May 2012

Ph.D. COMPREHENSIVE EXAMINATIONS
DEPARTMENT OF STATISTICS
UNIVERSITY OF TORONTO

PROBABILITY
COMPREHENSIVE EXAMINATION

May 30, 2012, 12:30 p.m. – 4:30 p.m.

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(# questions = 8); (#pages = 3 including cover page); (#points = 120)

1. **ATTEMPT ALL QUESTIONS.**
2. It is not necessary to completely solve every problem to achieve a good performance. Emphasize what you do know.
3. Please work neatly and legibly.
4. Start each question in a new book, with your name and the number of the question on the front cover. If there is more than one book for a question, then also indicate which is the first book and which second, e.g., Jane Smith, Question 5, Book 1 of 2.
5. The questions are not in any special order, nor are they all of equal difficulty. Point values for each question are given in [square-brackets].
6. The problems may be improperly phrased or may contain a misprint. Should this happen, reflect it in your discussion. Faculty members are not available to answer questions during the exam.
7. You are NOT permitted any aids (e.g., books, notes, etc.). **NO CALCULATORS ARE ALLOWED.**
8. Good luck!

1. [10] Prove or disprove the following statement: If X and Y are random variables defined jointly on some probability triple $(\Omega, \mathcal{F}, \mathbf{P})$, and $X \sim N(0,1)$, and $Y \sim N(0,1)$, and Covariance$(X, Y) = 0$, then X and Y must be independent. (Here $N(0,1)$ is the standard Normal distribution.)

2. [10] Let $X_1, X_2, \ldots$ be a sequence of random variables defined jointly on some probability triple $(\Omega, \mathcal{F}, P)$, <u>not</u> necessarily independent. Assume that $\mathbf{E}[X_n] = 2$ and $\mathbf{E}[(X_n)^2] = 9$ for all $n \in \mathbf{N}$. Compute $\mathbf{P}(X_n \geq n \text{ } i.o.)$. (Here "*i.o.*" stands for "infinitely often", i.e. for infinitely many different values of n.)

3. For $r \geq 0$, let ρ_r be the probability law having density function (with respect to Lebesgue measure on $[0,1]$) given by $f_r(x) = (r+1)x^r \mathbf{1}_{x \in [0,1]}$, and let δ_r be a point-mass at r. Let $\mu = (1/4)\rho_2 + (3/4)\delta_{9/10}$, and let $\nu = a\rho_3 + (1-a)\delta_b$ for some $a, b \in [0,1]$.

(a) [5] Determine (with explanation) all possible values of a and b which make $\mu \ll \nu$ (i.e., which make μ be absolutely continuous with respect to ν).

(b) [5] For the values of a and b found in part (a), compute (with explanation) the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$.

4. Let $S = \{6, 7, 8, 9, \ldots\}$ be a state space. Let $\{X_n\}$ be a Markov chain on S, with $X_0 = 10$, and with transition probabilities given by $p_{6,6} = 1$, and $p_{7,6} = p_{7,8} = 1/2$, and for $i \geq 8$, $p_{i,i-1} = p_{i,i-2} = 1/5$ and $p_{i,i+1} = 3/5$, with $p_{i,j} = 0$ otherwise.

(a) [5] Prove that $\{X_n\}$ is a martingale.

(b) [5] Compute $\mathbf{P}(X_3 = 10)$.

(c) [5] Compute $\mathbf{E}(X_3)$.

(d) [5] Let $T = \inf\{n \geq 0 : X_n = 6 \text{ or } 13\}$. Compute $\mathbf{P}(X_T = 13)$.

(e) [5] Determine whether or not the random variable $\lim_{n\to\infty} X_n$ exists with probability 1, and if so then what is its distribution.

