May 2010

Ph.D COMPREHENSIVE EXAMINATIONS
DEPARTMENT OF STATISTICS
UNIVERSITY OF TORONTO

THEORETICAL STATISTICS COMPREHENSIVE EXAMINATION

May 10, 2010, 12:30 p.m. – 4:30 p.m.

Sidney Smith Hall

(#questions = 6); (#pages = 4, including cover page)

1. ATTEMPT ALL OF QUESTIONS 1-2
2. Please work neatly and legibly, and start each question in a new book With your name and the number of the question, then also indicate which is the first book for a question, then also indicate which is the first book and which is the second.
3. It is not necessary to completely solve every problem to achieve a good Performance:
   emphasize what you know.
4. The questions are not in any special order, and they are not of equal difficulty.
5. Like typical real problems, the problems here may be improperly phased or may contain a misprint. Should this happen, reflect it in your discussion.
6. You are Not permitted any aids (e.g. books, notes, etc.) aside from a single non-programmable calculator.
7. Good Luck!

Question 1:

We observe the values of non-negative integers $X_1, X_2, \ldots, X_n$, which, given a positive real parameter λ, we model as being indepenent and identically distributed, with the distribution of each X_i being an equal mixture of a Poisson distribution with mean λ and a point mass at zero. In other words,

$$P(X_i = x_i \mid \lambda) = \begin{cases} (1/2) + (1/2)\exp(-\lambda) & \text{if } x_i = 0 \\ (1/2)\,(1/x_i!)\,\lambda^{x_i}\exp(-\lambda) & \text{if } x_i > 0 \end{cases}$$

a) Find a simple form for the minimal sufficient of this model, and prove that it is minimal sufficient.

b) Describe how one could compute the maximum likelihood estimate of λ for given values $x_1, x_2, \ldots, x_n$. You needn't give an explicit formula for the estimate, and you needn't discuss practical details of how the computation would be done.

c) Find the asymptotic variance of the maximum likelihood estimator for λ.

d) Find the asymptotic variance of the maximum likelihood estimator for $\exp(-\lambda)$.

e) Suppose that we do Bayesian inference with this model using a fairly vague prior for λ, without a singularity at $\lambda = 0$, such as a prior with density proportional to $1/(1+\lambda^2)$ or $\exp(-\lambda^{1/2})$. For moderately large values of n (for example, $n = 100$), find an approximate formula for the posterior standard deviation of λ.

f) Suppose we wish to produce both a point estimate for λ and an indication of uncertainty in this estimate. Consider two approaches to this task.

1) Use the maximum likelihood estimate for λ, and indicate the uncertainty in this estimate using a standard error derived from the asymptotic variance you found in (c), plugging in the maximum likelihood estimate for λ.

2) Use the posterior mean as the estimate for λ, based on a fairly vague prior as in (e), and indicate the uncertainty in this estimate by the posterior standard deviation.

Discuss the relative merits of these two approaches, when n is moderately large (for example, $n = 100$). Consider different possible true values of λ, such as $\lambda = 2$ and $\lambda = 20$.

$-3 \quad -2 \quad -1 \quad \theta \quad \theta+1 \quad 2 \quad 3$

Question 2:

We observe the values of integers $X_1, X_2, \ldots, X_n$ (collectively written as X), which, given an integer parameter θ, we model as being indepenent and identically distributed, with the distribution of each X_i being uniform on the set

$$\{\theta-3,\ \theta-2,\ \theta-1,\ \theta,\ \theta+1,\ \theta+2,\ \theta+3\}$$

$X_i = \theta$, $X_i = \theta - 3$, $\theta = X_i + 3$.

We aim to estimate θ by a real-valued estimate a, with loss function $L(\theta, a) = \lfloor|\theta - a|\rfloor$, where $\lfloor z\rfloor$ is the floor of z (ie, the largest integer less than or equal to z). For example, $L(8, 7.1) = 0$, $L(8, 10.1) = 2$, and $L(8, 1) = 7$.

a) Suppose that it is proposed to estimate θ using the rule $\delta_1(x) = \bar{x}$, where $\bar{x}$ is the sample mean of $x_1, \ldots, x_n$. Use the Rao-Blackwell theorem to find an estimator, $\delta_2(x)$, based on $\delta_1(x)$, that dominates $\delta_1(x)$.

b) Show that the estimator $\delta_2(x)$ that you found in (a) is inadmissible.

c) Consider the prior for θ in which $P(\theta = i) = c\,(999/1000)^{|i|}$, where c is the appropriate normalizing constant. Find all the formal Bayes rules for this prior, with the loss as defined above.

d) Determine the admissibility of the rules you found in (c).

