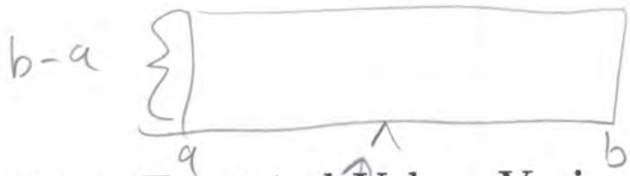


Sample Questions: Expected Value, Variance and Covariance

STA256 Fall 2018. Copyright information is at the end of the last page.

---

1. Let X have a continuous uniform distribution on (a, b). Calculate $E(X)$.

$$E(X) = \int_a^b x \frac{1}{b-a}\, dx = \frac{1}{b-a} \frac{x^2}{2}\Big|_a^b$$

$$= \frac{1}{2} \frac{1}{b-a} (b^2 - a^2) = \frac{1}{2} \frac{1}{\cancel{b-a}} (\cancel{b-a})(b+a)$$

$$= \frac{b+a}{2}$$

2. Let $X \sim \text{Poisson}(\lambda)$. Calculate $E(X)$.

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!} = \sum_{x=1}^{\infty} \cancel{x} \frac{e^{-\lambda}\lambda^{x}}{\cancel{x}(x-1)!}$$

$k = x-1$

$x = k+1$

x	$k = x-1$
∞	∞
1	0

$$= \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{k+1}}{k!} = \lambda \underbrace{\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{k}}{k!}}_{=1}$$

$$= \lambda$$

3. Let the continuous random variable X have density $f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$

(a) Verify that $f(x)$ integrates to one.

$$\int_1^\infty x^{-2}\,dx = \left.\frac{x^{-1}}{-1}\right|_1^\infty = (-1)\left.\frac{1}{x}\right|_1^\infty$$

$$= (-1)(0-1) = 1$$

(b) Calculate $E(X)$.

$$E(X) = \int_1^\infty x\,\frac{1}{x^2}\,dx = \int_1^\infty \frac{1}{x}\,dx$$

$$= \ln x \Big|_1^\infty = \infty$$

$E(X)$ does not exist

4. Let $X \sim N(\mu, \sigma)$. Calculate $E(X)$.

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$z = \frac{x-\mu}{\sigma} \qquad dz = \frac{1}{\sigma} dx$$

x	z
∞	∞
$-\infty$	$-\infty$

$$x - \mu = z\sigma$$
$$x = z\sigma + \mu$$

$$= \int_{-\infty}^{\infty} (z\sigma + \mu) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= \sigma \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz}_{=1}$$

$$= \mu + \sigma \int_{-\infty}^{0} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \sigma \int_{0}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$u = \frac{1}{2}z^2 \qquad du = z\,dz$$

=

x	u
0	0
$-\infty$	∞

x	$u = \frac{1}{2}z^2$
∞	∞
0	0

$$= \mu + \frac{\sigma}{\sqrt{2\pi}} \underbrace{\int_{\infty}^{0} e^{-u} du}_{-1} + \frac{\sigma}{\sqrt{2\pi}} \underbrace{\int_{0}^{\infty} e^{-u} du}_{=1} = \mu$$

5. Let X have a binomial distribution with parameters n and p. Calculate $E(X)$.

$$E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^{x}(1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \cancel{x} \frac{n!}{\cancel{x!}\,(x-1)!\,(n-x)!} p^{x}(1-p)^{n-x}$$

x	$k=x-1$
n	$n-1$
1	0

$k = x-1$
$x = k+1$

$$= \sum_{k=0}^{n-1} \frac{n(n-1)!}{k!\,(n-(k+1))!} p^{k+1}(1-p)^{n-(k+1)}$$

$$= np \underbrace{\sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} p^{k}(1-p)^{n-1-k}}_{=(p+(1-p))^{n} = 1}$$

$= np$

Toss fair coin 100 times. How many heads would you expect?