5. [10] Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability triple defined by $\Omega = \mathbf{N} = \{1, 2, 3, \ldots\}$, and $\mathcal{F}$ is all subsets of Ω, and $\mathbf{P}\{\omega\} = 2^{-\omega}$ for all $\omega \in \Omega$. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, \mathbf{P})$, such that $X_n \to 0$ w.p. 1 as $n \to \infty$. Identify (with proof) the set A of all positive real numbers $a > 0$ with the property that if $-2^{\omega}/\omega^a \leq X_n(\omega) \leq 2^{\omega/3}$ for all $n \in \mathbf{N}$ and all $\omega \in \Omega$, then we must have $\lim_{n\to\infty} \mathbf{E}(X_n) = 0$. (That is, specify a set A of positive real numbers, and for $a \in A$ prove that we must have $\lim_{n\to\infty} \mathbf{E}(X_n) = 0$, and for $a \notin A$ provide a counter-example to show that we might <u>not</u> have $\lim_{n\to\infty} \mathbf{E}(X_n) = 0$.)

6. Let X and Y be two jointly-defined real-valued random variables with finite means.

(a) [5] Define the conditional expectation $\mathbf{E}(Y|X)$.

(b) [5] Give an example to show that we might <u>not</u> have $\mathbf{E}(Y|X) = \mathbf{E}(Y)$ w.p. 1.

(c) [5] Suppose X and Y are independent. Prove that $\mathbf{E}(Y|X) = \mathbf{E}(Y)$ w.p. 1.

7. Let $\{B_t\}_{t\geq 0}$ and $\{C_t\}_{t\geq 0}$ be two independent standard Brownian motions, and let $0 < \rho < 1$ be a real number, and let $D_t = \rho\, B_t + \sqrt{1-\rho^2}\, C_t$ for $t \geq 0$.

(a) [5] Compute $\mathbf{E}[(B_3+1)(2B_5+3)]$.

(b) [5] Show that $\{D_t\}_{t\geq 0}$ is also a standard Brownian motion.

(c) [5] Compute $\mathbf{E}[D_t^2 B_s^2]$ for $0 < t < s$.

(d) [5] Let $\{X_t\}_{t\geq 0}$ be a diffusion, satisfying $X_0 = 8$ and $dX_t = 3\,dB_t - 2\,dt$. Compute $\mathbf{E}(X_5^2)$.

(e) [5] Let $\{Y_t\}_{t\geq 0}$ be a diffusion, satisfying $dY_t = 5dB_t - 3Y_t\,dt$. Let $Z_t = (Y_t)^2$ for all $t \geq 0$. Use Itô's Lemma (i.e., Itô's Formula) to derive a formula for dZ_t.

8. Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity 4, i.e. such that $N(0) = 0$, and $N(t) - N(s) \sim \text{Poisson}[4(t-s)]$ for $0 < s < t$, and $\{N(t_i) - N(s_i)\}_{i=1}^k$ are independent whenever $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq s_3 \leq \ldots \leq t_k$ for any $k \in \mathbf{N}$.

(a) [5] Compute (with explanation) $\mathbf{E}[N(3)\,(N(5) - N(1))]$.

(b) [5] Find (with explanation) constants a_m and b_m for $m \in \mathbf{N}$, such that

$$\lim_{m\to\infty} \mathbf{P}\Big(\frac{N(m) - a_m}{b_m} \leq x\Big) \;=\; \Phi(x)\,,$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\, du$ is the c.d.f. of a standard Normal distribution.

(c) [5] Let $Y_t = N(t) - 4t$ for $t \geq 0$. Prove that $\{Y_t\}_{t\geq 0}$ is a martingale.

May 2012 Probability Comprehensive Exam: SOLUTIONS

NOTE: These are just outlines of solutions; students should provide more details to get full marks. (And, these solutions may well contain errors!) If desired, further background/definitions/etc. are available in e.g. the book "A First Look at Rigorous Probability Theory" by J.S. Rosenthal.

