Bootstrapping point processes with some applications

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Abstract

One way to check asymptotic validity of the bootstrap is by looking at a point process in which points are observations. We study the weak limit behavior of some certain types of point processes by replacing the original observations by the bootstrap sample. The usual bootstrap fails asymptotically in cases for which there exist a Poisson point process or a fixed point measure for the limiting point process. In some cases, changing the resampling sample size to $m_n = o(n) \to \infty$ where n is the sample size this problem will be resolved and convergence holds in probability. If $m_n \log\log n/n \to 0$ then asymptotic results are valid almost surely. We present some cases that this does not resolve the asymptotic failure. Some applications of these representations are provided.

1 Introduction

Properties of the bootstrap for the mean of heavy tailed distributions have been the subject of much attention in statistical literature. See, for example, Athreya (1987), Knight (1989), and Hall (1990). It has been pointed out that the usual bootstrap (Efron (1979)) is not consistent for estimating the distribution of the mean, for general regression, and for first order autoregressive processes with infinite variance errors; see Davis and Wu (1994).

When the bootstrap limiting distribution differs from that of the original statistic, some authors have suggested changing the resampling sample size to $m_n = o(n)$ and $m_n \to \infty$. See for example, Wu *et al* (1990) and Davis and Wu (1994).

Instead of a specific statistic, here we study the weak limit behavior of the induced point process by the original sample and the induced point process by the bootstrap sample. By this, we mean to check if $\sum_{k=1}^{n} \varepsilon_{a_n^{-1}X_i^*}(\cdot) \stackrel{vd}{\to} \mu(\cdot)$ in some modes of convergent (notice that the probability measure induced by the bootstrap sample is a random measure and this random measure can be convergent almost surely, in probability or in distribution) where $\sum_{k=1}^{n} \varepsilon_{a_n^{-1}X_i}(\cdot) \stackrel{vd}{\to} \mu(\cdot)$. Note: "$\stackrel{vd}{\to}$" stands for convergence in distribution with respect to the vague topology and for any set A define

$$\varepsilon_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \in A^c. \end{cases}$$

In other words we get $\mu_n \stackrel{vd}{\to} \mu$ if and only if

$$(\mu_n(I_1), \mu_n(I_2), \ldots, \mu_n(I_k)) \Rightarrow (\mu(I_1), \mu(I_2), \ldots, \mu(I_k))$$

($\Rightarrow$ denotes convergence in distribution) for any $I_1, I_2, \ldots, I_k$ in $\mathcal{I}$ where $\mathcal{I}$ is a DC-semiring and

$$\mathcal{I} \subset \mathcal{B}_\mu = \{B \in \mathcal{B}: \quad \mu(\partial B) = 0 \quad \text{a.s.} \ \}$$

where $\mathcal{B}$ consists of all relatively compact sets (with compact closure). For most of our cases we can assume the family of finite intervals for a DC-semiring. For details see Kallenberg (1983). Notice that the probability measure induced by the bootstrap point process is a random measure and

$$\mu_n^*(\cdot) \stackrel{vd}{\to} \mu^*(\cdot) \quad \text{if and only if} \quad \int f dP_n^* \Rightarrow \int f dP^*$$

for all real valued, bounded and continuous functions on $(\mathcal{N}, d)$ where P_n^* and P^* are the random probability measure induced by μ_n^* and μ^* respectively, $\mathcal{N} = \mathcal{N}(\Re)$ is the set of point measures on $\Re$ (or in general, any Polish space $\mathcal{C}$) and d is the vague metric on $\mathcal{N}$. For details, see Kallenberg (1983). Throughout this paper we assume that m_n is the resampling sample size which is a nondecreasing sequence with $m_n \to \infty$.

2 Preliminary results

The asymptotic theory of point processes is heavily related to the following important Theorem.

Theorem 2.1 (Resnick(1987)) For each n suppose $\{X_{n,j}; j = 1, 2, \ldots\}$ are i.i.d. random elements of $(\mathcal{C}, \mathcal{B})$ where $\mathcal{C}$ is a Polish space and $\mathcal{B}$ is the Borel σ-algebra on $\mathcal{C}$. Assume μ is a Radon measure ($\mu(K) < \infty$ for any relatively compact set K). Define

$$\eta_n(\cdot) := \sum_{k=1}^{\infty} \varepsilon_{(k/n, X_{n,k})}(\cdot)$$

and suppose η is a Poisson random measure on $(0, \infty) \times \mathcal{C}$ with mean measure $dt \times d\mu$. We have

$$\eta_n \stackrel{vd}{\to} \eta \quad \text{if and only if} \quad nP\left[X_{n,1} \in \cdot\right] \stackrel{v}{\to} \mu(\cdot)$$

where "$\stackrel{v}{\to}$" stands for vague convergence. ($\int f d\mu_n \to \int f d\mu$ for all nonnegative and continuous function with compact support.)

