Monge-Kantorovich Ranks and Signs

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Unlike the real line, the real space $\mathbb{R}^K$, $K \geq 2$ is not "naturally" ordered. As a consequence, such fundamentals univariate concepts as quantile and distribution functions, ranks, signs, all order-related, do not straightforwardly extend to the multivariate context. Since no universal pre-existing order exists, each distribution, each data set, has to generate its own—the rankings behind sensible concepts of multivariate quantile, ranks, or signs, inherently will be distribution-specific and, in empirical situations, data-driven.

Many proposals have been made in the literature for such orderings—all extending some aspects of the univariate concepts, but failing to preserve the essential properties that make classical rank-based inference a major inferential tool in the analysis of semiparametric models where the density of some underlying noise remains unspecified: (i) exact distribution-freeness, and (ii) asymptotic semiparametric efficiency, see Hallin and Werker (2003).

Ranks and signs, and the resulting inference methods, are well understood and well developed, essentially, in two cases: one-dimensional observations, and elliptically symmetric ones. We start by establishing the close connection, in those two cases, between classical ranks and signs and measure transportation results, showing that the rank transformation there actually reduces to an empirical version of the unique gradient of convex function mapping a distribution to the uniform over the unit ball.

That fact, along with a result by McCann (1995), itself extending Brenier's celebrated *polar factorization Theorem*(Brenier 1991), is then exploited to define fully general concepts of ranks and signs—called the *Monge-Kantorovich ranks and signs* coinciding, in the univariate and elliptical settings, with the traditional concepts, and enjoying under completely unspecified (absolutely continuous) d-dimensional distributions, the essential properties that make traditional ranks and quantiles an essential part of the univariate semiparametric inference toolkit.

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