May 2010

Ph.D. COMPREHENSIVE EXAMINATIONS
DEPARTMENT OF STATISTICS
UNIVERSITY OF TORONTO

PROBABILITY COMPREHENSIVE EXAMINATION

May 17, 2010, 12:30 p.m. – 4:30 p.m.

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(#questions = 10; (#points per question = 10) (#pages = 3, including cover page)

1. ATTEMPT ALL QUESTIONS.
2. Please work neatly and legibly.
3. ***You may answer all the questions in the same book(s).*** If you use more than one book for this exam, then also indicate which is the first book and which second, etc, as well as the total number of books used. Example: Book 1/2, Book 2/2.
4. It is not necessary to completely solve every problem to achieve a good performance. Emphasize what you do know.
5. Occasionally, problems may be improperly phrased or may contain a misprint. Should this happen, reflect it in your discussion.
6. This exam is closed book and notes. **NO CALCULATORS ARE ALLOWED.**
7. Good luck!

Probability Comprehensive Exam

Each problem has equal weight

For each problem, give a convincing justification for your answer. It does not have to be a complete mathematical proof.

1. Let $Z_1, Z_2, \dots$ be independent random variables with Gaussian distribution, mean 0 and variance 1. Which of the following is the best estimate of
$$\max\{Z_1, Z_2, \dots, Z_n\}$$
   (a) $\sqrt{\log n}$
   (b) $\log n$
   (c) $\sqrt{n}$
   (d) n

2. Determine whether the following statement is true or false: *There exists a random variable X such that $E[e^{itX}] = e^{-t^4}$.*

3. Determine whether the following statement is true or false: *Let $X_1, X_2, \dots$ be independent random variables, with $E[X_n] = 0$, $E[X_n^2] = 1$ and $E[X_n X_m] \le 0$ for $n \neq m$. Then*
$$\frac{X_1 + \cdots + X_n - n}{n}$$
converges to zero in probability.

4. Determine whether the following statement is true or false: *Let $X_1, X_2, \dots$ be independent random variables, with $X_n \sim Gamma(k_n, \theta_n)$ where $k_n = 1 + \sin^2(n)$ and $\theta_n = 1/k_n$. Then*
$$\frac{X_1 + \cdots + X_n - n}{\sqrt{n}}$$
converges in distribution to a Gaussian. (Recall the density of a $Gamma(k, \theta)$ is $x^{k-1}\frac{e^{-x/\theta}}{\theta^k \Gamma(k)} 1_{x>0}$.)

5. For which $n \ge 1$ is the following statement true: *Let $X_1, X_2, \dots$ be a Markov chain on a state space $\{1, \dots, n\}$ with transition matrix*
$$\begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ & \ddots & \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$
(ie. every entry is $1/n$). Then the chain is irreducible and aperiodic.

6. Determine whether the following statement is true or false: *Let $X(t)$, $t \geq 0$ be a random continuous function from $[0, \infty)$ to the reals with $X(0) = 0$ and, for each $t > 0$, $X(t)$ is normally distributed with mean 0 and variance t. Then $X(t)$ is a Brownian motion.*

7. Let $X_0, X_1, X_2, \ldots$ be a simple symmetric random walk on the integers, with $X_0 = 0$. Let τ_A be the hitting time of A, ie.

$$\tau = \min\{n > 0 \ : \ X_n = A\}.$$

Which of the following is the best estimate of $E[\tau_A]$?

(a) 0
(b) $\sqrt{A}$
(c) A
(d) A^2
(e) ∞

8. Determine whether the following statement is true or false: *Let $X_0, X_1, X_2, \ldots$ be a martingale with uniformly bounded increments. Then $\lim_{n\to\infty} X_n$ exists almost surely on the set $\{\limsup_{n\to\infty} X_n < \infty\}$.*

9. Determine whether the following statement is true or false: *Let $X_0, X_1, X_2, \ldots$ be a martingale with $X_0 = 0$. Then*

$$\mathrm{Var}(X_n) = \sum_{m=1}^{n} \mathrm{Var}(X_m - X_{m-1}).$$

10. Determine if the following is true or false:

Let X_1, X_2 be an i.i.d sequence of real-valued random variables, and let $Y_n = \max(X_n, X_n + 1)$. Then the sequence Y_n is ergodic.