Remark. Notice that in this paper $X_{n,j} = a_n^{-1} X_j$ for $j \in \{1, 2, \ldots\}$ where $\{X_j\}$ is i.i.d. and $\{a_n\}$ is a sequence of real numbers such that $a_n \to \infty$ and in each circumstance must be chosen appropriately.

This theorem generalizes the concept of approximating a binomial distribution $bin(n, p)$ by a Poisson distribution when p is small and n is large. By using Theorem 2.1 we can get a variety of results for different cases. An example can be given as follows:

Consider an i.i.d. sample $\{X_k\}$ with distribution function F where $P(|X_1| \geq x) = x^{-\alpha} L(x)$, $\alpha > 0$ for some slowly varying function L at infinity and

$$\frac{P(X_k > x)}{P(|X_k| > x)} \to p \quad \text{and} \quad \frac{P(X_k \leq -x)}{P(|X_k| > x)} \to q$$

as $x \to \infty$ for $p \in [0,1]$ and $q = 1-p$. By a slowly varying function at infinity we mean a function L with $\lim_{t\to\infty} L(tx)/L(t) = 1$. This is equivalent to $\{X_k\}$ belonging to the domain of attraction of a stable law with index $\alpha \in (0,2)$. By Davis and Resnick (1985a) we have

$$\sum_{k=1}^{n} \varepsilon_{\left(k/n, a_n^{-1} X_i\right)}(\cdot) \xrightarrow{vd} \sum_{k=1}^{\infty} \varepsilon_{\left(t_k, \delta_k \Gamma_k^{-1/\alpha}\right)}(\cdot) \tag{1.1}$$

where $a_n = \sup\{x : P(|X_1| > x) \leq 1/n\}$, $\{\delta_k\}$ is an i.i.d. and independent from $\{\Gamma_k\}$ with $P(\delta_1 = 1) = p$ and $P(\delta_1 = -1) = 1-p$ and $\Gamma_k = E_1 + \ldots + E_k$, where $\{E_i\}$ are i.i.d. with exponential distribution with unit mean. Also we get

$$\sum_{k=1}^{n} \varepsilon_{a_n^{-1} X_i}(\cdot) \xrightarrow{vd} \sum_{k=1}^{\infty} \varepsilon_{\delta_k \Gamma_k^{-1/\alpha}}(\cdot).$$

As is explained in Theorem 2.1, this is due to the fact that

$$nP(a_n^{-1} X_1 \in \cdot) \xrightarrow{v} \nu(\cdot) \qquad (1.2),$$

where ν is the Lévy measure defined by

$$\nu(dx) = \alpha p I(x > 0) x^{-\alpha-1} dx + \alpha(1-p) I(x < 0) |x|^{-\alpha-1} dx$$

with $0 < \alpha < 2$ and $p \in [0,1]$ is the same as before. ($X_{n,j} = a_n^{-1} X_j$ for $j = 1, \ldots, n$.) These representations are widely used by many authors for studying the limiting behavior of different statistics involving infinite variance observations. See for example, Davis *et al* (1992) and Davis and Resnick (1985 a,b). In this paper we give a similar representation for the bootstrap point process. By this we mean the point process that has been created by replacing the original observations with the bootstrap sample.

3 Results

In this section, we give a relatively general result for the limiting distribution of the bootstrap point process in a certain case. This theorem shows that the usual bootstrap fails to recover the original distribution asymptotically. In the limiting point process a random multiplicity makes the limiting bootstrap point process different from original asymptotic point process. Changing the bootstrap sample size to $m_n = o(n)$ resolves this problem. By using this result we will present some examples and applications.

Theorem 3.1 (i) For $n \geq 1$, let $\{X_{n,1}, \ldots, X_{n,n}\}$ denote i.i.d. random elements of any Polish space $(\mathcal{C}, \mathcal{S})$ such that $nP(X_{n,1} \in \cdot) \xrightarrow{v} \mu(\cdot)$ where μ is a Radon measure (measure of compact sets are finite) on $\mathcal{C}$. Also assume that $\{X_{n,1}^*, \ldots, X_{n,n}^*\}$ (the bootstrap sample) are i.i.d. from the distribution

$$F_n^{(1)}(x) = \frac{1}{n} \sum_{i=1}^{n} I\left(X_{n,i} \leq x\right).$$

(ii) Under the assumption in (i) assume $\{X^*_{m_n,1},\ldots,X^*_{m_n,m_n}\}$ are i.i.d from the distribution

$$F_n^{(2)}(x)=\frac{1}{n}\sum_{i=1}^{n}I\left(X_{m_n,i}\le x\right)$$

with $m_n=o(n)\to\infty$. We have

$$m_n n^{-1}a_{m_n}^{-1}\sum_{i=1}^{n}\varepsilon_{X_{m_n,i}}(\cdot)\overset{vp}{\to}\mu(\cdot)$$

