Counting Methods for Computing Probabilities[1]

STA 256: Fall 2018

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Countable set

A set is said to be *countable* if it can be placed in one-to-one correspondence with the set of natural numbers $\mathbb{N} = \{1, 2, \ldots\}$.

If the sample space Ω is countable and $A \subseteq \Omega$,

$$P(A) = \sum_{\omega \in A} P(\omega\}$$

Example: Roll a fair die. What is the probability of an odd number?

$$P(\text{Odd}) = P\{1, 3, 5\} = P\{1\} + P\{3\} + P\{5\}$$

If all outcomes of an experiment are equally likely,

$$P(A) = \frac{\text{Number of ways for } A \text{ to happen}}{\text{Total number of outcomes}}$$

Need to count.

Multiplication Principle

Also called the Fundamental Principle of Counting

If there are p experiments and the first has n_1 outcomes, the second has n_2 outcomes, etc., then there are

$$n_1 \times n_2 \times \cdots \times n_p$$

outcomes in all.

Sample Question

If there are nine horses in a race, in how many ways can they finish first, second and third?

$$9 \times 8 \times 7 = 504$$

The number of *permutations* (ordered subsets) of n objects taken r at a time is

$$
\begin{aligned}
{}_nP_r &= n \times (n-1) \times \cdots \times (n-r+1) \\
&= \frac{n!}{(n-r)!}
\end{aligned}
$$

Combinations
Unordered subsets

The number of *combinations* (unordered subsets) of n objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Proof of $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$

Part of Proposition B in the text, p.12

Choose an unordered subset of r items from n. Then place them in order. By the Multiplication Principle,

$$
\begin{aligned}
& {}_nP_r = \binom{n}{r} \times r! \\
\Rightarrow \quad & \frac{n!}{(n-r)!} = \binom{n}{r} \times r! \\
\Rightarrow \quad & \binom{n}{r} = \frac{n!}{r!\,(n-r)!}
\end{aligned}
$$

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Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

The number of ways that n objects can be divided into r subsets with n_i objects in set i, $i = 1, \ldots, r$ is

$$\binom{n}{n_1 \cdots n_r} = \frac{n!}{n_1! \cdots n_r!}$$

$$(x_1 + \cdots x_r)^n = \sum_{\mathbf{n}} \binom{n}{n_1 \ \dots \ n_r} x_1^{n_1} \cdots x_r^{n_r}$$

where the sum is over all non-negative integers $n_1, \ldots, n_r$ such that $\sum_{j=1}^{r} n_j = n$.

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