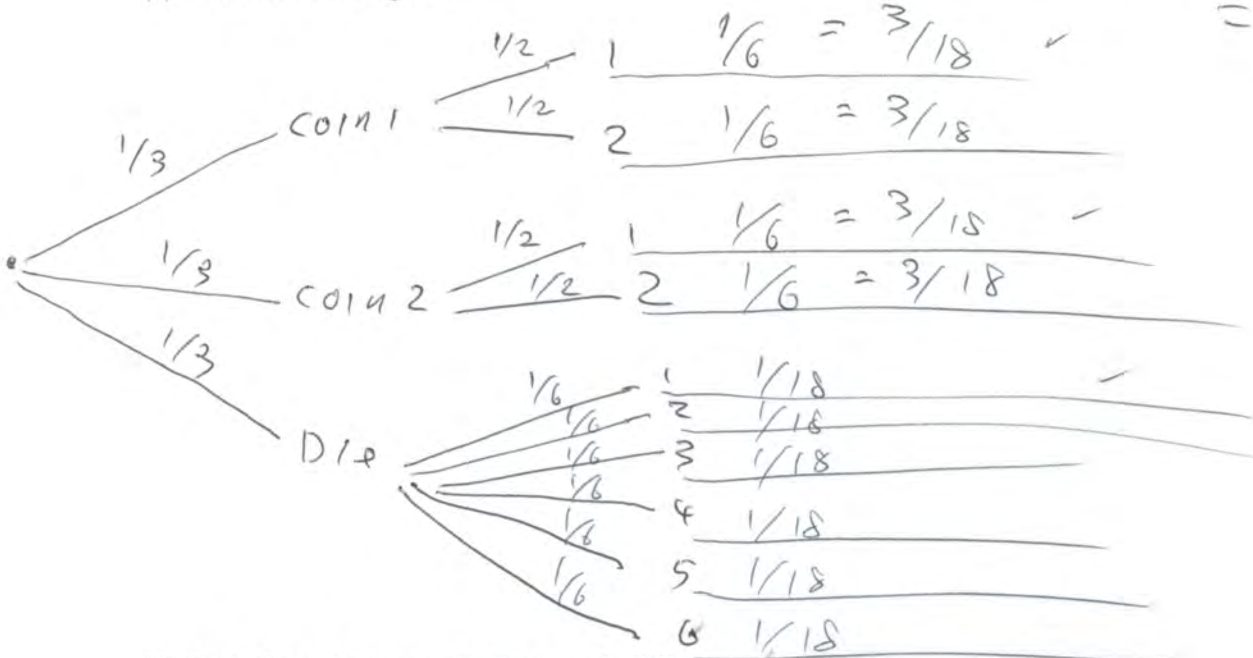
2. A jar contains two fair coins and one fair die. The coins have a "1" on one side and a "2" on the other side. Pick an object at random, roll or toss, and observe the number.

(a) What is $P(2 \cap C)$?

$$P(2 \cap C) = P(2|C)P(C) = \frac{1}{2} \cdot \frac{2}{3} = \cancel{\frac{1}{6}}\ \frac{1}{3}$$

(b) What is $P(6|C)$? $= 0 = \dfrac{P(6 \cap C)}{P(C)} = \dfrac{P(\emptyset)}{P(C)} = \dfrac{0}{P(C)} = 0$

(c) Make a tree diagram.



(d) List the outcomes with their probabilities.

1	2	3	4	5	6
$\frac{7}{18}$	$\frac{7}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$

(e) What is $P(C|2)$?

$$P(C|2) = \frac{P(C \cap 2)}{P(2)}$$

$$= \left(\frac{3}{18} + \frac{3}{18}\right) / \frac{7}{18} = \frac{6}{7}$$

3. Let $\Omega = \cup_{k=1}^{\infty} B_k$, disjoint, with $P(B_k) > 0$ for all k. Using the formula sheet and the tabular format illustrated in lecture, prove $P(A) = \sum_{k=1}^{\infty} P(A|B_k)P(B_k)$.

$A = A \cap \Omega$	Set logic
$= A \cap \bigcup_{k=1}^{\infty} B_k$	Substitution
$= \bigcup_{k=1}^{\infty} (A \cap B_k)$	Distributive law
Because the $(A \cap B_k)$ are disjoint	B_k are disjoint
$P(A) = P(\bigcup_{k=1}^{\infty} (A \cap B_k))$ $= \sum_{k=1}^{\infty} P(A \cap B_k)$	Axiom 3
$= \sum_{k=1}^{\infty} P(A \mid B_k) P(B_k)$	Multiplication Law

4. On the Titanic, 62.5% of the first-class passengers survived, 40.9% of the second class passengers survived, and 25.9% of the third class passengers survived. If 24% of the passengers were first class, 22% were second class and 54% were third class, what percent of the passengers survived overall?

$$P(S) = P(S|1)P(1) + P(S|2)P(2) + P(S|3)P(3)$$

$$= (.625)(.24) + (.409)(.22) + (.259)(.54)$$

$$= .38$$

5. Prove the following version of Bayes' Theorem. Let $\Omega = \cup_{k=1}^{\infty} B_k$, disjoint, with $P(B_k) > 0$ for all k. Then

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{k=1}^{\infty} P(A|B_k)P(B_k)}.$$

You may use anything from the formula sheet except Bayes' theorem itself.

$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} \quad \text{Def}$$

$$= \frac{P(A|B_j)P(B_j)}{P(A)} \quad \text{Mult Law}$$

$$= \frac{P(A|B_j)P(B_j)}{\sum_{k=1}^{\infty} P(A|B_k)P(B_k)} \quad \text{Law of total Prob.}$$

$\blacksquare$

6. Two balls are drawn in succession from a jar containing three red balls and four white balls. What is the probability that the first ball was white given that the second ball was red? The answer is a number. Circle your answer.

3 R
4 W

$$P(W_1 | R_2) = \frac{P(R_2 | W_1) P(W_1)}{P(R_2|W_1)P(W_1) + P(R_2|R_1)P(R_1)}$$

$$= \frac{\frac{1}{2} \cdot \frac{4}{\cancel{7}}}{\frac{1}{2} \cdot \frac{4}{\cancel{7}} + \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{7}}}$$

$$= \frac{2}{2+1} = \frac{2}{3}$$

7. This is an important real-world application of Bayes' Theorem. Suppose only one person in a thousand has some rare disease. We have a screening test for the disease, and it's a good test.

   - 90% of those with the disease test positive.
   - 95% of those without the disease test negative.

   Given a positive test, what is the probability that the person actually has the disease? The answer is a number. Circle your answer.

T = Test positive, D = Have disease

$P(D) = \frac{1}{1000}$, $P(D^c) = \frac{999}{1000}$

$P(T|D) = \frac{900}{1000}$, $P(T^c|D) = \frac{100}{1000}$

$P(T^c|D^c) = \frac{950}{1000}$, $P(T|D^c) = \frac{50}{1000}$

$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$

$$= \frac{\frac{900}{\cancel{1000}} \cdot \frac{1}{\cancel{1000}}}{\frac{900}{\cancel{1000}} \cdot \frac{1}{\cancel{1000}} + \frac{50}{\cancel{1000}} \cdot \frac{999}{\cancel{1000}}}$$

0.177