

Counting Methods for Computing Probabilities¹

STA 256: Fall 2018

¹This slide show is an open-source document. See last slide for copyright information.

Countable set

A set is said to be *countable* if it can be placed in one-to-one correspondence with the set of natural numbers $\mathbb{N} = \{1, 2, \dots\}$.

If the sample space Ω is countable and $A \subseteq \Omega$,

$$P(A) = \sum_{\omega \in A} P(\omega)$$

Example: Roll a fair die. What is the probability of an odd number?

$$P(\text{Odd}) = P\{1, 3, 5\} = P\{1\} + P\{3\} + P\{5\}$$

If all outcomes of an experiment are equally likely,

$$P(A) = \frac{\text{Number of ways for } A \text{ to happen}}{\text{Total number of outcomes}}$$

Need to count.

Multiplication Principle

Also called the Fundamental Principle of Counting

If there are p experiments and the first has n_1 outcomes, the second has n_2 outcomes, etc., then there are

$$n_1 \times n_2 \times \cdots \times n_p$$

outcomes in all.

Sample Question

If there are nine horses in a race, in how many ways can they finish first, second and third?

$$9 \times 8 \times 7 = 504$$

Permutations

Ordered subsets

The number of *permutations* (ordered subsets) of n objects taken r at a time is

$$\begin{aligned} {}_n P_r &= n \times (n - 1) \times \cdots \times (n - r + 1) \\ &= \frac{n!}{(n - r)!} \end{aligned}$$

Combinations

Unordered subsets

The number of *combinations* (unordered subsets) of n objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Proof of $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Part of Proposition B in the text, p.12

Choose an unordered subset of r items from n . Then place them in order. By the Multiplication Principle,

$$\begin{aligned} {}_n P_r &= \binom{n}{r} \times r! \\ \Rightarrow \frac{n!}{(n-r)!} &= \binom{n}{r} \times r! \\ \Rightarrow \binom{n}{r} &= \frac{n!}{r!(n-r)!} \end{aligned}$$



Binomial Theorem

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

Multinomial Coefficients

Proposition C in the text

The number of ways that n objects can be divided into r subsets with n_i objects in set i , $i = 1, \dots, r$ is

$$\binom{n}{n_1 \cdots n_r} = \frac{n!}{n_1! \cdots n_r!}$$

Multinomial Theorem

Nice to know

$$(x_1 + \cdots + x_r)^n = \sum_{\mathbf{n}} \binom{n}{n_1 \cdots n_r} x_1^{n_1} \cdots x_r^{n_r}$$

where the sum is over all non-negative integers n_1, \dots, n_r such that $\sum_{j=1}^r n_j = n$.

Copyright Information

This slide show was prepared by **Jerry Brunner**, Department of Statistical Sciences, University of Toronto. It is licensed under a **Creative Commons Attribution - ShareAlike 3.0 Unported License**. Use any part of it as you like and share the result freely. The L^AT_EX source code is available from the course website:

<http://www.utstat.toronto.edu/~brunner/oldclass/256f18>