Growth Curve Models

by

Muni S. Srivastava
Department of Statistics
University of Toronto

and

Dietrich von Rosen
Department of Mathematics
Uppsala University

Technical Report No. 9806, March 13, (1998)

TECHNICAL REPORT SERIES

University of Toronto
Department of Statistics

Growth Curve Models

By

Muni S. Srivastava
Dept. of Statistics
University of Toronto
Toronto, Canada. M5S 3G3
and
Dietrich von Rosen
Dept. of Mathematics
Uppsala University
Uppsala, Sweden

Abstract

In this chapter we consider generalized multivariate analysis of variance (MANOVA) models, known in the literature as growth curve models. In this model we have a matrix of unknown parameters connected with the mean of the observation matrix. The observation matrix is assumed to follow an additive model with the mean matrix and an error matrix. The columns of the error matrix are assumed to be independent and identically distributed as a multivariate normal with zero mean vector and an unknown positive definite covariance matrix. The maximum likelihood method is used to estimate the unknown parameters. Properties of these estimators are given. A test for the adequacy of the model against the alternative that it is a MANOVA model is obtained by the maximum likelihood method and its distribution given. The testing of a general bilinear hypotheses is considered. A generalized model known as nested or sum of profiles model is also considered along with its testing and estimation problems.

Keywords and Phrases: Estimation, MANOVA models, Multivariate normal, Nested Growth Curve Model, Testing of Models.

1 INTRODUCTION

In many experiments, the observations are taken over time on the same experimental unit to measure the effect of some drugs or in more common terminology to observe the growth of a subject. The usual null hypothesis of no change in growth or 'no effect' of the drug can easily be carried out by a test proposed by Rao (1948) when several subjects chosen at random are subjectd to the same treatment. In most cases we expect that the null hypothesis will be rejected unless the drug happened to be a placebo. Thus, either after the rejection of the hypothesis or more realistically, right in the begining, we may assume that the growth or the 'drug effect' depends on time, often to be a polynomial over time but mathematically it may be assumed that the expected value of the observation vector $\mathbf{x}$ is given by $B\boldsymbol{\xi}$, where $\mathbf{x}$ is the observation vector taken at p different time points and B is a known $p \times q$ matrix and $\boldsymbol{\xi}$ is an unknown q-vector. Obviously q should be less than p, otherwise the p means of the observation vector are not related. As noted above, in growth curve models B is usually a polynomial. For example, for $p = 4$, the highest degree of polynomial that can be fitted to the observation vector will be of order two if there is a constant term in the model. Thus for $p = 4$, we may have

$$E(\mathbf{x}_i) = B\boldsymbol{\xi} \quad , \quad i = 1, \ldots, N_1$$

where N_1 is the number of subjects (or observation vectors),

$$B = \begin{pmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \\ 1 & t_4 & t_4^2 \end{pmatrix} \quad \text{, and} \quad \boldsymbol{\xi}_1 = \begin{pmatrix} \xi_{01} \\ \xi_{11} \\ \xi_{21} \end{pmatrix} .$$

This model was introduced by Rao (1959) who also provided the statistical analysis. An alternative way of writing the above model is to write it as

$$E(X) = E(\mathbf{x}_1, \ldots, \mathbf{x}_{\mathbf{N}_1}) = B\boldsymbol{\xi}_1\mathbf{1}_1' \quad ,$$

where $\mathbf{1}_1' = (1, \ldots, 1)$ is a row vector of ones of length N_1. An advantage of writing in this manner is that if we have another observation matrix on a second treatment group, say $X_2 = (\mathbf{x}_1, \ldots, \mathbf{x}_{N_2})$ which has its expectation given by

$$E(X_2) = B\boldsymbol{\xi}_2 \mathbf{1}_2' \quad ,$$

where $\mathbf{1}_2' = (1, \ldots, 1)$ is again a row vector of ones but of length N_2, then we can write the two models together as

$$E(X) = E(X_1, X_2) = B(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \begin{pmatrix} \mathbf{1}_1' & \mathbf{0}' \\ \mathbf{0}' & \mathbf{1}_2' \end{pmatrix} \quad ,$$

and a comparison of $\boldsymbol{\xi}_1$ with $\boldsymbol{\xi}_2$ can be carried out ie., a comparison of two treatment groups. In most general form, a growth curve model can be written as

$$X = B\xi A + E$$

where X is a $p \times N$ matrix of observations, $B : p \times q$ and $A : m \times N$ are known matrices and $\xi : q \times m$ is a matrix of unknown parameters. It is assumed that the N columns of E are iid $N_p(\mathbf{0}, \Sigma)$.