In this case we have

$$a_{m_n}^{-1}\sum_{i=1}^{m_n}\varepsilon_{X^*_{m_n,i}}(\cdot)\Rightarrow\chi(\cdot)\ \text{ in probability}$$

where χ is a Poisson random measure with mean measure μ.

This is an example that the bootstrap fails in probability. Asymptotic failure in here is very different from the previous cases. As it is well understood for example, in finite variance or infinite variance i.i.d in the domain of attraction of the stable law with index $\alpha\in(0,2)$ with first moment $\bar{X}^*-\bar{X}\overset{P^*}{\to}0$ (almost surely in the finite variance case and in probability in infinite variance case) even though in the infinite variance case the bootstrap fails asymptotically. In our case even this property does not hold. This will not be resolved by taking $m_n=o(n)$ as usual.

4.4 Bootstrapping the extremes.

Let $\{X_n\}$ be i.i.d. with distribution F in the maximum domain of attraction of the distribution G. This means that there exist $a_n>0$ and b_n, $n\ge1$ such that

$$P\left(\bigvee_{i=1}^{n}X_i\le a_nx+b_n\right)=F^n\left(a_nx+b_n\right)\to G(x) \tag{4.1}$$

as $n\to\infty$ where $\bigvee_{i=1}^{n}X_i=\max(X_1,\ldots,X_n)$. It can be shown that (Galambos (1978)) G is of the type of one of the following three classes: $\Phi_\alpha(x)=\exp\{-x^{-\alpha}\}I(x\ge0)$ for some $\alpha>0$, $\Psi_\alpha(x)=I(x<0)\exp\{-(-x)^\alpha\}+I(x\ge0)$ for some $\alpha>0$ or $\Lambda(x)=\exp(-\exp(-x))$ for $x\in\Re$. For these types we have

(i) $G=\Lambda$ on $\mathcal{N}(-\infty,\infty]$ if and only if

$$\xi_n(\cdot)=\sum_{k=1}^{n}\varepsilon_{a_n^{-1}(X_k-b_n)}(\cdot)\Rightarrow\xi(\cdot)=\sum_{k=1}^{\infty}\varepsilon_{-log\Gamma_k}(\cdot)$$

where $\xi(\cdot)$ is a Poison random measure with a mean measure ν on $(-\infty,\infty]$ with $\nu(x,\infty]=\exp(-x)$,

(ii) $G=\Phi_\alpha$, $F(0)=0$ if and only if

$$\xi_n(\cdot)=\sum_{k=1}^{n}\varepsilon_{a_n^{-1}X_k}(\cdot)\Rightarrow\xi(\cdot)=\sum_{k=1}^{\infty}\varepsilon_{\Gamma_k^{-1/\alpha}}(\cdot)$$

where $\xi(\cdot)$ is a Poisson random measure on $(0,\infty]$ and with a mean measure $\nu(x,\infty]=x^{-\alpha}$ for $x>0$, and

(iii) $G = \Psi_\alpha$ if and only if

$$\xi_n(\cdot) = \sum_{k=1}^{n} \varepsilon_{(X_k - x_0)/(x_0 - \gamma_n)}(\cdot) \Rightarrow \xi(\cdot) = \sum_{k=1}^{\infty} \varepsilon_{-\Gamma_k^{1/\alpha}}(\cdot)$$

where $x_0 = \sup\{x: \ F(x) < 1\}$ and $\gamma_n = \inf\{s: \ F(s) > 1 - 1/n\}$. These are an easy consequence of the Theorem 2.1. In other words by using (4.1) we can show that

