Simplified Bounds on the Tails of Compound Distributions

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Abstract

Upper and lower bounds are derived for the tail probabilities of compound distributions using simple properties of the claim size distribution. General bounds are then obtained for various classes of claim size distributions. Some examples are given.

Keywords: failure rate; Lundberg inequality, ruin probability, IFR, DFR, NBU, NWU

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1 Introduction and Notation

Let $N$ be a counting random variable with

$$p_n = Pr(N = n); \quad n = 0, 1, 2, \ldots,$$  \hspace{1cm} (1.1)

and

$$a_n = Pr(N > n) = \sum_{k=n+1}^{\infty} p_k; \quad n = 0, 1, 2, \ldots.$$  \hspace{1cm} (1.2)

Also, let $\{Y_1, Y_2, \ldots\}$ be a sequence of independent and identically distributed positive random variables independent of $N$ with common distribution function (df)

$$F(y) = 1 - \overline{F}(y) = Pr(Y \leq y); \quad y \geq 0.$$  \hspace{1cm} (1.3)

Let $\overline{F}^{*n}(y) = Pr(Y_1 + Y_2 + \ldots + Y_n > y)$ for $n = 1, 2, 3, \ldots$, and

$$\overline{G}(x) = \sum_{n=1}^{\infty} p_n \overline{F}^{*n}(x); \quad x \geq 0.$$  \hspace{1cm} (1.4)

Then $\overline{G}(x) = Pr(Y_1 + Y_2 + \ldots + Y_N > x)$ is the tail of a compound distribution. Lin (1996) derived upper and lower bounds on the tail $\overline{G}(x)$ in terms of new worse than used (NWU) and new better than used (NBU) df’s. The df $B(y) = 1 - \overline{B}(y)$ is NWU (NBU) if $B(x)\overline{B}(y) \leq (\geq) B(x + y)$ for all $x \geq 0, y \geq 0$. A subclass of the NWU (NBU) class is the decreasing failure rate or DFR (increasing failure rate or IFR) class. The df $1 - \overline{B}(y)$ is DFR (IFR) if and only if $\overline{B}(y + t)/\overline{B}(y)$ is an increasing (decreasing) function of $y$ for all $t \geq 0$. See Barlow and Proschan (1975, p. 159).

In this paper simpler bounds than those of Lin (1996) are derived, and applications are given in various situations. For related results, see Willmot(1994).

2 Upper bounds

Suppose that $\phi < 1$ satisfies

$$a_{n+1} \leq \phi a_n; \quad n = 0, 1, 2, \ldots,$$  \hspace{1cm} (2.1)
and the df $1 - \overline{B}(y)$ is NWU and satisfies

$$\int_0^\infty \left\{ \overline{B}(y) \right\}^{-1} dF(y) = \phi^{-1}. \quad (2.2)$$

A detailed discussion of (2.1) is given by Willmot and Lin (1994). Then Lin (1996) showed that

$$\overline{G}(x) \leq \frac{1-p_0}{\phi} \Delta(x), \quad x \geq 0, \quad (2.3)$$

where

$$\{\Delta(x)\}^{-1} \leq \inf_{0 \leq z \leq x, \overline{F}(z) > 0} \left\{ \overline{F}(z) \right\}^{-1} \int_z^\infty \left\{ \overline{B}(x + y - z) \right\}^{-1} dF(y). \quad (2.4)$$

We have the following result.