This model was introduced by Potthoff and Roy (1964) and the maximum likelihood solution was given by Khatri (1966). The above model includes Rao's (1959) model for $m = 1$ and the two group case considered above for $m = 2$. Further properties and analysis of the above model was carried out by Rao (1965, 1966, 1967), Grizzle and Allen (1969), Gleser and Olkin (1970), Khatri (1973), Khatri and Srivastava (1975, 1976), Srivastava (1997 b), Srivastava and Khatri (1979), Srivastava and Carter (1977, 1983), Kariya (1978), Marden (1983) and Hooper (1983). For some interesting applications of this model, see Potthoff and Roy (1964), Ware and Bowden (1977), Zerke and Jones (1980), and Carter and Hubert (1984). Bayesian prediction and analysis were carried out by Lee and Geisser (1972, 1975). A review of the literature has appeared in von Rosen (1991) and a book that consider this model exclusively is by Kshirsager and Smith (1995). Robustness of test procedures were investigated by Khatri (1988). The missing observation situation was considered by Kleinbaum (1973), Srivastava, J.N and Mc Donald (1974), Liski (1985), Srivastava (1985) and Tsai & Koziol (1988).

The objective of this writing is to cover the basic traditional estimation and testing problems to make it more accessible by practitioners of the Growth Curve models. The organization of the paper is as follows. In Section 2, we give the maximum likelihood estimates of the unknown parameters. Section 3 and 4 discuss and propose tests for the adequacy of the growth curve model and whether a MANOVA model may be more appropriate. In Section 5 we test a general hypothesis in the growth curve model. The sum of profiles or nested model introduced by Srivastava and Khatri (1979) is considered in Section 6.

2 Maximum Likelihood Estimation.

In this section we give the maximum likelihood estimates of the unknown parameters in the growth curve model

$$X = B\xi A + E \quad , \tag{2.1 }$$

where the columns of the $p \times N$ matrix E are iid $N_p(\mathbf{0}, \Sigma)$, where the unknown covariance matrix Σ is positive define and ξ is a $q \times m$ matrix of unknown parameters. The matrices $B : p \times q, q \leq p$, and $A : m \times N, m < N - p$ are known matirces of ranks q and m respectively. Throughout this paper, we shall write $(\mathbf{x} - \boldsymbol{\mu})(\ )'$, for $(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})'$ or $(\mathbf{x} - \boldsymbol{\mu})A(\ )'$ for $(\mathbf{x} - \boldsymbol{\mu})A(\mathbf{x} - \boldsymbol{\mu})'$. In the following theorem we state the result.

Theorem 2.1 *The maximum likelihood estimate of ξ and Σ are respectively given by*

$$\hat{\xi} = B(B'V^{-1}B)^{-1}B'V^{-1}XA'(AA')^{-1} \tag{2.2 }$$

and

$$N\hat{\Sigma} = V + (I - KV^{-1})V_1(\ )' \quad , \tag{2.3 }$$

where

$$V = X[I - H]X' \quad , \quad H = A'(AA')^{-1}A \tag{2.4 }$$

$$V_1 = XHX' \quad , \tag{2.5 }$$

$$K = B(B'V^{-1}B)^{-1}B' \quad . \tag{2.6 }$$

Proof. The likelihood function is given by

$$L(\xi, \Sigma) = (2\pi)^{-\frac{1}{2}pN}|\Sigma|^{-\frac{1}{2}N} etr - \frac{1}{2}\Sigma^{-1}[(X - B\xi A)(X - B\xi A)'].$$

