May 2015

Ph.D. COMPREHENSIVE EXAMINATIONS
DEPARTMENT OF STATISTICAL SCIENCES
UNIVERSITY OF TORONTO

PROBABILITY COMPREHENSIVE EXAMINATION

May 4, 2015, 12:30 p.m. - 4:30 p.m.

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(#questions = 10; (#points per question = 10) (#pages = 5, including cover page)

1. ATTEMPT ALL QUESTIONS.
2. Please work neatly and legibly.
3. **You may answer all the questions in the same book(s).** If you use more than one book for this exam, then also indicate which is the first book and which second, etc, as well as the total number of books used. Example: Book 1/2, Book 2/2.
4. It is not necessary to completely solve every problem to achieve a good performance. Emphasize what you do know.
5. Occasionally, problems may be improperly phrased or may contain a misprint. Should this happen, reflect it in your discussion.
6. This exam is closed book and notes. NO CALCULATORS ARE ALLOWED.
7. Good luck!

Problem 1 *(a). Let X be a random variable such that $xP(|X| > x) \to 0$ as $x \to \infty$. Prove that $E|X|^p < \infty$ for any $p \in (0,1)$. Meanwhile, give a counterexample to show that $E|X|^p < \infty$ does not necessarily hold for $p = 1$.*

(b). Let $X_1, X_2, \cdots$ be a infinite sequence of random variables on $(\Omega, \mathcal{F}, P)$. Let $X = \sum_{i=1}^{\infty} X_i$ if $\sum_{i=1}^{\infty} X_i$ exists and is finite. Otherwise let $X = 0$. Prove that X is a random variable.

Problem 2 *(a). Let* $X_1, X_2, \ldots$ *be i.i.d. with* $E|X_i| < \infty$. *Let* $Y_i = \sum_{j=0}^{i} a^j X_j$, *where* $a \in (-1, 1)$ *is a constant. Prove that* Y_i *converges almost surely to a finite random variable* Y.

(b). Suppose that $(X_i)_{i=1}^{\infty}$ *is a monotonic sequence of random variables such that* $X_i \to X$ *in probability. Will* $X_i \to X$ *almost surely? Prove it if your answer is yes and give a counterexample if your answer is no.*

(c). Let $X_1, X_2, \ldots$ *be independent random variables such that* $P(X_i > x) \geq 1/(ix)$ *for any* $i \geq 1$ *and any* $x \geq 2$. *Let* $Y_i = \max_{1 \leq j \leq i} X_j$. *Prove that* $Y_i \to \infty$ *almost surely.*

(d). Let $X_1, X_2, \ldots$ *be i.i.d. random variables. Suppose that* X_i*'s are symmetric in the sense that* $P(X_i > x) = P(X_i < -x)$ *for any* $x \in R$. *Further assume that* $xP(|X_1| > x) \to c$ *for some positive and finite constant* c *as* $x \to \infty$. *Prove that*

$$\frac{\sum_{i=1}^{n} X_i}{l_n} \to 0$$

in probability for any increasing sequence l_n *such that* $l_n/n \to \infty$.

Problem 3 *(a). Let* $X_1, X_2, \ldots$ *be independent identically distributed random variables with mean* 0 *and variance* 1*. True or false: The distribution of* $\frac{1}{n}\sum_{i=1}^{n}\sqrt{n-i+1}\;X_i$ *is asymptotically Gaussian. (Either way, you must completely justify your answer.)*

(b). Let $X_1, X_2, \ldots$ *be random variables with* $E[X_i] = 0$ *for all* i *and*

$$E[X_i X_j] \leq e^{-|i-j|}$$

. *True or false:* $\frac{1}{n}\sum_{i=1}^{n} X_i \to 0$ *in distribution. (Either way, you must completely justify your answer.)*

Problem 4 *(a). Let $X_1, X_2, \ldots$ be independent identically distributed random variables with the Holtsmark density function,*

$$f(x) = \frac{1}{\pi}\Gamma(5/3)\,{}_2F_3(5/12, 11/12; 1/3, 1/2, 5/6; -4x^6/729) \tag{1}$$

$$- \frac{x^2}{3\pi}\,{}_3F_4(3/4, 1, 5/4; 2/3, 5/6, 7/6, 4/3; -4x^6/729) \tag{2}$$

$$+ \frac{7x^4}{81\pi}\Gamma(4/3)\,{}_2F_3(13/12, 19/12; 7/6, 3/2, 5/3; -4x^6/729), \tag{3}$$

where

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!}$$

are hypergeometric functions, and defined via the Pochhammer symbol:

$$(a)_0 = 1, \tag{4}$$

$$(a)_n = a(a+1)(a+2)\ldots(a+n-1), \qquad n \geq 1 \tag{5}$$

Give a formula for $P(X_1 + \cdots + X_n \geq \lambda)$ using only a single integration of $f(x)$.

(You may find the following formula useful: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx - |t|^{3/2}}\,dt$.)

(b). Let $S_n = X_1 + \cdots + X_n$ be the simple symmetric random walk on $\mathbb{Z}$ starting at $S_0 = 0$, i.e. $X_1, X_2, \ldots$ are independent with $P(X_i = 1) = P(X_i = -1) = 1/2$. Let $\tau = \min\{n \geq 1 : S_n = 1\}$ be the time it takes the walk to hit 1. Compute the moment generating function $\varphi(\lambda) = E[e^{-\lambda\tau}]$ of τ. (It is acceptable in the formula to use a non-explicit inverse of an explicit function.)