May 2013

Ph.D COMPREHENSIVE EXAMINATIONS
DEPARTMENT OF STATISTICS
UNIVERSITY OF TORONTO

PROBABILITY COMPREHENSIVE EXAMINATION

MAY 31, 2013, 12:30 p.m.-4:30 p.m.

Sidney Smith Hall

(#questions = 10) (#points per question = 20)
(#pages = 4, including cover page)

1. ATTEMPT ANY FIVE (5) QUESTIONS. $[5\times 20 = 100]$

2. Please work neatly and legibly. Neatness counts.

3. ***You may answer all the questions in the same book(s).*** If you use more than one book for this exam, then also indicate which is the first book and which is the second, etc., as well as the total number of books used. Example: Book 1/2, Book 2/2.

4. It is not necessary to solve every problem to achieve a good performance. Emphasize what you do know.

5. Should a problem seem improperly phrased, or contain a possible misprint, reflect this in your discussion.

6. This exam is closed book and no notes. NO CALCULATORS ARE ALLOWED.

7. Good Luck!

Probability comprehensive exam
May 2013

Do any five (5) questions.

1. For an $\mathbb{R}$-valued random variable, X
   a) Explain what is meant by EX and thus what is $L = L_1$
      (you need only cite the essential facts)
   b) Explain, simply, why it is that $|X| \le Y,\ Y \in L \ \Rightarrow\ X \in L$.
   c) Verify that $X \in L \Leftrightarrow E|X|I(|X| > n) \to 0$ as $n \to \infty$.
   d) Verify that $X \in L \Rightarrow nP(|X| > n) \to 0$ as $n \to \infty$,
      but the converse is *false*.

2. For an $\mathbb{R}$-valued random variable, X
   a) Show that the *monotone convergence theorem* is equivalent to the 'super linearity' property:

$$0 \le Z_n \uparrow Z \ \Rightarrow\ 0 \le EZ_n \uparrow EZ$$

$$\equiv \quad Z_n \ge 0 \ \Rightarrow\ E\sum_{n=1}^{\infty} Z_n = \sum_{n=1}^{\infty} EZ_n.$$

   b) Verify the criterion: $X \in L \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} P(|X| > n) < \infty$.

3. For an $\mathbb{R}$-valued random sequence $X_n, n = 1, 2, \ldots$
   a) Give *any* example for which convergence occurs with probability $1/2$.
   b) Define $wP1$-convergence, $X_n \overset{wP1}{\to} X$, and P-convergence, $X_n \overset{P}{\to} X$, and give any example for which P-convergence occurs but $wP1$-converence fails.
   c) *Verify* that $\rho(X,Y) = E|X-Y| \wedge 1$ defines a metric on $\mathcal{R}$
      for which $X_n \overset{P}{\to} X \ \Leftrightarrow\ \rho(X_n, X) \to 0$. $[X \wedge Y = \min(X,Y))]$

4. For an $\mathbb{R}$-valued random sequence $X_n, n = 1, 2, \ldots$ let $L = L_1$
   a) Define what is meant by uniform integrability $[UI]$ and *verify* that
   $|X_n| \leq Y \;\; w. \, Y \in L \quad \Rightarrow \quad X_n \; UI.$
   b) Give any example for which $X_n \xrightarrow{L} X$ yet $\nexists \, Y \in L \;\; w. \, |X_n| \leq Y.$
   c) Give any example for which $X_n \xrightarrow{wP1} X, \; X_n, X \in L$ yet $X_n \not\xrightarrow{L} X.$
   d) If $X_n \; UI$ does it follow that $g(X_n) \; UI$ if g is continuous?