For given ξ, the MLE of Σ is given by

$$N\hat{\Sigma} = (X - B\xi A)(X - B\xi A)' \quad .$$

To obtain the MLE of ξ, we need to minimize the determinant of $(X - B\xi A)(X - B\xi A)'$ with respect to ξ. Using Lemma A.2 of the appendix, we find that

$$\hat{\xi} = (B'V^{-1}B)^{-1}B'V^{-1}XA'(AA')^{-1} \quad ,$$

and

$$B\hat{\xi}A = KV^{-1}XH \quad .$$

Hence, the MLE of Σ is given by

$$\begin{aligned} N\hat{\Sigma} &= (X - KV^{-1}XH)(I - H)(X - KV^{-1}XH)' + (X - KV^{-1}XH)H(X - KV^{-1}XH)' \\ &= X(I - H)X' + (I - KV^{-1})XHX'(I - V^{-1}K) \\ &= V + (I - KV^{-1})V_1(I - V^{-1}K). \end{aligned}$$

The maximum likelihood estimates of ξ and Σ can also be obtained by directly differentiating the likelihood function as has been done in von Rosen (1989) and Srivastava (1997 a).

In the next theorem, we state the property of the estimator $\hat{\xi}$.

Theorem 2.2 *The maximum likelihood estimator $\hat{\xi}$ is an unbiased estimator of ξ, and the covariance matrix of $vec(\hat{\xi})$ is given by*

$$cov.[vec(\hat{\xi})] = \frac{N - m - 1}{N - m - p + q - 1}(AA')^{-1} \otimes (B'\Sigma^{-1}B)^{-1} \quad ,$$

where $vec(G) \equiv vec(\mathbf{g}_1, \ldots, \mathbf{g}_m) = (\mathbf{g}_1', \ldots, \mathbf{g}_m')'$, and $F \otimes G$ is the kronecker product defined by $(f_{ij}G)$ where $F = (f_{ij})$. For brevity of notation, we shall write $cov.(\hat{\xi})$

for $cov.(vec(\hat{\boldsymbol{\xi}}))$.

<u>Proof</u>. From normal theory V and XA' are independently distributed. Hence

$$\begin{aligned} E(\hat{\xi}) &= E[E\left((B'V^{-1}B)^{-1}B'V^{-1}XA'(AA')^{-1}|V\right)] \\ &= E[(B'V^{-1}B)^{-1}B'V^{-1}B\xi AA'(AA')^{-1}] \\ &= \xi \end{aligned}$$

Thus $\hat{\xi}$ is an unbiased estimator of ξ.

From the definition of vec of a matrix given in the theorem, we find that $vec(\hat{\xi}) \equiv vec(\hat{\boldsymbol{\xi}}_1, \ldots, \hat{\boldsymbol{\xi}}_m) \equiv (\hat{\boldsymbol{\xi}}_1', \ldots, \hat{\boldsymbol{\xi}}_m')'$. That is $vec(\hat{\xi})$ is a qm vector since $\hat{\boldsymbol{\xi}}_i's$ are q-vectors. From the vec definition, it can also be verified that for any three matrices A, B and C such that the product ABC is defined,

$$vec(ABC) = (C' \otimes A)vec(B) \quad ,$$

Since

$$\begin{aligned} \hat{\xi} &= (B'V^{-1}B)^{-1}B'V^{-1}XA'(AA')^{-1} \quad , \\ vec(\hat{\xi}) &= [(AA')^{-1} \otimes P]vec(XA') \quad , \end{aligned}$$

where

$$P = (B'V^{-1}B)^{-1}B'V^{-1}.$$

From normal theory it follows that

$$cov.[vec(XA')] = (AA') \otimes \Sigma.$$

Letting

$$\boldsymbol{\theta} = E[vec\ XA'],$$

we find that

$$\begin{aligned} Cov[vec(\hat{\xi})] &= E((AA')^{-1}\otimes P)(vec(XA')-\boldsymbol{\theta})(vec(XA)-\boldsymbol{\theta})'((AA')^{-1}\otimes P') \\ &= E[E\{(AA')^{-1}\otimes P)(vec(XA')-\boldsymbol{\theta})(vec(XA')-\boldsymbol{\theta})'((AA')^{-1}\otimes P')|V\}] \\ &= E[((AA')^{-1}\otimes P)((AA')\otimes\Sigma)((AA')^{-1}\otimes P')] \\ &= E[(AA')^{-1}\otimes(P\Sigma P')] \\ &= E[(AA')^{-1}\otimes(B'V^{-1}B)^{-1}B'V^{-1}\Sigma V^{-1}B(B'V^{-1}B)^{-1}] \end{aligned}$$

since under the normality assumption V and XA' are independently distributed, where $V \sim W_p(\Sigma, N-m)$.