1. [10] Prove or disprove the following statement: If X and Y are random variables defined jointly on some probability triple $(\Omega, \mathcal{F}, \mathbf{P})$, and $X \sim N(0,1)$, and $Y \sim N(0,1)$, and Covariance$(X,Y) = 0$, then X and Y must be independent. (Here $N(0,1)$ is the standard Normal distribution.)

Solution. *This is false! For example, let X and Z be independent, with $X \sim N(0,1)$, and $\mathbf{P}(Z = +1) = \mathbf{P}(Z = -1) = 1/2$, and set $Y = XZ$ (i.e., the product of X and Z). Then clearly $Y \sim N(0,1)$, and Covariance$(X,Y) = \mathbf{E}(XY) = \mathbf{E}(X^2)\,\mathbf{E}(Z) = (1)(0) = 0$. But $\mathbf{P}(|X| = |Y|) = 1$, so clearly X and Y are not independent.*

NOTE: If X and Y are assumed to be jointly absolutely continuous, then they must jointly follow the bivariate normal distribution, and then they must indeed be independent. But we have not assumed this! If students make this mistake then they should receive essentially no credit; I suggest a mark of 1/10.

2. [10] Let $X_1, X_2, \ldots$ be a sequence of random variables defined jointly on some probability triple $(\Omega, \mathcal{F}, P)$, not necessarily independent. Assume that $\mathbf{E}[X_n] = 2$ and $\mathbf{E}[(X_n)^2] = 9$ for all $n \in \mathbf{N}$. Compute $\mathbf{P}(X_n \geq n \text{ i.o.})$. (Here "*i.o.*" stands for "infinitely often", i.e. for infinitely many different values of n.)

Solution. *By Chebychev's inequality, $\mathbf{P}(X_n \geq n) \leq \frac{\mathrm{Var}(X_n)}{n^2} = \frac{9+2^2}{n^2} = 13n^{-2}$.*

Hence, $\sum_n \mathbf{P}(X_n \geq n) \leq \sum_n 13n^{-2} < \infty$.

Hence, by the Borel-Cantelli Lemma, $\mathbf{P}(X_n \geq n \text{ i.o.}) = 0$.

3. For $r \geq 0$, let ρ_r be the probability law having density function (with respect to Lebesgue measure on $[0,1]$) given by $f_r(x) = (r+1)x^r \mathbf{1}_{x \in [0,1]}$, and let δ_r be a point-mass at r. Let $\mu = (1/4)\rho_2 + (3/4)\delta_{9/10}$, and let $\nu = a\rho_3 + (1-a)\delta_b$ for some $a, b \in [0,1]$.

(a) [5] Determine (with explanation) all possible values of a and b which make $\mu \ll \nu$ (i.e., which make μ be absolutely continuous with respect to ν).

Solution. *Recall that $\mu \ll \nu$ iff $\nu(A) > 0$ whenever $\mu(A) > 0$. Since $\mu\{9/10\} = 3/4 > 0$, we need $\nu\{9/10\} > 0$, i.e. $b = 9/10$ and $a < 1$. Since $\mu([0,1/2]) = 1/2 > 0$, we need $\nu([0,1/2]) > 0$, i.e. $a > 0$. Then (check!) if $b = 9/10$ and $0 < a < 1$, then $\nu(A) > 0$ whenever $\mu(A) > 0$, so $\mu \ll \nu$.*

(b) [5] For the values of a and b found in part (a), compute (with explanation) the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$.

Solution. *Here* $\frac{d\mu}{d\nu}(9/10) = \frac{3/4}{1-a} = \frac{3}{4-4a}$, *and for* $x \neq 9/10$, $\frac{d\mu}{d\nu}(x) = \frac{(1/4)3x^2}{a4x^3} = \frac{3}{16x}$. *(Note that* $\frac{d\mu}{d\nu}(0)$ *can be defined arbitrarily, since* $\nu\{0\} = 0$.*) Then (check!) for any measurable subset* $A \subseteq [0,1]$, $\mu(A) = \int_A \frac{d\mu}{d\nu}(x)\, \nu(dx)$, *as required.*

4. Let $S = \{6, 7, 8, 9, \ldots\}$ be a state space. Let $\{X_n\}$ be a Markov chain on S, with $X_0 = 10$, and with transition probabilities given by $p_{6,6} = 1$, and $p_{7,6} = p_{7,8} = 1/2$, and for $i \geq 8$, $p_{i,i-1} = p_{i,i-2} = 1/5$ and $p_{i,i+1} = 3/5$, with $p_{i,j} = 0$ otherwise.

