

Confirmatory Factor Analysis Part One¹

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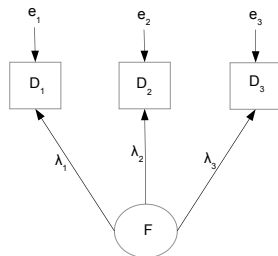
A confirmatory factor analysis model

One Factor: Starting simply

$$D_1 = \lambda_1 F + e_1$$

$$D_2 = \lambda_2 F + e_2$$

$$D_3 = \lambda_3 F + e_3$$



- $Var(F) = 1$
- $Var(e_j) = \omega_j$
- F, e_1, e_2, e_3 all independent.

Calculate Σ

$$\begin{aligned} D_1 &= \lambda_1 F + e_1 \\ D_2 &= \lambda_2 F + e_2 \\ D_3 &= \lambda_3 F + e_3 \end{aligned} \quad \Sigma = \begin{array}{c|ccc} & D_1 & D_2 & D_3 \\ \hline D_1 & \lambda_1^2 + \omega_1 & \lambda_1 \lambda_2 & \lambda_1 \lambda_3 \\ D_2 & & \lambda_2^2 + \omega_2 & \lambda_2 \lambda_3 \\ D_3 & & & \lambda_3^2 + \omega_3 \end{array}$$

Are the parameters identifiable? What if just one λ is zero?

Suppose no factor loadings equal zero

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \lambda_1^2 + \omega_1 & \lambda_1 \lambda_2 & \lambda_1 \lambda_3 \\ & \lambda_2^2 + \omega_2 & \lambda_2 \lambda_3 \\ & & \lambda_3^2 + \omega_3 \end{pmatrix}$$

$$\lambda_1^2 = \frac{\sigma_{12}\sigma_{13}}{\sigma_{23}} = \frac{\lambda_1 \lambda_2 \lambda_1 \lambda_3}{\lambda_2 \lambda_3}$$

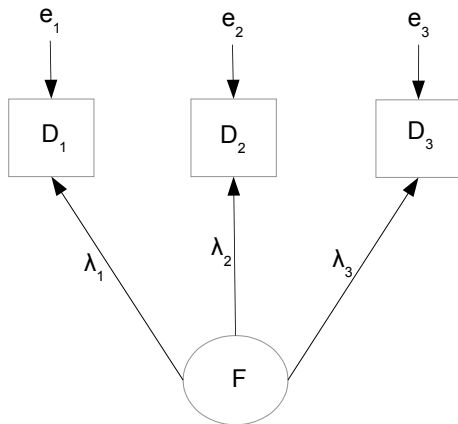
$$\lambda_2^2 = \frac{\sigma_{12}\sigma_{23}}{\sigma_{13}}$$

$$\lambda_3^2 = \frac{\sigma_{13}\sigma_{23}}{\sigma_{12}}$$

- Squared factor loadings are identifiable, but not the loadings.
- Replace all λ_j with $-\lambda_j$, get same Σ
- Likelihood function will have two maxima, same height.
- Which one you find depends on where you start.

Solution: Decide on the sign of one loading

Based on *meaning*



- Is F math ability or math *inability*? You decide.
- It's just a matter of naming the factors.

If $\lambda_1 > 0$

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \lambda_1^2 + \omega_1 & \lambda_1 \lambda_2 & \lambda_1 \lambda_3 \\ & \lambda_2^2 + \omega_2 & \lambda_2 \lambda_3 \\ & & \lambda_3^2 + \omega_3 \end{pmatrix}$$

- Signs of λ_2 and λ_3 can be recovered right away from Σ .
- And all the parameters are identified.

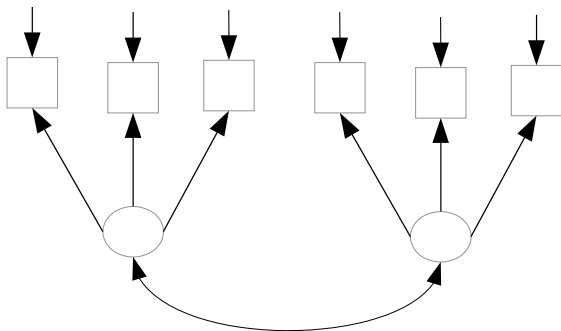
Add another variable: $D_4 = \lambda_4 F + e_4$

$$\Sigma = \begin{pmatrix} \lambda_1^2 + \omega_1 & \lambda_1 \lambda_2 & \lambda_1 \lambda_3 & \lambda_1 \lambda_4 \\ & \lambda_2^2 + \omega_2 & \lambda_2 \lambda_3 & \lambda_2 \lambda_4 \\ & & \lambda_3^2 + \omega_3 & \lambda_3 \lambda_4 \\ & & & \lambda_4^2 + \omega_4 \end{pmatrix}$$

- Parameters will all be identifiable as long as 3 out of 4 loadings are non-zero, and one sign is known.
- For example, if only $\lambda_1 = 0$ then the top row = 0, and you can get $\lambda_2, \lambda_3, \lambda_4$ as before.
- For 5 variables, two loadings can be zero, etc.
- How many equality restrictions? $6 - 4 = 2$.

Now add another factor

$$\text{Var}(F_1) = \text{Var}(F_2) = 1$$



$$D_1 = \lambda_1 F_1 + e_1$$

\vdots

$$D_6 = \lambda_6 F_2 + e_6$$

Covariance matrix of observable variables

$$\Sigma = \begin{pmatrix} \lambda_1^2 + \omega_1 & \lambda_1 \lambda_2 & \lambda_1 \lambda_3 & \lambda_1 \lambda_4 \phi_{12} & \lambda_1 \lambda_5 \phi_{12} & \lambda_1 \lambda_6 \phi_{12} \\ & \lambda_2^2 + \omega_2 & \lambda_2 \lambda_3 & \lambda_2 \lambda_4 \phi_{12} & \lambda_2 \lambda_5 \phi_{12} & \lambda_2 \lambda_6 \phi_{12} \\ & & \lambda_3^2 + \omega_3 & \lambda_3 \lambda_4 \phi_{12} & \lambda_3 \lambda_5 \phi_{12} & \lambda_3 \lambda_6 \phi_{12} \\ & & & \lambda_4^2 + \omega_4 & \lambda_4 \lambda_5 & \lambda_4 \lambda_6 \\ & & & & \lambda_5^2 + \omega_5 & \lambda_5 \lambda_6 \\ & & & & & \lambda_6^2 + \omega_6 \end{pmatrix}$$

- Identify $\lambda_1, \lambda_2, \lambda_3$ from set One (assuming one sign is known).
- Identify $\lambda_4, \lambda_5, \lambda_6$ from set Two (lower right).
- Identify ϕ_{12} from any unused covariance.
- What if you added more variables?
- What if you added more factors?

Three-variable identification rule

For standardized factors

For a factor analysis model, the parameters will be identifiable provided

- Errors are independent of one another and of the factors.
- Variances of all factors equal one.
- Each observed variable is a function of only one factor.
- There are at least three observable variables with non-zero loadings per factor.
- The sign of one non-zero loading is known for each factor.

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<http://www.utstat.toronto.edu/~brunner/oldclass/2101f19>