Let

$$B_1 = \Sigma^{-\frac{1}{2}}B, \quad W_1 = \Sigma^{-\frac{1}{2}}V\Sigma^{-\frac{1}{2}}, \quad W = \Gamma' W_1 \Gamma$$

and

$$W^{-1} = \begin{pmatrix} W^{11} & W^{12} \\ W^{12'} & W^{22} \end{pmatrix},$$

where $\Sigma^{-\frac{1}{2}}$ is a symmetric square root of $\Sigma^{-1} = \Sigma^{-\frac{1}{2}}\Sigma^{-\frac{1}{2}}$, and Γ is an orthogonal matrix given by $\Gamma = (B_1(B_1'B_1)^{-\frac{1}{2}}, C)$ for some suitable chosen C. Then, $W \sim W_p(I, N-m)$ and

$$\begin{aligned} &(B_1'V^{-1}B)^{-1}(B'V^{-1}\Sigma V^{-1}B)(B'V^{-1}B)^{-1} \\ &= (B_1'W_1^{-1}B_1)^{-1}(B_1'W_1^{-2}B_1)(B_1'W_1^{-1}B)^{-1} \\ &= (B_1'\Gamma W^{-1}\Gamma' B_1)^{-1}(B_1'\Gamma W^{-2}\Gamma' B_1)(B_1'\Gamma W^{-1}\Gamma' B_1)^{-1} \\ &= [(B_1'B_1)^{\frac{1}{2}}W^{11}(B_1'B_1)^{\frac{1}{2}}]^{-1}[(B_1'B_1)^{\frac{1}{2}}(W^{11}W^{11'}+W^{12}W^{12'})(B_1'B_1)^{\frac{1}{2}}][(B_1'B_1)^{\frac{1}{2}}W^{11}(B_1'B_1)^{\frac{1}{2}}]^{-1} \\ &= (B_1'B_1)^{-\frac{1}{2}}[\mathrm{I}+(W^{11})^{-1}W^{12}W^{12'}(W^{11^{-1}}](B_1'B_1)^{-\frac{1}{2}} \\ &= (B_1'B_1)^{-\frac{1}{2}}[\mathrm{I}+W_{12}W_{22}^{-2}W_{12}'](B_1'B_1)^{-\frac{1}{2}}, \end{aligned}$$

since $(W^{11})^{-1}W^{12} = -W_{12}W_{22}^{-1}$. From Srivastava and Khatri (1979, p.79) we know that $W_{12}W_{22}^{-\frac{1}{2}}$ is $N_{q,p-q}(0, I, I)$ and is independently distributed of W_{22}. Hence,

$$E(\mathrm{I} + W_{12}W_{22}^{-2}W_{12}') = [\mathrm{I} + (N-m-p+q-1)^{-1}E(W_{12}W_{22}^{-1}W_{12}')]$$
$$= [1 + (p-q)(N-m-p+q-1)^{-1}]\ \mathrm{I}.$$

This gives,

$$E[(B'V^{-1}B)^{-1}B'V^{-1}\Sigma V^{-1}B(B'V^{-1}B)^{-1}] = \frac{N-m-1}{N-m-p+q-1}(B'\Sigma^{-1}B)^{-1}.$$

Thus, $cov[vec(\hat{\xi})]$ is as stated in the theorem.

Following the above steps, it can be shown that

$$E(\hat{\Sigma}) = \Sigma - (\frac{m}{N})(1 - \frac{p-q}{N-m-p+q-1})B(B'\Sigma^{-1}B)^{-1}B'$$

Hence the maximum likelihood estimator of Σ is not an unbiased estimator. However, since (again following the above steps),

$$E(B'V^{-1}B)^{-1} = (N-m-p+q)(B'\Sigma^{-1}B)^{-1} \quad ,$$

an unbiased estimator of Σ can be obtained. We can also obtain an unbiased estimator of the $cov(\hat{\xi})$. These results are stated in the following

