May 2014

# Ph.D. COMPREHENSIVE EXAMINATIONS
## DEPARTMENT OF STATISTICAL SCIENCES
## UNIVERSITY OF TORONTO

## PROBABILITY COMPREHENSIVE EXAMINATION

May 2, 2014, 12:30 p.m. 4:30 p.m.

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(#questions = 12; (#points per question = 10) (#pages = 6, including cover page)

1. ATTEMPT ALL QUESTIONS.
2. Please work neatly and legibly.
3. **You may answer all the questions in the same book(s).** If you use more than one book for this exam, then also indicate which is the first book and which second, etc, as well as the total number of books used. Example: Book 1/2, Book 2/2.
4. It is not necessary to completely solve every problem to achieve a good performance. Emphasize what you do know.
5. Occasionally, problems may be improperly phrased or may contain a misprint. Should this happen, reflect it in your discussion.
6. This exam is closed book and notes. NO CALCULATORS ARE ALLOWED.
7. Good luck!

**Problem 1** *Let $\epsilon_n$ be i.i.d. with $\mathbb{E}\epsilon_n = 0$ and $\mathbb{E}\epsilon_n^4 < \infty$. For $\rho \in (-1, 1)$, define $X_n = \sum_{j=0}^{\infty} \rho^j \epsilon_{n-j}$ so that $X_n = \rho X_{n-1} + \epsilon_n$.*

*(a). Show that $X_n$ is well-defined, that is, $\sum_{j=0}^{m} \rho^j \epsilon_{n-j}$ converges almost surely as $m \to \infty$.*

*(b). Show that the least square estimator $\widehat{\rho} = \operatorname{argmin}_{\rho} \sum_{k=1}^{n} (X_k - \rho X_{k-1})^2$ converges to $\rho$ either in probability or almost surely.*

**Problem 2** *Let* $X_1, X_2, \ldots$ *be i.i.d.*

*(a). Find a necessary and sufficient condition for* $\max_{1 \le i \le n} X_i/n \to 0$ *in probability as* $n \to \infty$.

*(b). Show that* $\max_{1 \le i \le n} X_i/n \to 0$ *almost surely if and only if* $\mathbb{E}(\max(0, X_1)) < \infty$.

**Problem 3** *Let* $X_1, X_2, \ldots$ *be i.i.d. standard normal random variables and* $M_n = \max(X_1, \ldots, X_n)$.

*(a). Show that* $M_n/b_n \to 1$ *almost surely for* $b_n = \sqrt{2 \log n}$.

*(b). Find a diverging sequence* $a_n$ *so that* $a_n(M_n - a_n)$ *converges in distribution.*

**Problem 4** *(a). We say that a random variable $X$ is decided by its moments if (i): $E(X^k)$ exists and is finite for all positive integer $k$; (ii): if $E(Y^k) = E(X^k)$, $k = 1, 2, \cdots$ for some $Y$, then $Y$ has the same distribution as $X$. Suppose a sequence of random variables $\{X_i\}$, $i = 1, 2, \cdots$ satisfies $E(X_i^k) < \infty$ for all positive integers $i$ and $k$ and $\lim_{i\to\infty} E(X_i^k) = E(X^k)$ for all positive integers $k$. Furthermore, $X$ is decided by its moments. Prove that $X_i \to X$ in distribution.*

*(b). Suppose that $X_i$, $i = 1, 2, \cdots$ are i.i.d. random variables with $E(X_i) = 0$ and $Var(X_i) = 1$. Suppose $p \geq 1$ is an integer and $t_1 < t_2 \cdots < t_p$ are $p$ real numbers in $(0, 1)$. For each positive integer $n$, define $S_{n,i} = \sum_{j=1}^{\lfloor nt_i \rfloor} X_j$, $i = 1, 2, \cdots, p$, where $\lfloor x \rfloor$ denotes the largest integer smaller or equal to $x$. Let $S_n = (S_{n,1}, S_{n,2}, \cdots, S_{n,p})^\top$. Prove that*

$$S_n/\sqrt{n} \to N(0, \Sigma) \text{ in distribution as } n \to \infty$$

*and find the matrix $\Sigma$.*

**Problem 5** *(a). Suppose that* $X_n \to X$ *in distribution and* $Y_n \to c$ *in distribution and* $c$ *is a constant. Suppose* $f : \mathcal{R}^2 \to \mathcal{R}$ *is a continuous function. Prove that* $f(X_n, Y_n) \to f(X, c)$ *in distribution.*

*(b). Let* $X_n, Y_n$, $n \geq 1$ *be random variables. Suppose that (i):* $X_n \to a$ *in probability for some constant* $a$*; (ii): the sequences* $\{Y_n\}$ *and* $\{X_n Y_n\}$ *are uniformly integrable; (iii):* $E(Y_n) \to 1$*. Prove that* $E(X_n Y_n) \to a$*.*

*(c). Let* $\{X_n, \mathcal{F}_n\}$, $n \geq 1$ *be a submartingale. Define the submartingale differences* $D_1 = X_1$ *and* $D_i = X_i - X_{i-1}$, $i \geq 2$*. Suppose that* $E(\max_i |D_i|) < \infty$*. Let* $\{A_i\}$, $i \geq 2$ *be a sequence of decreasing events such that (i) :* $A_i \in \mathcal{F}_{i-1}$ *and (ii) :* $|X_{i-1}| \leq 2$ *if* $A_i$ *is true,* $i = 2, 3, \cdots$*. Define* $Y_1 = X_1$ *and* $Y_i = D_1 + \sum_{j=2}^{i} D_j I(A_j)$, $j \geq 2$*. Let* $M \leq N$ *be two stopping times with respect to* $\mathcal{F}_n$*. Prove that* $E(Y_M) \leq E(Y_N)$*.*

*(d). For each positive integer* $n$*, let* $X_{n,i}$, $1 \leq i \leq n$ *be a sequence of martingale differences with respect to filtrations* $\mathcal{F}_{n,i}$ *(i.e.* $E(X_{n,i}|\mathcal{F}_{n,i-1}) = 0$, $i = 2, 3, \cdots, n$*). Suppose that* $E(\max_{j \leq n} |X_{n,j}|) \to 0$ *as* $n \to \infty$ *and* $\sum_{i=1}^{n} X_{n,i}^2 \to \sigma^2$ *in probability as* $n \to \infty$*. Here* $\sigma^2 \in (0, \infty)$*. Prove that*

$$\sum_{i=1}^{n} X_{n,i} \to N(0, \sigma^2) \text{ in distribution as } n \to \infty.$$

*(Hint: You may want to use* $\exp(ix) = (1 + ix)\exp(-x^2/2 + r(x))$*, where the error term satisfies* $|r(x)| \leq |x|^3$ *for all* $x$ *and parts (a)-(c) of the problem.)*