(a) [5] Prove that $\{X_n\}$ is a martingale.

Solution. *(i) We must have* $6 \leq X_n \leq 10 + n$, *so* $\mathbf{E}|X_n| \leq 10 + n < \infty$.

(ii) $\mathbf{E}(X_{n+1} \,|\, X_n) = (1/5)(X_n - 1) + (1/5)(X_n - 2) + (3/5)(X_n + 1) = X_n$ *for* $X_n \geq 8$, *and* $\mathbf{E}(X_{n+1} \,|\, X_n) = (1/2)(6) + (1/2)(8) = 7 = X_n$ *for* $X_n = 7$, *and* $\mathbf{E}(X_{n+1} \,|\, X_n) = (1)(6) = 6 = X_n$ *for* $X_n = 6$, *so in any case* $\mathbf{E}(X_{n+1} \,|\, X_n) = X_n$.

Facts (i) and (ii) show that $\{X_n\}$ *is a martingale.*

(b) [5] Compute $\mathbf{P}(X_3 = 10)$.

Solution. *The paths of positive probability leading from 10 to 10 in three steps are* $10 \to 11 \to 12 \to 10$, $10 \to 11 \to 9 \to 10$, *and* $10 \to 8 \to 9 \to 10$. *So,* $\mathbf{P}(X_3 = 10) = (3/5)(3/5)(1/5) + (3/5)(1/5)(3/5) + (1/5)(3/5)(3/5) = 27/125$.

(c) [5] Compute $\mathbf{E}(X_3)$.

Solution. *Since* $\{X_n\}$ *is a martingale,* $\mathbf{E}(X_3) = \mathbf{E}(X_0) = \mathbf{E}(10) = 10$.

NOTE: This can also be computed directly, but it's messier.

(d) [5] Let $T = \inf\{n \geq 0 : X_n = 6 \text{ or } 13\}$. Compute $\mathbf{P}(X_T = 13)$.

Solution. *Let* $p = \mathbf{P}(X_T = 13)$. *Then* $\mathbf{P}(X_T = 6) = 1 - p$. *Furthermore, since* $\{X_n\}$ *is bounded (between 6 and 13) up to time* T, *by the Optional Stopping [or, Sampling] Theorem, we must have* $\mathbf{E}(X_T) = \mathbf{E}(X_0) = 10$. *Hence,* $10 = p(13) + (1-p)(6) = 6 + 7p$. *So,* $p = 4/7$.

(e) [5] Determine whether or not the random variable $\lim_{n\to\infty} X_n$ exists with probability 1, and if so then what is its distribution.

Solution. *Here $\{X_n\}$ is a non-negative martingale, so by the Martingale Convergence Theorem, $\lim_{n\to\infty} X_n$ exists with probability 1.*

But if $\lim_{n\to\infty} X_n$ exists, then since the state space S is discrete, the process $\{X_n\}$ must eventually stop moving. This is only possible at the state 6. Hence, we must have $\lim_{n\to\infty} X_n = 6$ with probability 1.