**Theorem 1**

Suppose $\phi < 1$ satisfies (2.1), and $1 - \overline{B}(y)$ is a NWU df satisfying (2.2). If $H_x(y) = 1 - \overline{H}_x(y)$ is a df satisfying $H_x(0) = 0$ and

$$\overline{H}_x(y) \leq \inf_{0 \leq z \leq x} \overline{F}(z + y)/\overline{F}(z), \quad y \geq 0, \quad (2.5)$$

then

$$\overline{G}(x) \leq \frac{1-p_0}{\phi} \left\{ \int_0^\infty \frac{dH_x(y)}{\overline{B}(x + y)} \right\}^{-1}, \quad x \geq 0. \quad (2.6)$$

Proof: One has

$$\frac{\overline{F}(x)}{\overline{B}(x)} = \int_x^\infty \frac{dF(y)}{\overline{B}(x + y)} \leq \int_x^\infty \frac{dF(y)}{\overline{B}(y) \overline{B}(x + y)}. $$

The right hand side of this inequality must go to 0 as $x$ goes to infinity for (2.2) to hold. Thus, $\lim_{x \to \infty} \overline{F}(x)/\overline{B}(x) = 0$. Also, since $1 - \overline{B}(y)$ is NWU,

$$\frac{\overline{F}(y)}{\overline{B}(y + x - z)} \leq \left\{ \overline{B}(x - z) \right\}^{-1} \frac{\overline{F}(y)}{\overline{B}(y)}$$
for $x \geq z$, implying that $\lim_{y \to \infty} \frac{F(y)}{B(y + x - z)} = 0$. Then integration by parts yields

$$
\int_{z}^{\infty} \frac{\{B(x+y-z)\}^{-1}}{F(x)} dF(y) = \frac{1}{B(x)} + \int_{0}^{\infty} \frac{F(y+z)}{F(x)} d\left\{B(y+x)\right\}^{-1}
$$

$$
\geq \frac{1}{B(x)} + \int_{0}^{\infty} \overline{H}_x(y) d\left\{B(y+x)\right\}^{-1}.
$$

But from (2.5), $\frac{\overline{H}_x(y)}{B(x-y)} \leq \left\{\overline{B}(x)\right\}^{-1} \frac{F(y)}{B(y)}$, implying that $\lim_{y \to \infty} \frac{\overline{H}_x(y)}{B(x+y)} = 0$, and integration by parts again yields

$$
\int_{0}^{\infty} \left\{\overline{B}(y+x)\right\}^{-1} dH_x(y) = \frac{1}{B(x)} + \int_{0}^{\infty} \overline{H}_x(y) d\left\{B(x+y)\right\}^{-1}.
$$

Thus (2.3) holds with $\Delta(x) = \left\{\int_{0}^{\infty} \left\{\overline{B}(y+x)\right\}^{-1} dH_x(y)\right\}^{-1}$. \hfill \Box

As noted by Lin (1996), (2.5) holds with $\overline{H}_x(y) = \overline{F}(y)$ if $F(y)$ is NWU. In this and some other applications $\overline{H}_x(y)$ may be chosen independently of $x$ (see example 1). This is not the case, however, in the following corollary.

**Corollary 1**

Suppose $\phi < 1$ satisfies (2.1), and $1 - \overline{B}(y)$ is a NWU df satisfying (2.2). If $F(y)$ is IFR then

$$
\overline{G}(x) \leq \frac{1 - p_0}{\phi} \left\{\int_{x}^{\infty} \frac{dF(y)}{B(y)}\right\}^{-1} \overline{F}(x), \ x \geq 0.
$$

(2.7)

Proof: Since $F(y)$ is IFR, $\frac{F(y + z)}{\overline{F}(z)}$ is decreasing in $z$ and so

$$
\frac{F(y + x)}{\overline{F}(x)} = \inf_{0 \leq z \leq x} \frac{F(y + z)}{\overline{F}(z)}.
$$

Thus $\overline{H}_x(y) = \frac{F(y + x)}{\overline{F}(x)}$ and theorem 1 applies. \hfill \Box

**Corollary 2**

5
Suppose that \( \phi < 1 \) satisfies (2.1), and \( 1 - \overline{B}(y) \) is a NWU df satisfying (2.2). If \( F(y) \) is absolutely continuous with failure rate \( \mu_F(y) = -\frac{d}{dy} \ln \overline{F}(y) \) satisfying \( \mu_F(y) \leq \mu_F < \infty \), then

\[
\overline{G}(x) \leq \frac{1 - \frac{p_n}{\phi}}{\phi} \left\{ \int_{0}^{\infty} \left\{ \overline{B}(y + x) \right\}^{-1} \mu_F e^{-\mu_F y} dy \right\}^{-1}, \quad x \geq 0.
\] (2.8)

