May 2011

Ph.D. COMPREHENSIVE EXAMINATIONS
DEPARTMENT OF STATISTICS
UNIVERSITY OF TORONTO

PROBABILITY COMPREHENSIVE EXAMINATION

May 24, 2011, 12:30 p.m. – 4:30 p.m.

Sidney Smith Hall

(#questions = 10; (#points per question = 10) (#pages = 3, including cover page)

1. ATTEMPT ALL QUESTIONS.

2. Please work neatly and legibly.

3. ***You may answer all the questions in the same book(s).*** If you use more than one book for this exam, then also indicate which is the first book and which second, etc, as well as the total number of books used. Example: Book 1/2, Book 2/2.

4. It is not necessary to completely solve every problem to achieve a good performance. Emphasize what you do know.

5. Occasionally, problems may be improperly phrased or may contain a misprint. Should this happen, reflect it in your discussion.

6. This exam is closed book and notes. **NO CALCULATORS ARE ALLOWED.**

7. Good luck!

Probability Comprehensive Exam May 2011

All problems have equal weight

You may base your proofs on well known theorems if they are stated correctly and completely (but there will be no credit for saying that this is something we proved in class)

1. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$. Prove or disprove: There exist countably many $x_i \in \mathbb{R}$ and $a_i > 0$ and a probability density function f such that $P(X = x_i) = a_i$, $i = 1, 2, \ldots$ and for any $A \subset \mathbb{R} - \cup_{i=1}^{\infty}\{x_i\}$,
$$P(X \in A) = \int_A f(x)dx.$$

2. Let $f(x) = 1$ for $-1/2 \le x \le 1/2$ and $f(x) = 0$ elsewhere. Define $f_1(x) = f(x)$ and, for $n = 2, 3, \ldots$, $f_n(x) = \int_{-\infty}^{\infty} f(x-y) f_{n-1}(y)dy$. Find a_n, b_n and c_n so that $a_n f_n(b_n x + c_n)$ has a limiting probability density.

3. Let $X_1, X_2, \ldots$ be independent random variables, Poisson distributed with parameter $\lambda = 1$. Prove or disprove: There exist κ_n and μ_n so that $\kappa_n \sum_{\ell=1}^{n} X_\ell \log(\ell) - \mu_n$ has an asymptotically normal distribution.

4. Prove or disprove: If for each n, $X_{n,m}$, $m = 1, \ldots, n$ are random variables with mean $E[X_{n,m}] = 0$, $E[X_{n,m}^2] = 1$ and, for any $i \ne j$, $E[X_{n,i}X_{n,j}] = 1/n$, then $\frac{X_{n,1} + \cdots + X_{n,n}}{n} \to 0$ in probability.

5. Prove or disprove:
   (a) $e^{-|t|}$ is a characteristic function of a non-degenerate probability distribution.
   (b) $e^{-|t|^3}$ is a characteristic function of a non-degenerate probability distribution.
   (c) If $\varphi(t)$ is a characteristic function of a non-degenerate probability distribution, then $e^{\varphi(t)-1}$ is as well.

6. For each n, let $X_{n,m}$, $m = 1, \ldots, n$ be independent and identically distributed random variables with $P(X_{n,1} = k) = (1 - \frac{1}{n})n^{-k}$, $k = 0, 1, 2, \ldots$. Let $\phi_n(t) = E[e^{it(X_{n,1} + \cdots + X_{n,n})}]$. Compute
$$\phi(t) = \lim_{n \to \infty} \phi_n(t).$$

7. Prove or disprove: Suppose that $X_1, X_2, \ldots$ are independent with $P(X_i = 1) = P(X_i = -1) = 1/2$ and $a_1, a_2, \ldots$ is a non-random sequence of numbers. Let $S_n = a_1 X_1 + \cdots + a_n X_n$. Then if $E[S_n^2] \le 10$, the limit $S = \lim_{n\to\infty} S_n$ 1. exists almost surely; 2. defines a random variable; 3. satisfies $0 < E[S^2] < \infty$.

8. Let M_n, $n = 1, 2, \ldots$ be a martingale with respect to a filtration $\mathcal{F}_n$, $n = 1, 2, \ldots$ with $E[M_n^2] < \infty$ for each $n = 1, 2, \ldots$. Let $0 \le n_1 \le n_2 \le n_3 \le n_4 < \infty$. Prove or disprove:
   (a) $M_{n_4} - M_{n_3}$ and $M_{n_2} - M_{n_1}$ are uncorrelated.
   (b) $M_{n_4} - M_{n_3}$ and $M_{n_2} - M_{n_1}$ are independent.

9. Let X_n be the Markov chain with state space $\mathbb{Z}/L\mathbb{Z}$ (i.e. the integers $\{0, 1, \ldots, L-1\}$ with the rule that $(L-1)+1 = 0$) and transition probabilities $p(n, n+1) = p$, $p(n, n-1) = 1-p$, and $p(n, m) = 0$ if $m \neq n+1$ or $n-1$. Here $0 \leq p \leq 1$ is fixed.

   (a) Find an invariant probability measure (= stationary distribution).

   (b) Prove that there is at most one invariant probability measure.

   (c) For which value(s) of p does the invariant measure satisfy the detailed balance condition (i.e. is it reversible)?

10. Let $B(t)$, $t \geq 0$ be a Brownian motion starting at $B(0) = 0$ and let $X(t) = t^{1/4}B(t^{1/2})$, $t \geq 0$.

   (a) Compute the mean $m(t) = E[X(t)]$ and variance $v(t) = E[X^2(t)]$.

   (b) Prove or disprove: $X(t)$, $t \geq 0$ is a Brownian motion.