5. [10] Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability triple defined by $\Omega = \mathbf{N} = \{1, 2, 3, \ldots\}$, and $\mathcal{F}$ is all subsets of Ω, and $\mathbf{P}\{\omega\} = 2^{-\omega}$ for all $\omega \in \Omega$. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, \mathbf{P})$, such that $X_n \to 0$ w.p. 1 as $n \to \infty$. Identify (with proof) the set A of all positive real numbers $a > 0$ with the property that if $-2^{\omega}/\omega^a \le X_n(\omega) \le 2^{\omega/3}$ for all $n \in \mathbf{N}$ and all $\omega \in \Omega$, then we must have $\lim_{n\to\infty} \mathbf{E}(X_n) = 0$. (That is, specify a set A of positive real numbers, and for $a \in A$ prove that we must have $\lim_{n\to\infty} \mathbf{E}(X_n) = 0$, and for $a \notin A$ provide a counter-example to show that we might <u>not</u> have $\lim_{n\to\infty} \mathbf{E}(X_n) = 0$.)

Solution. *Here $A = (1, \infty) = \{a \in \mathbf{R} : a > 1\}$.*

Then if $a \in A$, i.e. $a > 1$, then define the random variable Y by $Y(\omega) = (2^{\omega}/\omega^a) + 2^{\omega/3}$. Then since $-2^{\omega}/\omega^a \le X_n(\omega) \le 2^{\omega/3}$, therefore $|X_n(\omega)| \le \max\left[2^{\omega}/\omega^a,\ 2^{\omega/3}\right] \le (2^{\omega}/\omega^a) + 2^{\omega/3}\ Y(\omega)$, i.e. $|X_n| \le Y$ for all n. Furthermore,

$$\mathbf{E}(Y) \;=\; \sum_{\omega=1}^{\infty}(2^{-\omega})[(2^{\omega}/\omega^a) + 2^{\omega/3}] \;=\; \sum_{\omega=1}^{\infty}[\omega^{-a} + 2^{-2\omega/3}] \;<\; \infty$$

since $a > 1$. Hence, by the Bounded Convergence Theorem, since $X_n \to 0$ w.p. 1, therefore $\mathbf{E}(X_n) \to \mathbf{E}(0) = 0$.

On the other hand, if $a \notin A$, i.e. $0 < a \le 1$, then define X_n by $X_n(\omega) = -(2^{\omega}/\omega^a)\mathbf{1}_{\omega \ge n}$ for all $\omega \in \Omega$. Then $X_n(\omega) = 0$ whenever $n > \omega$, so $X_n \to 0$ w.p. 1. On the other hand,

$$\mathbf{E}(X_n) \;=\; \sum_{\omega=1}^{\infty}(2^{-\omega})(-2^{\omega}/\omega^a) \;=\; \sum_{\omega=1}^{\infty} -\omega^{-a} \;=\; -\infty$$

for all n, since $a \le 1$, so $\mathbf{E}(X_n) \not\to 0$.

6. Let X and Y be two jointly-defined real-valued random variables with finite means.

(a) [5] Define the conditional expectation $\mathbf{E}(Y|X)$.

Solution. $\mathbf{E}(Y|X)$ *is a <u>random variable</u>, defined on the same probability triple as are X and Y, satisfying the properties that (i) it is $\sigma(X)$-measurable, and (ii) $\mathbf{E}[\mathbf{E}(Y|X)\,\mathbf{1}_{X\in A}] = \mathbf{E}[Y\,\mathbf{1}_{X\in A}]$ for all Borel subsets $A \subseteq \mathbf{R}$. (This random variable is unique only up to sets of measure 0.)*

(b) [5] Give an example to show that we might <u>not</u> have $\mathbf{E}(Y|X) = \mathbf{E}(Y)$ w.p. 1.

Solution. *Suppose* $\mathbf{P}(X=3, Y=5) = \mathbf{P}(X=3, Y=7) = \mathbf{P}(X=4, Y=6) = \mathbf{P}(X=4, Y=8) = 1/4$. *Then* $\mathbf{E}(Y|X)(\omega) = 6$ *for all* ω *such that* $X(\omega)=3$, *and* $\mathbf{E}(Y|X)(\omega) = 7$ *for all* ω *such that* $X(\omega) = 4$. *By contrast,* $\mathbf{E}(Y) = 6.5$ *which is different. (Of course, we still have* $\mathbf{E}[\mathbf{E}(Y|X)] = 6.5 = \mathbf{E}(Y)$, *as it must.)*

NOTE: This is of course just one of many possible examples.