Proof: \( \overline{F}(y + z)/\overline{F}(z) = e^{-\int_{z}^{z+y} \mu_F(t) dt} \geq e^{-\int_{z}^{z+y} \mu_F dt} = e^{-\mu_F y} \), and theorem 1 applies with \( \overline{H}_x(y) = e^{-\mu_F y} \). \( \square \)

Lin (1996) has considered the case when \( B(y) \) has failure rate bounded from below. Other applications of theorem 1 are given in section 4. We now consider lower bounds.

3 Lower bounds

Suppose that \( \phi < 1 \) satisfies

\[
a_{n+1} \geq \phi a_n, \quad n = 0, 1, 2, \ldots,
\] (3.1)

and the df \( 1 - \overline{B}(y) \) is NBU and satisfies (2.2). Then Lin (1996) showed that

\[
\overline{G}(x) \geq \frac{1 - p_0}{\phi} \Delta(x), \quad x \geq 0,
\] (3.2)

where

\[
\{\Delta(x)\}^{-1} \geq \sup_{0 \leq z \leq x, \overline{F}(z) > 0} \left\{ \overline{F}(z) \right\}^{-1} \int_{z}^{\infty} \left\{ \overline{B}(x + y - z) \right\}^{-1} dF(y).
\] (3.3)

In this case we have the following result.

Theorem 2
Suppose $\phi < 1$ satisfies (3.1), and $1 - \overline{B}(y)$ is a NBU df satisfying (2.2). If $H_x(y)$ is a df satisfying $\lim_{y\to\infty} \overline{H}_x(y)/\overline{B}(x + y) = 0$ for any $x \geq 0$ and

$$\overline{H}_x(y) \geq \sup_{0 \leq z \leq x} \frac{\overline{F}(z + y)}{\overline{F}(z)}, \quad y \geq 0,$$

(3.4)

then

$$\overline{G}(x) \geq \frac{1 - p_0}{\phi} \left\{ \int_0^\infty \frac{dH_x(y)}{\overline{B}(x + y)} \right\}^{-1}, \quad x \geq 0.$$  \hspace{1cm} (3.5)

Proof: For $x \geq z$ one has from (3.4)

$$\frac{\overline{F}(y)}{\overline{B}(x + y - z)} \leq \frac{\overline{H}_x(y)}{\overline{B}(x + y)}$$

and so $\lim_{y \to \infty} \overline{F}(y)/\overline{B}(y + x - z) = 0$. Then integration by parts yields

$$\int_x^\infty \left\{ \frac{\overline{B}(x + y - z)}{\overline{B}(x)} \right\}^{-1} d\overline{F}(y) = \frac{1}{\overline{B}(x)} + \int_0^\infty \frac{\overline{F}(y + x)}{\overline{F}(x)} d\left\{ \overline{B}(y + x) \right\}^{-1}$$

$$\leq \frac{1}{\overline{B}(x)} + \int_0^\infty \overline{H}_x(y) d\left\{ \overline{B}(y + x) \right\}^{-1}$$

$$= \int_0^\infty \left\{ \overline{B}(y + x) \right\}^{-1} d\overline{H}_x(y)$$

since (3.4) implies that $\overline{H}_x(0) = 1$. Thus (3.2) is satisfied by

$$\Delta(x) = \left\{ \int_0^\infty \left\{ \overline{B}(y + x) \right\}^{-1} d\overline{H}_x(y) \right\}^{-1}.$$ \hspace{1cm} \Box

Lin (1996) noted that one may choose $\overline{H}_x(y) = \overline{F}(y)$ if $F(y)$ is NBU and theorem 2 then applies if $\lim_{y \to \infty} \overline{F}(y)/\overline{F}(x + y) = 0$. Other applications of theorem 2 are now given.

