STA 256f18 Assignment Nine[1]

Please read Sections 4.3-4.5 in the text. The following homework problems are not to be handed in. They are preparation for Test 3 and the final exam. All textbook problems are from Chapter Four. Use the formula sheet to do the problems. On tests and the final exam, you may use anything on the formula sheet unless you are being directly asked to prove or derive it.

1. The discrete random variables X and Y have joint probability mass function $p_{xy} = xy/18$ for $x = 1, 2, 3$, $y = 1, 2$, and zero otherwise.
   (a) Find $cov(X, Y)$. My answer is 0.
   (b) Find $E(Y|X = 1)$ My answer is $\frac{5}{3}$.
   (c) Are X and Y independent?
2. The continuous random variables X and Y have joint density function $f_{xy} = 1$ for $0 < x < 1$ and $2x < y < 2$, and zero otherwise.
   (a) Find $cov(X, Y)$. My answer is $\frac{1}{18}$.
   (b) Find $E(X|Y = y)$ My answer is $\frac{y}{4}$.
   (c) Are X and Y independent?
   (d) Find $E(X)$ using double expectation: $E(X) = E(E[X|Y])$. Compare to the $E(X) = \frac{1}{3}$ you got using the definition of expected value.
3. Starting with the definitions on the formula sheet, prove the following facts about variances and covariances. Use expected value rules rather than integration or summation.
   (a) $Cov(X, Y) = E(XY) - E(X)E(Y)$
   (b) $Cov(a + X, b + Y) = Cov(X, Y)$
   (c) $Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$
   (d) $Cov(aW+bX, cY+dZ) = acCov(W, Y)+adCov(W, Z)+bcCov(X, Y)+bdCov(X, Z)$
   (e) $Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$
   (f) $Var\left(\sum_{i=1}^n a_iX_i\right) = \sum_{i=1}^n a_i^2Var(X_i) + \sum\sum_{i\neq j} a_ib_jCov(X_i, X_j)$. How many covariances are you adding up in the last term?
4. Do Problem 44 in the text.
5. Do Problem 50 in the text.

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6. The correlation between two random variables X and Y is defined as

$$Corr(X,Y) = \rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

   Find a formula for $Corr(a+bX, c+dY)$, where b and d are both non-zero. Show your work.

7. Let the pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$ be selected independently from a joint distribution with $E(X_i) = \mu_x$, $E(Y_i) = \mu_y$, $Var(X_i) = \sigma_x^2$, $Var(Y_i) = \sigma_y^2$, and $Cov(X_i, Y_i) = \sigma_{xy}$. Independence means that X_i and Y_i are independent of X_j and Y_j for $i \neq j$.

   The sample mean (average) of the X values is $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and the sample mean of the Y values is $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$

   (a) Show $Cov(X_j, \overline{X}) = \sigma_x^2/n$.

   (b) Show $Cov(\overline{X}, \overline{Y}) = \sigma_{xy}^2/n$

   (c) The formula for correlation is not on the formula sheet, but it is given in Problem 6. Show $Corr(\overline{X}, \overline{Y}) = Corr(X_i, Y_i)$.

8. The double expectation formula says $E(Y) = E[E(Y|X)]$. Prove it for the case where X and Y are discrete random variables.

9. Do Problem 70 in the text. Consider the continuous and discrete cases separately.

10. Do Problem 73 in the text. Try denoting the number of heads in the first n tosses by N_1, and the number of heads in the second set of tosses by N_2. You want $E(N_1 + N_2)$. Use double expectation, conditioning on N_1.

11. Do Problem 75 in the text. Just find $E(U)$. For the variance you would use Theorem B on page 151, which we skipped.

12. Let X have a moment-generating function $M_x(t)$ and let a be a constant. Show $M_{a+x}(t) = e^{at}M_x(t)$.

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14. Let X and Y be independent, (continuous) random variables. Show $M_{x+y}(t) = M_x(t)\,M_y(t)$.

15. In the following table, derive the moment-generating functions (given on the formula sheet), and then use them to obtain the expected values and variances. To make the task shorter, notice that the Bernoulli is a special case of the binomial, and that the exponential and chi-squared distributions are special cases of the gamma. Do the general case first and then just write the answer for the special cases.

Distribution	**MGF** $M_x(t)$	$E(X)$	$Var(X)$
Bernoulli (p)			
Binomial (n,p)			
Poisson (λ)			
Uniform (a,b)			
Exponential (λ)			
Gamma (α,λ)			
Normal (μ,σ)			
Chi-squared (ν)			

16. Do Problem 83 in the text.

17. Do Problem 88 in the text.

18. Do Problem 89 in the text.

19. Do Problem 93 in the text. By a "geometric sum of exponential random variables," they mean that first N is sampled from a geometric distribution, and then $X_1,\ldots,X_N$ are sampled independently from an exponential distribution – independently of N as well as each other. The sum is $S = \sum_{i=1}^N X_i$. To find the distribution of S, find its moment-generating function. Obtain $M_s(t) = E(e^{St})$ using double expectation. Condition on N.