$np = 100 \cdot \frac{1}{2} = 50$

6. Let X have a Gamma distribution with parameters α and λ. Calculate $E(X^k)$.

$$E(X^k) = \int_0^\infty x^k \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}\,dx$$

$$= \frac{\cancel{\lambda^\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\cancel{\lambda^\alpha}\lambda^k} \underbrace{\int_0^\infty \frac{\lambda^{\alpha+k}}{\Gamma(\alpha+k)} e^{-\lambda x} x^{(\alpha+k)-1}\,dx}_{=1}$$

$$= \frac{\Gamma(\alpha+k)}{\lambda^k\,\Gamma(\alpha)}$$

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$$

For ~~ex~~ $k=1$

$$\frac{\Gamma(\alpha+1)}{\lambda^1\,\Gamma(\alpha)} = \frac{\alpha\,\cancel{\Gamma(\alpha)}}{\lambda\,\cancel{\Gamma(\alpha)}} = \frac{\alpha}{\lambda}$$

7. Let X and Y be independent (continuous) random variables. Show $E(XY) = E(X)E(Y)$.

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{XY}(x,y)\, dx\, dy$$

$$\overset{\text{ind}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} y\, f_Y(y) \Bigg(\underbrace{\int_{-\infty}^{\infty} x\, f_X(x)\, dx}_{E(X)} \Bigg)\, dy$$

$$= E(X) \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = E(X)E(Y)$$

$$E(a+X) = a + E(X) = a + \overset{\overset{E(X)}{\shortparallel}}{\mu}$$

8. Prove $Var(a+X) = Var(X)$.

$$Var(a+X) = E\{(a+X-(a+\mu))^2\}$$

$$= E\{(\cancel{a}+X-\cancel{a}-\mu)^2\}$$

$$= E\{(X-\mu)^2\} = Var(X)$$

$$E(bX) = b\mu$$

9. Prove $Var(bX) = b^2 Var(X)$.

$$Var(bX) = E\{(bX - b\mu)^2\}$$

$$= E\{(b(X-\mu))^2\} = E\{b^2(X-\mu)^2\}$$

$$= b^2 E\{(X-\mu)^2\} = b^2 Var(X)$$

10. Show $Var(X) = E(X^2) - [E(X)]^2$. $E(X) = \mu$

$Var(X) = E\{(X-\mu)^2\}$

$= E\{X^2 - 2X\mu + \mu^2\}$

$= E(X^2) - 2\mu E(X) + E(\mu^2)$

$= E(X^2) - 2\mu\mu + \mu^2$

$= E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$

11. Let $X \sim$ Uniform(0,1). Calculate $Var(X)$.

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X) = \int_0^1 x \cdot 1 \, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$E(X^2) = \int_0^1 x^2 \, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$E(X^2) - (E(X))^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4}$$

$$= \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$$

12. Let X have density e^{-x} for $x \geq 0$ and zero otherwise. Calculate $Var(X)$.

$$\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt, \quad \Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$$

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$$

$$E(X) = \int_0^\infty x\, e^{-x}\, dx = \int_0^\infty e^{-x}\, x^{2-1}\, dx = \Gamma(2) = 1\,\Gamma(1) = 1$$

$$E(X^2) = \int_0^\infty x^2 e^{-x}\, dx = \int_0^\infty e^{-x}\, x^{3-1}\, dx = \Gamma(3) = 2\,\Gamma(2) = 2$$

$$Var(X) = E(X^2) - (E(X))^2 = 2 - 1^2 = 1$$

13. Let $X \sim N(\mu, \sigma)$. Calculate $Var(X)$.

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\,dt$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$$

$$Var(X) = E\{(X-\mu)^2\}$$

$$= \sigma^2 \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sigma^2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad z = \frac{x-\mu}{\sigma}, \quad dz = \frac{1}{\sigma}\,dx$$

x	z
∞	∞
$-\infty$	$-\infty$

$$= \sigma^2 \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz$$

symmetric around $z = 0$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^\infty z\, e^{-\frac{1}{2}z^2}\, \underbrace{z\,dz}$$

$z = \sqrt{2}\sqrt{u}$

$u = \frac{1}{2}z^2 \qquad du = z\,dz$

z	u
∞	∞
0	0

$$= \frac{2\sigma^2}{\cancel{\sqrt{2}}\sqrt{\pi}} \int_0^\infty \cancel{\sqrt{2}}\, u^{\frac{1}{2}} e^{-u}\,du$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-u} u^{\frac{3}{2}-1}\,du = \frac{2\sigma^2}{\sqrt{\pi}} \underline{\Gamma\left(\tfrac{3}{2}\right)}$$

$$= \frac{\cancel{2}\sigma^2}{\cancel{\sqrt{\pi}}} \cancel{\tfrac{1}{2}} \cancel{\Gamma\left(\tfrac{1}{2}\right)} = \sigma^2$$