**Corollary 3**
Suppose $\phi < 1$ satisfies (3.1), and $1 - \overline{B}(y)$ is a NBU df satisfying (2.2). If $F(y)$ is DFR then
\[
\overline{G}(x) \geq \frac{1 - p_0}{\phi} \left\{ \int_x^\infty \frac{dF(y)}{B(y)} \right\}^{-1} F(x), \quad x \geq 0.
\] (3.6)

Proof: Since $F(y)$ is DFR, $F(y + z)/\overline{F}(z)$ is increasing in $z$, and
\[
\overline{F}(y + x)/\overline{F}(x) = \sup_{0 \leq z \leq x} \overline{F}(y + z)/\overline{F}(z).
\]
Thus choose $\overline{H}_x(y) = \overline{F}(y + x)/\overline{F}(x)$. As noted in the proof of theorem 1, if (2.2) holds then $\lim_{y \to \infty} \overline{F}(y)/\overline{B}(y) = 0$. Thus
\[
\lim_{y \to \infty} \overline{H}_x(y)/\overline{B}(x + y) = \left\{ \overline{F}(x) \right\}^{-1} \lim_{y \to \infty} \overline{F}(y + x)/\overline{B}(y + x) = 0.
\]
and theorem 2 applies. \hfill \Box

**Corollary 4**

Suppose that the conditions of theorem 2 hold. If, in addition, $1 - \overline{B}(y)$ is absolutely continuous with failure rate $\mu_B(y) = -\frac{d}{dy} \ln \overline{B}(y)$ which satisfies $\sup_{y \geq x} \mu_B(y) \leq \kappa_x < \infty$, then
\[
\overline{G}(x) \geq \frac{1 - p_0}{\phi} \left\{ \int_0^\infty e^{\kappa_x y} dH_x(y) \right\}^{-1} \overline{B}(x), \quad x \geq 0.
\] (3.7)

Proof:
\[ \int_0^\infty \{ \overline{B}(y + x) \}^{-1} dH_x(y) = \frac{1}{B(x)} \int_0^\infty \int e^{\int_s^t \mu_B(t) dt} \, dH_x(y) \]
\[ \leq \frac{1}{B(x)} \int_0^\infty e^{\int s^t \kappa_x dt} \, dH_x(y) \]
\[ = \frac{1}{B(x)} \int_0^\infty e^{\kappa_x y} dH_x(y) . \]

The result then follows from (3.5). \[ \square \]

If \( 1 - \overline{B}(y) \) is absolutely continuous and IFR, then corollary 4 may be used with
\[ \kappa_x = \lim_{y \to \infty} \mu_B(y) \text{ if } \lim_{y \to \infty} \mu_B(y) < \infty . \]

We now construct a general lower bound in terms of \( F(y) \) itself when \( F(y) \) is an absolutely continuous NBU df with bounded failure rate.

**Corollary 5**

Suppose that \( \phi < 1 \) satisfies (3.1). Suppose also that \( F(y) \) is an absolutely continuous NBU df with failure rate \( \mu_F(y) = -\frac{d}{dy} \ln F(y) \) which satisfies \( \mu_F(y) \leq \mu_F < \infty \). Then
\[ \overline{C}(x) \geq \frac{1 - p_0}{\phi \int_0^\infty e^{\mu_F(1-\phi)y} dF(y)} \left\{ \overline{F}(x) \right\}^{1-\phi}, \quad x \geq 0 . \tag{3.8} \]