Theorem 2.3 *An unbiased estimator of Σ and $Cov(\hat{\xi})$ are respectively given by*

$$\hat{\Sigma} + m\frac{N-m-2p+2q-1}{(N-m-p+q)(N-m-p+q-1)}B(B'V^{-1}B)^{-1}B'$$

and

$$\widehat{cov}(\hat{\xi}) = \frac{N-m-1}{(N-m-p+q-1)(N-m-p+q)}(A'A)^{-1} \otimes (B'V^{-1}B)^{-1}.$$

3 Adequacy of the model.

Consider the multivariate regression model in which

$$X = \beta A + E \quad ,$$

where β is a $p \times m$ matrix of unknown multivariate regression parameters, and the columns of E are iid $N_p(\mathbf{0}, \Sigma)$. The growth curve model is a special case of it in which we assume that $\beta = B\xi$ for some known $p \times q$ matrix B, $p \geq q$. That is, the number of unknown parameters required to describe the model is fewer than pm. However, it may be desirable to check if this indeed is true before using this model. Thus, we wish to test the hypothesis

$$H_1 : \beta = B\xi \quad , \tag{3.1 }$$

against the alternative $A_1 \neq H_1$.

Let C be a $(p-q) \times p$ matrix of rank $(p-q)$ such that $CB = 0$. Then the hypothesis that $\beta = B\xi$ is equivalent to testing the hypothesis that $C\beta = 0$. The likelihood ratio test for this hypothesis is given by

$$\lambda_1 \equiv U_{(p-q),m,n} = \frac{|CVC'|}{|CVC' + CV_1C'|} \quad , \quad n = N - m \quad , \tag{3.2 }$$

in the notation of Srivastava and Carter (1983).The asymptotic distribution of λ_1 is given by

$$\begin{aligned} &P\{-[n - \frac{1}{2}(p - q - m + 1)]log\lambda_1 \geq z\} \\ &= \quad P\{\chi_f^2 \geq z\} + n^{-2}\gamma_2\{P(\chi_{f+4}^2 \geq z) - P(\chi_f^2 \geq z)\} + O(n^{-3}) \ , \end{aligned} \tag{3.3 }$$

where $f = (p-q)m$ and $\gamma_2 = f[(p-q)^2 + m - 5]/48$.

This test was given by Srivastava and Carter (1983), and later by Chinchilli and Elswick (1985).

In terms of the original variables, it is given by

$$\lambda_1 = \frac{|V|}{|V + V_1|} \frac{|B'V^{-1}B|}{|B'(V + V_1)^{-1}B|} \quad , \tag{3.4 }$$

see Srivastava (1997 b) for details. Thus, (3.4) does not require us to find a matrix C such that $CB = 0$, although C can be chosen as $M'[I - B(B'B)^{-1}B']$, where M is any $p \times (p-q)$ matrix such that (B, M) is nonsingular. We may chose M from $\Gamma = (L, M)$ such that $\Gamma'[I - B(B'B)^{-1}B']\Gamma = diag(0, \ldots, 0, 1, \ldots, 1)$, the last $(p-q)$

diagonal elements as one. Thus, we get the following

Theorem 3.1 *The likelihood ratio test for the adequacy of the model, that is, the growth curve model is tenable is given by (3.2) or (3.4). The asymptotic distribution is given in (3.3).*

4 Growth Curve Model as a Conditional MANOVA model.

Let $B_0 = (B_1, B_2)$ be a nonsingular $p \times p$ matrix such that $B_2'B_1 = 0$, $B_1 = B(B'B)^{-1} : p \times q$ and $B_2 : p \times (p-q)$. Then $B_2'B = 0$ and $B_1'B = I_q$. Hence

$$E(B_0'X) = E\begin{pmatrix} B_1'X \\ B_2'X \end{pmatrix} = \begin{pmatrix} \xi A \\ 0 \end{pmatrix} \quad , \tag{4.1}$$

since $E(X) = B\xi A$. Define

$$Y = B_0'X, \quad Y_1 = B_1'X \quad \text{and} \quad Y_2 = B_2'X \quad . \tag{4.2}$$

Then the N columns of the random matrix Y are independently normally distributed with covariance matrix

$$\Lambda \equiv B_0'\Sigma B_0 = \begin{pmatrix} B_1'\Sigma B_1 & B_1'\Sigma B_2 \\ B_2'\Sigma B_1 & B_2'\Sigma B_2 \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}' & \lambda_{22} \end{pmatrix} \quad . \tag{4.3}$$