$$nP\left(a_n^{-1}(X_k - b_n) \in (\cdot)\right) \xrightarrow{v} \nu(\cdot).$$

Thus by using Theorem 3.1 we get all three types of representation for the bootstrap limiting point processes. For case (i) the limiting point process will be $\xi^*(\cdot) = \sum_{k=1}^{\infty} M_k^* \varepsilon_{-log\Gamma_k}(\cdot)$; for the case (ii) we get $\xi^*(\cdot) = \sum_{k=1}^{\infty} M_k^* \varepsilon_{\Gamma_k^{-1/\alpha}}(\cdot)$, and finally for case (iii) we get $\xi^*(\cdot) = \sum_{k=1}^{\infty} M_k^* \varepsilon_{-\Gamma_k^{1/\alpha}}(\cdot)$. While by using $m_n = o(n) \to \infty$ sample results the same limiting Poisson point processes as the original point processes. This indicates that the usual bootstrap fails asymptotically in these cases but by changing the resampling sample size to $m_n = o(n) \to \infty$ this problem will be resolved. Now, consider the functional $\mathcal{T}$, $\mathcal{T}: \mathcal{N}(\mathcal{C}) \longrightarrow \Re$ in either of three cases by defining $\mathcal{T}(\sum_{i=1}^{n} \varepsilon_{U_i}(\cdot)) = \bigvee_{i=1}^{n} U_i$. $\mathcal{T}$ is continuous mapping (Resnick (1987)). Therefore

$$a_n^{-1}\left(\bigvee_{i=1}^{n} X_i^* - b_n\right) \Rightarrow \mathcal{M}^*$$

where $\mathcal{M}^*$ has a random distribution

$$G^*(x) = P^*(\mathcal{M}^* \leq x) = \exp(-\xi(x, \infty]).$$

Notice convergence holds in distribution with respect to the vague topology. By using $m_n = o(n)$ resampling sample size we get

$$a_{m_n}^{-1}\left(\bigvee_{i=1}^{m_n} X_i^* - b_{m_n}\right) \Rightarrow \mathcal{M} \text{ in probability}$$

where $P(\mathcal{M} \leq x) = G(x)$ and G is the same as (4.1). It is not difficult to observe a direct approach for these results. For example, for an appropriate $x \in \Re$ we get

$$P\left(a_n^{-1}\left(\bigvee_{i=1}^{n} X_i^* - b_n\right) \leq x\right) = \left\{1 - n^{-1}\sum_{i=1}^{n} \varepsilon_{a_n^{-1}(X_i - b_n)}\left((x, \infty]\right)\right\}^n.$$

Since

$$\sum_{i=1}^{n} \varepsilon_{a_n^{-1}(X_i - b_n)}\left((x, \infty]\right) \Rightarrow \xi\left((x, \infty]\right)$$

we have

$$P\left(a_n^{-1}\left(\bigvee_{i=1}^{n} X_i^* - b_n\right) \leq x\right) \to \exp\left(-\xi\left((x, \infty]\right)\right)$$

Remark. As we mentioned in Theorem 3.1 random variables are living in a very general space therefore as long as a limiting Poisson point process exists results will follow consequently. For example, in extreme value theory the class of max-stable distribution can be obtained by a Poisson point process (see Resnick (1987)) and this leads us to investigate bootstrapping extremes in any Euclidean space. Obviously the usual bootstrap will fail asymptotically but the bootstrap will be valid asymptotically with $m_n = o(n)$ in probability and by taking a regular sequence m_n with $m_n \log\log n/n \to 0$ the bootstrap will be valid almost surely. This idea can be generalized to the mean as well.

Appendix.

Proof of Theorem 3.1:

For (i), note that the space of random measures with the vague topology is complete and separable thus it is metrizable and by using Skorohod representation we may find a probability space and random measures π_n and π with the same distribution as $\xi_n(\cdot) = n^{-1}\sum_{i=1}^{n}\varepsilon_{X_{n,j}}(\cdot)$ and $\xi(\cdot) = \sum_{i=1}^{\infty}\varepsilon_{U_i}(\cdot)$ respectively such that $n\pi_n \xrightarrow{a.s.} \pi$ with respect to the vague topology. Now, since $P^*\left(X_{n,1}^* \in \cdot\right) = \xi_n(\cdot)$ thus $nP^*\left(X_{n,1}^* \in \cdot\right) \xrightarrow{v} \xi(\cdot)$. On this space, for any continuous, nonnegative function with compact support on C we have

$$\Psi_n^*(f) = E\left[\exp\left(-\int f d\xi_n^*\right)\right] = \left(1 - \frac{\int (1-\exp(-f(x)))\, nP^*[X_{n,1}^* \in dx]}{n}\right)^n$$

$$\longrightarrow \Psi^*(f) = E\left[\exp\left(-\int f d\xi^*\right)\right].$$

Therefore convergence holds in distribution in the original space.