**Proof:** Since \( F(y) \) is absolutely continuous and NBU so is \( 1 - \overline{B}(y) \) where \( \overline{B}(y) = \left\{ \overline{F}(y) \right\}^{1-\phi} \), and \( \overline{B}(y) \) satisfies (2.2). Also, (3.4) is satisfied by \( \overline{H}_x(y) = \overline{F}(y) \) since \( F(y) \) is
NBU. Furthermore,

\[
\frac{H_x(y)}{B(x+y)} = \frac{F(y)}{F(x+y)} \left\{ \frac{F(x+y)}{F(x)} \right\}^\phi = e^{\int_y^{y\phi} \mu_F(t)dt} \left\{ \frac{F(x+y)}{F(x)} \right\}^\phi \\
\leq e^{\int_y^{y\phi} \mu_F dt} \left\{ \frac{F(x+y)}{F(x)} \right\}^\phi = e^{\mu_F x} \left\{ \frac{F(x+y)}{F(x)} \right\}^\phi.
\]

Since \( \lim_{y \to \infty} \left\{ \frac{F(y+x)}{F(x)} \right\}^\phi = 0 \), it follows that \( \lim_{y \to \infty} \frac{H_x(y)}{B(x+y)} = 0 \). Thus, theorem 2 holds, and \( 1 - B(y) \) has failure rate \( (1 - \phi)\mu_F(y) \leq (1 - \phi)\mu_F \), and the result follows from corollary 4.

It can be shown that (3.8) is an equality when \( F(y) = e^{-\mu y} \) and \( p_n = (1 - p_0)(1 - \phi)^{n-1} \); \( n = 1, 2, 3, \ldots \). In the next section we consider examples where \( \frac{F(x+y)}{F(x)} \) is not monotone in \( x \) and where \( F(y) \) is a mixture of a class of distributions.

4 Further applications

In this section we consider construction of bounds for certain choices of \( F(y) \).

Example 1

Suppose that we wish to find an upper bound when

\[
F(y) = \left( \frac{\mu^2}{\mu^2 + y^2} \right)^\alpha, \quad y \geq 0.
\]

where \( \mu > 0, \alpha > 0 \). This is the tail of a Burr distribution (e.g. Hogg and Klugman, 1984, p. 220). Moments exist up to order \( 2\alpha \). Thus, a convenient choice of \( B(y) \) is the Pareto, i.e. \( B(y) = (1 + \kappa y)^{-r}, \quad y \geq 0 \) where \( 0 < r < 2\alpha \) and \( \kappa > 0 \) are chosen to satisfy (2.2), i.e.

\[
\frac{1}{\phi} = \int_0^{\infty} (1 + \kappa y)^r dF(y).
\]
If $r$ is a positive integer then (4.2) is a polynomial in $\kappa$. The failure rate is

$$
\mu_F(y) = - \frac{d}{dy} \ln F(y) = \frac{2\alpha y}{\mu^2 + y^2}, \quad y > 0.
$$

Since

$$
\mu_F'(y) = \frac{2\alpha (\mu^2 - y^2)}{(\mu^2 + y^2)^2},
$$

it is clear that $\mu_F(0) = \mu_F(\infty) = 0$, and that $\mu_F(y)$ increases to its maximum value of $\alpha/\mu$ at $y = \mu$, then decreases to 0. Thus $\mu_F(y)$ is not monotone.

We wish to find a df $1 - \overline{H}_x(y)$ satisfying (2.5). One approach uses corollary 2 with $\mu_F(y) \leq \mu_F = \alpha/\mu$. Then one obtains from (2.8)

$$
\overline{G}(x) \leq \frac{1 - \rho_0}{\phi} \left( \int_0^\infty (1 + \kappa x + \kappa y)^\alpha \frac{\alpha}{\mu} e^{-2y} dy \right)^{-1}, \quad x \geq 0.
$$

A superior approach is to fix $y \geq 0$ and consider

$$
Q_y(z) = \frac{F(y + z)}{F(z)} = \left( \frac{\mu^2 + z^2}{\mu^2 + (y + z)^2} \right)^\alpha.
$$

Then

$$
Q_y'(z) = \frac{2\alpha y (\mu^2 + z^2)^\alpha - \alpha z^2 (\mu^2 + (y + z)^2) - 2z^2 y^2 \mu^2}{(\mu^2 + (y + z)^2)^{\alpha+1}}.
$$