14. The discrete random variables x and y have joint distribution

	$x = 1$	$x = 2$	$x = 3$	
$y = 1$	3/12	1/12	3/12	7/12
$y = 2$	1/12	3/12	1/12	5/12
	$\frac{4}{12}$	$\frac{4}{12}$	$\frac{4}{12}$	

(a) What is $E(X|Y = 1)$?

$$= \sum_x x\, P(X=x \mid Y=1)$$

$$= 1 \cdot \frac{3/12}{7/12} + 2\, \frac{1/12}{7/12} + 3\, \frac{3/12}{7/12}$$

$$= \frac{1}{7}(3+2+9) = \frac{14}{7} = 2$$

(b) What is $E(Y^2|X = 2)$?

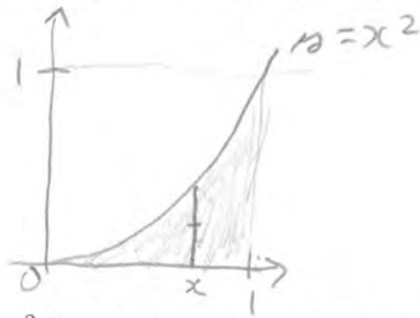
$$= \sum_y y^2\, P(Y=y \mid X=2)$$

$$= 1^2\, \frac{1/12}{4/12} + 2^2\, \frac{3/12}{4/12}$$

$$= \frac{1}{4} + \frac{12}{4} = \frac{13}{4}$$

	$x = 1$	$x = 2$	$x = 3$
$y = 1$	3/12	1/12	3/12
$y = 2$	1/12	3/12	1/12

15. Let $f_{x,y}(x, y) = 3$ for $0 < x < 1$ and $0 < y < x^2$, and zero otherwise.



(a) Using $f_x(x) = 3x^2$ for $0 < x < 1$, what is $f_{y|x}(y|x)$? Don't forget the support.

$$f_{Y|X}(y|x) = \begin{cases} \frac{\cancel{3}}{\cancel{3}x^2} & 0 \le y \le x^2 \\ 0 & \text{otherwise} \end{cases}$$

when $0 < x < 1$

(b) Find $E(Y|X = x)$, where x is a fixed constant between zero and one.

$$E(Y|X=x) = \int_0^{x^2} y \frac{1}{x^2}\, dy = \frac{1}{x^2}\int_0^{x^2} y\, dy$$

$$= \frac{1}{x^2} \frac{y^2}{2}\Big|_0^{x^2} = \frac{1}{x^2}\frac{1}{2}x^4 = \frac{x^2}{2}$$

(c) Find $E(Y)$ by double expectation. $E(Y) = E(E(Y|X))$

$$= E\left(\frac{X^2}{2}\right) = \int_0^1 \frac{x^2}{2}\, 3x^2\, dx$$

$$= \frac{3}{2}\int_0^1 x^4\, dx = \frac{3}{2}\, \frac{x^5}{5}\Big|_0^1 = \frac{3}{10}$$

$$= \int_{\sqrt{y}}^1 3\, dx$$

(d) Using $f_y(y) = 3(1 - y^{1/2})$ for $0 < y < 1$, calculate $E(Y)$ directly.

$$E(Y) = \int_0^1 3y(1 - y^{1/2})\, dy$$

$$= \int_0^1 3(y - y^{3/2})\, dy$$

$$= \frac{3y^2}{2}\Big|_0^1 - \frac{3y^{5/2}}{5/2}\Big|_0^1 = 3\left(\frac{1}{2} - \frac{2}{5}\right) = 3\left(\frac{5}{10} - \frac{4}{10}\right) = \frac{3}{10}$$

16. Show that $Cov(X, Y) = E(XY) - E(X)E(Y)$

$$Cov(X,Y) \overset{def}{=} E\{(X-\mu_x)(Y-\mu_y)\}$$

$$= E\{XY - X\mu_y - \mu_x Y + \mu_x \mu_y\}$$

$$= E(XY) - E(X)\mu_y - \mu_x E(Y) + \mu_x \mu_y$$

$$= E(XY) - \mu_x \mu_y \cancel{- \mu_x \mu_y + \mu_x \mu_y}$$

$$= E(XY) - E(X)E(Y)$$

17. Show that if X and Y are independent, $Cov(X, Y) = 0$.

Have seen if X & Y are independent $E(XY) = E(X)E(Y)$

So $Cov(X,Y) = E(XY) - E(X)E(Y)$

$$\overset{ind}{=} E(X)E(Y) - E(X)E(Y) = 0$$

18. The discrete random variables x and y have joint distribution

	$x=1$	$x=2$	$x=3$	
$y=1$	3/12	1/12	3/12	7/12
$y=2$	1/12	3/12	1/12	5/12
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

(a) Find $Cov(X,Y)$.