(c) [5] Suppose X and Y are independent. Prove that $\mathbf{E}(Y|X) = \mathbf{E}(Y)$ w.p. 1.

Solution. *Since* $\mathbf{E}(Y)$ *is just a real number (constant), it is of course* $\sigma(X)$*-measurable. So, it suffices to show that* $\mathbf{E}[\mathbf{E}(Y)\,\mathbf{1}_{X\in A}] = \mathbf{E}[Y\,\mathbf{1}_{X\in A}]$ *for all Borel subsets* $A \subseteq \mathbf{R}$. *But since* $\mathbf{E}(Y)$ *is a constant,* $\mathbf{E}[\mathbf{E}(Y)\,\mathbf{1}_{X\in A}] = \mathbf{E}(Y)\mathbf{E}[\mathbf{1}_{X\in A}] = \mathbf{E}(Y)\mathbf{P}[X\in A]$. *Also, since* X *and* Y *are independent, so are* Y *and* $\mathbf{1}_{X\in A}$, *and hence* $\mathbf{E}[Y\,\mathbf{1}_{X\in A}] = \mathbf{E}[Y]\,\mathbf{E}[\mathbf{1}_{X\in A}] = \mathbf{E}(Y)\mathbf{P}[X\in A]$, *which is the same expression. This completes the proof.*

7. Let $\{B_t\}_{t\ge 0}$ and $\{C_t\}_{t\ge0}$ be two independent standard Brownian motions, and let $0<\rho<1$ be a real number, and let $D_t = \rho\, B_t + \sqrt{1-\rho^2}\, C_t$ for $t \ge 0$.

(a) [5] Compute $\mathbf{E}[(B_3+1)(2B_5+3)]$.

Solution. $\mathbf{E}[(B_3+1)(2B_5+3)] = 2\mathbf{E}[B_3B_5] + 2\mathbf{E}[B_5] + 3\mathbf{E}[B_3] + 3 = 2(3) + 2(0) + 3(0) + 3 = 9$, *since* $\mathbf{E}[B_t] = 0$ *for all* t, *and* $\mathbf{E}[B_sB_t] = \min(s,t)$.

(b) [5] Show that $\{D_t\}_{t\ge0}$ is also a standard Brownian motion.

Solution. *Firstly,* $D_0 = \rho\, B_0 + \sqrt{1-\rho^2}\, C_0 = \rho\,(0) + \sqrt{1-\rho^2}\,(0) = 0$, *as it must. We then compute that* $\mathbf{E}[D_t] = \rho\mathbf{E}[B_t] + \sqrt{1-\rho^2}\,\mathbf{E}[C_t] = \rho(0) + \sqrt{1-\rho^2}\,(0) = 0$, *and* $\mathbf{E}[D_t^2] = \rho^2\mathbf{E}[B_t^2] + (\sqrt{1-\rho^2})^2\,\mathbf{E}[C_t^2] + 2\rho\sqrt{1-\rho^2}\,\mathbf{E}[B_tC_t] = \rho^2(t) + (1-\rho^2)\,(t) + 2\rho\sqrt{1-\rho^2}\,(0)(0) = t$, *as required.*

Furthermore, for $0 \le s < t$, $D_t - D_s$ *is a linear combination of the independent normal random variables* $B_t - B_s$ *and* $C_t - C_s$, *and hence is also normal.*

Finally, since $\{B_t\}$ *and* $\{C_t\}$ *have independent increments, this implies (check!) that* $\{D_t\}$ *has independent increments too.*

(c) [5] Compute $\mathbf{E}[D_t^2B_s^2]$ for $0<t<s$.