Let $c(y) = \left( \sqrt{y^2 + 4\mu^2} - y \right) / 2$ and $Q_y'(z) \leq 0$ for $z \leq c(y)$, $Q_y'(z) \geq 0$ for $z \geq c(y)$. Thus, $Q_y(z)$ attains a minimum at $z = c(y)$, and so

$$
\frac{F(y + z)}{F(z)} \geq \frac{F(y + c(y))}{F(c(y))} = \left\{ 1 - \frac{2y}{y + \sqrt{y^2 + 4\mu^2}} \right\}^\alpha.
$$

It is not hard to see that $2y \left\{ y + \sqrt{y^2 + 4\mu^2} \right\}^{-1}$ is a df, so one may choose

$$
\overline{H}_x(y) = \left\{ 1 - \frac{2y}{y + \sqrt{y^2 + 4\mu^2}} \right\}^\alpha = \left\{ \frac{2\mu}{y + \sqrt{y^2 + 4\mu^2}} \right\}^{2\alpha}, \quad y \geq 0.
$$

Hence, the associated failure rate is

$$
\mu_H(y) = - \frac{d}{dy} \ln \overline{H}_x(y) = \frac{2\alpha}{\sqrt{y^2 + 4\mu^2}}.
$$
Clearly, $\mu_H(y) \leq \alpha/\phi$, implying that $\overline{H}_n(y) \geq e^{-\frac{\mu y}{\phi}}$. Since $\{\overline{B}(y + x)\}^{-1}$ is an increasing function of $y$, it follows that

$$
\overline{G}(x) \leq \frac{1 - p_0}{\phi} \left\{ \int_0^\infty \left( 1 + \kappa x + \kappa y \right)^r dH_n(y) \right\}^{-1}, \quad x \geq 0 ,
$$

(4.5)

and (4.5) is a tighter bound than (4.3) (e.g., Ross, 1983, p.252). From (4.4), one has $\lim_{y \to \infty} y^{2\alpha} \overline{H}_n(y) = \mu^{2\alpha}$, implying that $H_n(y)$ also has moments up to $2\alpha$. For $n < 2\alpha$, let

$$
m_n = \int_0^\infty y^ndH_n(y) = n \int_0^\infty \left( \frac{2\mu}{\sqrt{y^2 + 4\mu^2}} \right)^{2\alpha} dy.
$$

(4.6)

Change variable from $y$ to $t = \sqrt{y^2 + 4\mu^2} + y$. Then $(t - y)^2 = y^2 + 4\mu^2$, $y = \frac{1}{2} - \frac{2\mu^2}{t}$ and $dy = \left( \frac{1}{2} + \frac{2\mu^2}{t^2} \right) dt$. Thus, (4.6) becomes

$$
m_n = n2^{-n}(2\mu)^{2\alpha} \int_2^{\infty} (t^2 - 4\mu^2)^{n-1}(t^2 + 4\mu^2)t^{-2\alpha-n-1} dt.
$$

(4.7)

If $n$ is a positive integer, one has after a binomial expansion and some rearrangement,

$$
m_n = 2n\mu^n \sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n-1}{j} \frac{2\alpha + n - 2j - 1}{(2\alpha + n - 2j - 1)^2 - 1}.
$$

(4.8)

For $n = 1$ and 2, one has $m_1 = 4\mu\alpha/(4\alpha^2 - 1)$ and $m_2 = 2\mu^2/(\alpha^2 - 1)$. Finally, since $r$ in (4.2) is less than $2\alpha$, if $r$ is an integer (4.5) becomes

$$
\overline{G}(x) \leq \frac{1 - p_0}{\phi} \left\{ \sum_{n=1}^r \binom{r}{n} \left( 1 + \kappa x \right)^{r-n} \kappa^nm_n \right\}^{-1}, \quad x \geq 0 .
$$