Note that in answering these questions, you need not provide things that are not asked for if they aren't necessary to answer the question.

Question 3:

$\ell_\eta = T(X) - d'(\eta)$

Suppose $X_1, X_2, \cdots, X_n$ is an iid sample of size n from the 1-parameter exponential family

$$f_\eta(x) = \exp\{\eta T(x) - d(\eta) + S(x)\}, \quad -\infty < x < \infty,$$

$\ell = \eta T(x) - d(\eta) + S(x)$

where $-\infty < \eta < \infty$ is an unknown parameter. $E(T) = d'(\eta)$

$\varphi(EX) \le E(\varphi(X))$

a) Evaluate the Fisher Information *per observation* for η.

b) Evaluate the Kullback-Leibler distance $E_{f_{\eta_0}} \log\{f_{\eta_0}(X)/f_{\eta_1}(X)\}$ of f_{η_1} from f_{η_0}.

c) Show that (given a sample of size n) the MLE for η is equivalent to a method of moments estimator.

d) Making the *fewest* possible assumptions, give a *short* but *rigorous* proof that the MLE is consistent for this family.

e) Give the UMP tests for $H_0 : \eta = \eta_0$ versus $H_1 : \eta > \eta_0$, and prove they are UMP.

Question 4:

Let $X_1, \ldots, X_n$ be i.i.d. with the power series distribution having p.m.f. $f(x|\theta) = \theta^x a(x)/c(\theta)$, where $x = 0, 1, 2, \ldots$ and $\theta \in (0, 1)$.

a) Show that $T = \sum_{i=1}^{n} X_i$ is complete and sufficient for θ, and that its distribution has the p.m.f. $f(t|\theta) = \theta^t b(t)/[c(\theta)]^n$ for some function $b(\cdot)$, with $t = 0, 1, 2, \ldots$.

b) Obtain a UMVUE of θ^r, where r is a positive integer.

c) Show that there does not exist an unbiased estimator of $\theta^{1/2}$.

Question 5:

Let $X_1, \ldots, X_n$ be i.i.d. with the p.d.f./p.m.f. $f(x|\theta)$ with respect to some measure ν, where $\theta \in \Theta \subseteq \Re$. Assume that $\{x : f(x|\theta) > 0\}$ does not depend on θ. Assume also that $\mu(\theta) = E_\theta X = \int x f(x|\theta) d\nu(x)$ can be differentiated with respect to θ and passed through the integral sign, and $\mu'(\theta) \neq 0$. Assume that $\sigma^2(\theta) = \text{var}_\theta(X_i) > 0$.

a) Derive the asymptotic distribution of the method of moment estimator (MOM) $\hat{\theta}_M$ for θ. Justify all the steps.

b) Show that the asymptotic variance of $\sqrt{n}(\hat{\theta}_M - \theta)$ is at least as large as $I^{-1}(\theta)$, where $I(\theta)$ is the Fisher information of one observation.

c) Derive the necessary and sufficient conditions such that the minimal asymptotic variance in (b) is attained. When this minimum is attained, what is the relationship between the MLE of θ and the MOM $\hat{\theta}_M$?

Question 6:

Let $X_1, \ldots, X_n$ be i.i.d. continuous random variables with common p.d.f. $f(x|\theta) = c(\theta) x^\theta h(x)$, where $0 < x < 1$ and $\theta > 0$.

a) Find the UMP unbiased level α test for $H : \theta = \theta_0$ versus $K : \theta \neq \theta_0$. Justify all the steps.

b) Assume $c(\theta) = \theta + 1$ and $h(x) \equiv 1$. Show that the test obtained in (a) can be simplified such that the relevant constants are determined by a Chi-square distribution. Find this simplified form and specify the rejection region.

c) Still assume $c(\theta) = \theta + 1$ and $h(x) \equiv 1$. Consider the case of large sample size, determine the reject region in (b) based on the asymptotic normal approximation. Denote the power function by $\beta(\cdot)$ and use the asymptotic rejection region, obtain $\lim_{n\to\infty} \beta(\theta_n)$ for a sequence of contiguous alternatives $K_n : \theta = \theta_n = \theta_0 + t/\sqrt{n}$, $t \neq 0$.