For (ii), notice that

$$E\left[\exp\left(\theta n^{-1} m_n \sum_{i=1}^{n}\varepsilon_{X_{m_n,i}}(I)\right)\right] = \left[P\left(X_{m_n,1} \in I\right)\left(\exp\left(\theta n^{-1} m_n\right) - 1\right) + 1\right]^n \to \exp(\theta\mu(I)).$$

This is due to the fact that $m_n P\left(X_{m_n,1} \in \cdot\right) \xrightarrow{v} \mu(\cdot)$. Therefore to prove $\xi_n^{**}(\cdot) \Rightarrow \xi_\mu(\cdot)$ in probability, we show for any $I_1, \ldots, I_k$ in the DC-semiring $(\xi_n^{**}(I_1), \ldots, \xi_n^{**}(I_k)) \Rightarrow (\xi_\mu(I_1), \ldots, \xi_\mu(I_k))$ in probability. (See Kallenberg (1983) for details.) For simplicity we assume $k = 2$. We show the joint generating function for $(\xi_n^{**}(I_1), \xi_n^{**}(I_2))$ converges to the joint generating function of $(\xi_\mu(I_1), \xi_\mu(I_2))$. We have

$$g_n(s,t) = E\left[s^{\xi_n^{**}(I_1)} t^{\xi_n^{**}(I_2)}\right] = \left[E\left(s^{\varepsilon_{X_{m_n,1}^*}(I_1)} t^{\varepsilon_{X_{m_n,1}^*}(I_2)}\right)\right]^{m_n}$$

$$= \left[1 + n^{-1}\left(s\sum_{i=1}^{n}\varepsilon_{X_{m_n,i}}(I_1 \backslash I_2) + t\sum_{i=1}^{n}\varepsilon_{X_{m_n,i}}(I_2 \backslash I_1) + st\sum_{i=1}^{n}\varepsilon_{X_{m_n,i}}(I_1 \cap I_2) - \sum_{i=1}^{n}\varepsilon_{X_{m_n,i}}(I_1 \cup I_2)\right)\right]^{m_n}$$

$$\to \exp\left[(s-1)\mu(I_1) + (t-1)\mu(I_2) + (s-1)(t-1)\mu(I_1 \cap I_2)\right].$$

This is the joint generating function for $(\xi_\mu(I_1), \xi_\mu(I_2))$ and proof is complete.

For (iii), If we show that $n^{-1} m_n \sum_{i=1}^{n}\varepsilon_{X_{m_n,i}}(\cdot) \xrightarrow{v} \mu(\cdot)$ a.s. then we get $m_n P^*(X_{m_n,1}^* \in \cdot) \xrightarrow{v} \mu(\cdot)$ a.s. therefore by Theorem 2.1 result follows. To show this notice that from Proposition 3.12

in Resnick (1987) for a sequence of random measures δ_n, $\delta_n(\cdot) \to \mu(\cdot)$ if and only if $\delta_n(B) \to \mu(B)$ for all relatively compact set B with $\mu(\partial(B)) = 0$. Since $m_n P(X_{m_n,n} \in \cdot) \xrightarrow{v} \mu(\cdot)$ we show

$$n^{-1} m_n \sum_{i=1}^{n} \varepsilon_{X_{m_n,n}}(B) - m_n P(X_{m_n,n} \in B) = n^{-1} m_n \sum_{i=1}^{n} \left(\varepsilon_{X_{m_n,n}}(B) - P(X_{m_n,n} \in B)\right) \xrightarrow{a.s.} 0.$$

Similar case was proved in Arcones and Giné (1989) in Theorem 3.4.

Proof for Theorem 4.1: For (i) Using the Skorohod representation theorem and a similar argument to the Proposition 5.3 of Resnick (1986) the result follows easily.

For (ii) we can follow exactly the same method as in proof of the Theorem 3.1, (ii). For any set I in DC-semiring we have

$$E\left[\exp\left(\theta n^{-1} a_n^{-1} m_n \sum_{i=1}^{n} \varepsilon_{X_{m_n,i}}(I)\right)\right] = \left[P\left(X_{m_n,1} \in I\right)\left(\exp\left(\theta n^{-1} a_n^{-1} m_n\right) - 1\right) + 1\right]^n \to \exp(\theta\mu(I)).$$

This is due to the fact that $m_n a_n^{-1} P\left(X_{m_n,1} \in \cdot\right) \xrightarrow{v} \mu(\cdot)$. Therefore by using the same argument in the proof of the Theorem 3.1 we have

$$n^{-1} a_n^{-1} m_n \sum_{i=1}^{m_n} \varepsilon_{X^*_{m_n,i}}(\cdot) \xrightarrow{vd} \chi(\cdot) \text{ in distribution}$$

where $\chi(\cdot)$ is a Poisson random measure with mean measure $\mu(\cdot)$.

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