(4.9)

We now consider mixed distributions. Let

$$
\overline{I}'(y) = \int_\Theta \overline{I}'(y|\theta)g(\theta)d\theta ,
$$

(4.10)

where $g(\theta)$, $\theta \in \Theta$ is a mixing density function and $\overline{I}'(y|\theta)$ is the conditional df given $\theta$. Then,

$$
\overline{F}(y) = \int_\Theta \overline{F}(y|\theta)g(\theta)d\theta .
$$
Assume that for each $\theta \in \Theta$, there is a df $H_x(y|\theta)$ such that
\[
\overline{H}_x(y|\theta) \leq \inf_{0 \leq z \leq x} \frac{F(y + z|\theta)}{F(z|\theta)}.
\] (4.11)

Then for $0 \leq z \leq x$,
\[
\frac{F(y + z)}{F(z)} = \frac{\int_{\Theta} \frac{F(y + z|\theta)}{F(z|\theta)} F(z|\theta) g(\theta) d\theta}{\int_{\Theta} \frac{F(z|\theta)}{F(z|\theta)} g(\theta) d\theta} \geq \inf_{\theta \in \Theta} \overline{H}_x(y|\theta).
\] (4.12)

Let $\overline{H}_x(y) = \inf_{\theta \in \Theta} \overline{H}_x(y|\theta)$. If $1 - \overline{H}_x(y)$ is a df, Theorem 1 applies. In particular, if $F(y)$ is a two point mixture, i.e. $F(y) = pF_1(y) + (1 - p)F_2(y)$, $\overline{H}_x(y) = \min\{\overline{H}_{1,x}(y), \overline{H}_{2,x}(y)\}$, where $H_{i,x}$ satisfies (2.5) with $F$ being $F_i$.

In the next example, we apply this argument to a mixed Weibull distribution.

**Example 2**

Let $\Theta = \{\theta_1, \theta_2\}$ and let $F(y|\theta) = e^{-\lambda y^\theta}$ be the tail of a Weibull distribution.

(i) $1 \leq \theta_1 \leq \theta \leq \theta_2 < \infty$.

$F(y|\theta)$ is IFR in this case. Thus,
\[
\overline{H}_x(y|\theta) = e^{-\lambda (x+y)^{\theta_2-\theta}}.
\]

It is not hard to show that $\overline{H}_x(y|\theta)$ is decreasing in $\theta$. Hence,
\[
\overline{H}_x(y) = \inf_{\theta \in \Theta} \overline{H}_x(y|\theta) = \overline{H}_x(y|\theta_2).
\]

(ii) $0 < \theta_1 \leq \theta \leq \theta_2 \leq 1$.

$F(y|\theta)$ is DFR in this case. Thus,
\[
\overline{H}_x(y|\theta) = e^{-\lambda y^\theta}.
\]
Since $H_y(y|\theta)$ is increasing in $\theta$ when $y < 1$ and decreasing in $\theta$ when $y \geq 1$,

$$H_y(y) = \inf_{\theta \in \Theta} H_y(y|\theta) = \begin{cases} e^{-\lambda y^{\theta_1}}, & y < 1 \\ e^{-\lambda y^{\theta_2}}, & y \geq 1. \end{cases}$$

It is worth noting that $F(y)$ is DFR in this case and so $F(y)$ itself can be used as $H_y(y)$. However, the choice above enables us to choose $H_x(y)$ for a wider range of $\theta$ (see (iii) below).

(iii) $0 < \theta_1 \leq \theta \leq \theta_2 < \infty$. It follows from (i) and (ii) that

$$H_y(y) = \min(H_1(y), H_2(y)),$$

where

$$H_1(y) = e^{-\lambda [(x+y)^{\theta_2} - x^{\theta_2}]},$$

and

$$H_2(y) = \begin{cases} e^{-\lambda y^{\theta_1}}, & y < 1 \\ e^{-\lambda y}, & y \geq 1. \end{cases}$$

References