$E(X)=2$, $E(Y)=\sum_y y\,p(y) = 1\cdot\frac{7}{12}+2\cdot\frac{5}{12} = \frac{17}{12}$

y \ x	1	2	3
1	1	2	3
2	2	4	6

$$E(XY) = \frac{1}{12}\left(3+2+9+2+12+6\right) = \frac{34}{12}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{34}{12} - 2\cdot\frac{17}{12}$$

$$= \frac{34}{12} - \frac{34}{12} = 0$$

(b) Are X and Y independent?

$$p(1,1) \neq P_X(1)\,P_Y(1) = \frac{1}{3}\,\frac{7}{12} = \frac{7}{36}$$

$p(1,1) = \frac{3}{12}$

NO, NOT ind.

19. Show $Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$

$E(aX+bY) = a\mu_x + b\mu_y$

$Var(aX+bY) \overset{def}{=} E\{(aX+bY-(a\mu_x+b\mu_y))^2\}$

$= E\{(a(X-\mu_x)+b(Y-\mu_y))^2\}$

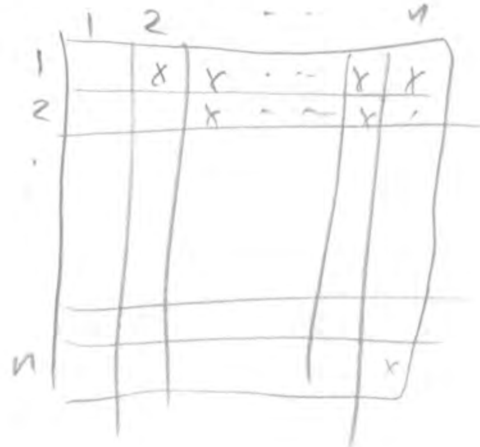
$= E\{a^2(X-\mu_x)^2 + b^2(Y-\mu_y)^2 + 2ab(X-\mu_x)(Y-\mu_y)\}$

$= a^2E(X-\mu_x)^2 + b^2E(Y-\mu_y)^2 + 2abE\{(X-\mu_x)(Y-\mu_y)\}$

$= a^2Var(X) + b^2Var(Y) + 2ab\,Cov(X,Y)$

$$\sum\sum_{i \neq j}$$

20. Prove that $Var\left(\sum_{i=1}^n a_iX_i\right) = \sum_{i=1}^n a_i^2 Var(X_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^n a_ib_jCov(X_i,X_j)$.



$$E\left(\sum_{i=1}^n a_iX_i\right) = \sum_{i=1}^n a_i\mu_i$$

$$Var\left(\sum_{i=1}^n a_iX_i\right) \stackrel{def}{=} E\left\{\left(\sum_{i=1}^n a_iX_i - \sum_{i=1}^n a_i\mu_i\right)^2\right\}$$

$$= E\left\{\left(\sum_{i=1}^n a_i(X_i-\mu_i)\right)^2\right\}$$

$$= E\{(a_1(X_1-\mu_1)+a_2(X_2-\mu_2)+\cdots+a_n(X_n-\mu_n))\cdot(a_1(X_1-\mu_1)+a_2(X_2-\mu_2)+\cdots+a_n(X_n-\mu_n))\}$$

$$= E\left\{\sum_{i=1}^n\sum_{j=1}^n a_ia_j(X_i-\mu_i)(X_j-\mu_j)\right\}$$

$$= \sum_{i=1}^n\sum_{j=1}^n a_ia_j E\{(X_i-\mu_i)(X_j-\mu_j)\}$$

$$= \sum_{i=1}^n a_i^2 Var(X_i) + \sum\sum_{i\neq j} a_ia_j Cov(X_i,X_j) \checkmark$$