Solution. *Note first that since* $B_t \sim N(0,t)$, *therefore* $t^{-1/2}B_t \sim N(0,1)$, *and so* $\mathbf{E}[B_t^4] = t^2\mathbf{E}[(t^{-1/2}B_t)^4] = t^2(3) = 3t^2$.

Then, $\mathbf{E}[B_t^2B_s^2] = \mathbf{E}[B_t^2(B_t+B_s-B_t)^2] = \mathbf{E}[B_t^2(B_t^2+(B_s-B_t)^2+2B_t(B_s-B_t)] = \mathbf{E}[B_t^4] + \mathbf{E}[B_t^2(B_s-B_t)^2] + 2\mathbf{E}[B_t^3(B_s-B_t)] = 3t^2 + (t)(s-t) + 2(0)(0) = 2t^2 + st$.

Hence, $\mathbf{E}[D_t^2B_s^2] = \rho^2\mathbf{E}[B_t^2B_s^2] + (\sqrt{1-\rho^2})^2\,\mathbf{E}[C_t^2B_s^2] + 2\rho\sqrt{1-\rho^2}\,\mathbf{E}[B_tC_tB_s^2] = \rho^2\mathbf{E}[B_t^2B_s^2] + (1-\rho^2)\,\mathbf{E}[C_t^2]\,\mathbf{E}[B_s^2] + 2\rho\sqrt{1-\rho^2}\,\mathbf{E}[B_tB_s^2]\,\mathbf{E}[C_t] = \rho^2(2t^2+st) + (1-\rho^2)(t)(s) + 2\rho\sqrt{1-\rho^2}\,\mathbf{E}[B_tB_s^2]\,(0) = \rho^2(2t^2+st) + (1-\rho^2)(t)(s) = 2\rho^2t^2 + st$.

(d) [5] Let $\{X_t\}_{t\geq 0}$ be a diffusion, satisfying $X_0 = 8$ and $dX_t = 3\,dB_t - 2\,dt$. Compute $\mathbf{E}(X_5^2)$.

Solution. *Here* $X_t = X_0 + \int_0^t dX_s = 8 + 3(B_t - B_0) - 2(t-0) = 8 + 3B_t - 2t$. *Hence,* $\mathbf{E}[X_5^2] = \mathbf{E}[(8 + 3B_5 - 10)^2] = \mathbf{E}[(3B_5 - 2)^2] = \mathbf{E}[9(B_5)^2 + 4 - 12B_5] = 9(5) + 4 - 12(0) = 49$.

(e) [5] Let $\{Y_t\}_{t\geq 0}$ be a diffusion, satisfying $dY_t = 5dB_t - 3Y_t\,dt$. Let $Z_t = (Y_t)^2$ for all $t \geq 0$. Use Itô's Lemma (i.e., Itô's Formula) to derive a formula for dZ_t.

Solution. *Here* $\{Y_t\}$ *is a diffusion with* $\sigma(y) = 5$, *and* $\mu(y) = -3y$. *Also* $Z_t = f(Y_t)$ *where* $f(y) = y^2$, *so* $f'(y) = 2y$ *and* $f''(y) = 2$. *Hence, by Itô's Lemma,*

$$dZ_t \;=\; d(f(Y_t)) \;=\; f'(Y_t)\,\sigma(Y_t)\,dB_t + \Big(f'(Y_t)\,\mu(Y_t) + \frac{1}{2}f''(Y_t)\,\sigma^2(Y_t)\Big)\;dt$$

$$=\; 2Y_t(5)\,dB_t + \Big(2Y_t(-3Y_t) + \frac{1}{2}(2)(5^2)\Big)\;dt \;=\; 10\,Y_t\,dB_t + \Big(-6\,Y_t^2 + 25\Big)\;dt\,.$$

8. Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity 4, i.e. such that $N(0) = 0$, and $N(t) - N(s) \sim \text{Poisson}[4(t-s)]$ for $0 < s < t$, and $\{N(t_i) - N(s_i)\}_{i=1}^k$ are independent whenever $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq s_3 \leq \ldots \leq t_k$ for any $k \in \mathbf{N}$.