5. *L-convergence & scheffé's theoerm*

   a) For $X, X_n \in L$
   $$X_n \xrightarrow{L} X \;\; \Leftrightarrow \;\; EX_nI_A \to EXI_A \;\; \textit{uniformly in } A$$
   b) For $X, X_n \in L$
   $$X_n \xrightarrow{L} X, \; P(A\Delta A_n) \to 0 \;\; \Rightarrow \;\; EX_nI_{A_n} \to EXI_A$$
   c) For $X, X_n \in L^+, \; EX_n = EX = 1 \;\; \forall \, n$
   $$X_n \xrightarrow{P} X \;\; \Leftrightarrow \;\; X_n \xrightarrow{L} X.$$

6. The n-dimensional random variable, $\boldsymbol{X}$, is said to be *rotationally invariant* if and only if its *distribution* is left completely unchanged by any *orthogonal transformation* $\boldsymbol{A}$:

   $$\boldsymbol{X} \textit{ rot. inv.} \quad \textit{iff} \qquad \boldsymbol{AX} \overset{d}{=} \boldsymbol{X} \quad \forall \, \boldsymbol{A} \in \mathbb{R}^n_n \;\; st \;\; \boldsymbol{A}'\boldsymbol{A} = \boldsymbol{I}$$

   a) If $\boldsymbol{X}$ is *rotationally invariant*, prove that, $X_i \overset{d}{=} X_1, \; i = 1, \ldots, n.$
   b) For $\boldsymbol{X}$ *rotationally invariant* and $c(t)$ the characteristic function for X_1, verify that $c_{\boldsymbol{X}}(\boldsymbol{t}) = c(|\boldsymbol{t}|) \;\; \forall \, \boldsymbol{t} \in \mathbb{R}^n.$
   c) Suppose, in addition, that the coordinates of $\boldsymbol{X}$ are *statistically independent.* In *that* case, verify that $\Pi_1^n \, c(t_i) = c(|\boldsymbol{t}|) \;\; \forall \, \boldsymbol{t} \in \mathbb{R}^n.$
   d) Use the results of a), b) & c) to establish the general result

   $$\boldsymbol{X} \text{ is } \textit{rot. invar. \& stat. ind.}$$
   $$\Leftrightarrow \quad X_1, \ldots, X_n \;\; IID \;\; N(0, \sigma^2) \;\; \text{for some } \sigma > 0.$$

7. By definition $U \sim unif[0,1]$ *iff* $P(U \leq u) = u \; \forall \, 0 \leq u \leq 1$
   a) Verify that $U \sim unif[0,1] \;\; \Leftrightarrow \;\; [nU] \sim unif\{0,\ldots,n{-}1\} \;\; \forall \, n \in \mathbb{N}$
      [the *greatest integer function* satisfies $[x] \leq x \leq [x]+1$]
   b) Prove that $U_n = [nU]/n \to U$ as $n \to \infty$
      and thus that $EU_n^p \to EU^p$ for any $p > 0$.
   c) Compute $P(U_{n+1} > U_n)$ for each $n \in \mathbb{N}$ to show that
      the convergence in b) is not monotone.

8. In $L_2 = \{ \, X \in L \mid X^2 \in L \, \}$
   a) Verify that $X, Y \in L_2 \;\Rightarrow\; XY \in L = L_1$
   b) Demonstrate that $\langle X, Y \rangle = EXY$ determines a (pseudo)
      inner product.
   c) Define and explain the concept of conditional expectation in L_2.

9. Bernoulli Variables, Martingales and stopping times
   a) Let X_i be *IID* Bernoulli(p) and set $Y_i = I(X_i \neq X_{i+1}), i = 1, 2, \ldots$
      Prove that $(Y_1 + \cdots + Y_n)/n$ converges $as/wP1$ to a limit
      and compute its value.
   b) Find a martingale that converges $as/wP1$ but not in L_1
   c) Let X_i be standard exponential random variables ($exp(1)$)
      and let T_n be the first t such that $X_1 + \cdots + X_t > n$.
      Show that $(T_n - ET_n)/\sqrt{n}$ converges to a normal random variable.

10. Brownian motion and random walks
   a) Let I_t be the image of $[0,t]$ under standard brownian motion.
      Compute the expected length of I_t.
   b) Let G be a finite connected graph, and let $e_x(n)$ be the
      expected number of visits to vertex x for the random
      walk on G up to time n. Show that
      $e_x(n)/n \to \pi_x$ (the stationary distribution).