(a) [5] Compute (with explanation) $\mathbf{E}[N(3)\,(N(5) - N(1))]$.

Solution. *Recall that since* $N(t) - N(s) \sim \text{Poisson}[4(t-s)]$, *therefore* $\mathbf{E}[N(t) - N(s)] = 4(t-s)$, *and* $\text{Var}[N(t) - N(s)] = 4(t-s)$, *and* $\mathbf{E}[(N(t) - N(s))^2] = 4(t-s) + [4(t-s)]^2$.
Hence, $\mathbf{E}[N(3)\,(N(5)-N(1))] = \mathbf{E}[N(3)\,(N(5)-N(3)+N(3)-N(1))] = \mathbf{E}[N(3)\,(N(5) - N(3))] + \mathbf{E}[N(3)\,(N(3) - N(1))] = \mathbf{E}[N(3)]\,\mathbf{E}[N(5) - N(3)] + \mathbf{E}[(N(3) - N(1) + N(1))\,(N(3) - N(1))] = (4(3))\,(4(5-3)) + \mathbf{E}[(N(3) - N(1))^2] + \mathbf{E}[N(1)]\,\mathbf{E}[N(3) - N(1)] = (12)(10) + 4(3-1) + [4(3-1)]^2 + 4(1)\,4(3-1) = 120 + 8 + 64 + 32 = 224$.

(b) [5] Find (with explanation) constants a_m and b_m for $m \in \mathbf{N}$, such that

$$\lim_{m\to\infty} \mathbf{P}\Big(\frac{N(m) - a_m}{b_m} \leq x\Big) \;=\; \Phi(x)\,,$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\,e^{-u^2/2}\,du$ is the c.d.f. of a standard Normal distribution.

Solution. *Here* $N(m) = \sum_{i=1}^m X_i$ *where* $X_i = N(i) - N(i-1)$. *Hence, the* $\{X_i\}$ *are i.i.d. with mean 4 and variance 4. Hence, if* $a_m = 4m$, *and* $b_m = \sqrt{4m} = 2\sqrt{m}$, *then by the classical Central Limit Theorem (CLT),*

$$\lim_{m\to\infty} \mathbf{P}\Big(\frac{N(m) - a_m}{b_m} \leq x\Big) \;=\; \lim_{m\to\infty} \mathbf{P}\Big(\frac{N(m) - 4m}{2\sqrt{m}} \leq x\Big) \;=\; \Phi(x)\,.$$

(c) [5] Let $Y_t = N(t) - 4t$ for $t \geq 0$. Prove that $\{Y_t\}_{t \geq 0}$ is a martingale.

Solution. *Firstly,* $\mathbf{E}|Y_t| \leq \mathbf{E}[N(t)] + 4t = 4t + 4t = 8t < \infty$.

Secondly, if $0 \leq s < t$, *then the conditional distribution of* $N(t) - N(s)$ *given* $N(s)$ *is* Poisson$(4(t-s))$, *so* $\mathbf{E}[N(t) - N(s) \mid N(s)] = 4(t-s)$, *so* $\mathbf{E}[N(t) \mid N(s)] = 4(t-s) + N(s)$.

Hence, $\mathbf{E}[Y_t \mid N(s)] = \mathbf{E}[N(t) - 4t \mid N(s)] = 4(t-s) + N(s) - 4t = N(s) - 4s = Y_s$.

(It then also follows that $\mathbf{E}[Y_t \mid Y_s] = \mathbf{E}\Big(\mathbf{E}[Y_t \mid N(s)] \,\Big|\, Y_s\Big) = \mathbf{E}\Big(Y_s \,\Big|\, Y_s\Big) = Y_s$, *too.)*

These two facts show that $\{Y_t\}$ *is a martingale.*