Hence, conditionally given Y_2,

$$E[Y_1|Y_2] = \xi A + \Lambda_{12}\Lambda_{22}^{-1}Y_2 = \beta A^* \quad , \tag{4.4}$$

where $\beta = (\xi, \Lambda_{12}\Lambda_{22}^{-1})$ and $A^* = (A', Y_2')'$. Also, conditionally given Y_2, the columns of Y_1 are independently normally distributed with covariance matrix

$$\begin{aligned} \Lambda_{11} - \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{12}' &= B_1'\Sigma B_1 - B_1'\Sigma B_2(B_2'\Sigma B_2)^{-1}B_2'\Sigma B_1 \\ &= B_1'[\Sigma - \Sigma B_2(B_2'\Sigma B_2)^{-1}B_2'\Sigma]B_1 \end{aligned} \tag{4.5}$$

$$= B_1'B_1(B_1'\Sigma^{-1}B_1)^{-1}B_1'B_1 \text{ from Lemma A.1 } .$$

$$= (B'\Sigma^{-1}B)^{-1} \ .$$

Thus, conditionally given Y_2, it is a MANOVA (multivariate regression model) model. And, if $\Lambda_{12} = B_1'\Sigma B_2 = 0$, it will be an unconditional MANOVA model.

Rao (1967) has shown that $\Lambda_{12} = 0$ if and only if Σ is of the form

$$\Sigma = B\Gamma_1 B' + B_2\Gamma_2 B_2' \ , \qquad (4.6\ )$$

where Γ_1 and Γ_2 are arbitrary unknown positive definite matrices and $B_2'B = 0$ where (B, B_2) is a nonsingular matrix.

Thus, before embarking on the analysis of growth curve models, it may be desirable to test the hypothesis that $\Lambda_{12} = 0$ or equivalently H : Σ is of the form (4.6) against the alternative A $\neq$ H. The likelihood ratio test for this hypothesis has been given by Lee and Geisser (1972). The likelihood ratio test is based on the statistics which is the ratio of the determinants of the two MLE's of Λ, one under the alternative and the other under the hypothesis.

From (2.2), it can be shown that

$$B'\hat{\Sigma}^{-1}B = NB'V^{-1}B.$$

Hence, under the alternative hypothesis

$$\begin{aligned} |\hat{\Lambda}| &= |B_0'\hat{\Sigma}B_0| = |\hat{\Lambda}_{11} - \hat{\Lambda}_{12}\hat{\Lambda}_{22}^{-1}\hat{\Lambda}_{12}'||\hat{\Lambda}_{22}| \\ &= |B'\hat{\Sigma}^{-1}B|^{-1}|B_2'\hat{\Sigma}B_2| \\ &= |NB'V^{-1}B|^{-1}|N^{-1}Y_2Y_2'| \ . \end{aligned}$$

Under the hypothesis, Y_1 and Y_2 are independently distributed. Hence, the MLE of Λ_{11} is

$$N\hat{\Lambda}_{11H} = Y_1[I - H]Y_1' = B_1'X[I - H]X'B_1 = B_1'VB_1.$$

Thus,

$$\begin{aligned}|\hat{\Lambda}_H| &= |\hat{\Lambda}_{11H}||\hat{\Lambda}_{22H}| \\ &= |B_1'VB_1||B_2'\hat{\Sigma}B_2| \\ &= |B'B|^{-2}|B'VB||N^{-1}Y_2Y_2'| \ .\end{aligned}$$

Hence, the likelihood ratio test is given by

$$\begin{aligned}\tilde{\lambda} &= U_{p-q,q,n-q-1} \\ &= \frac{|B'B|^2}{|B'VB||B'V^{-1}B|} \ .\end{aligned}$$

The asymptotic distribution of $\tilde{\lambda}$ is given by

$$P\{-[n-q-1-\frac{1}{2}(p-2q+1)log\tilde{\lambda} \geq z\} = P\{\chi^2_{\tilde{f}} \geq z\}+n^{-2}\tilde{\gamma}_2\{P(\chi^2_{\tilde{f}+4} \geq z)-P\{(\chi^2_{\tilde{f}} \geq z)\}+O(n^{-3})$$

where

$$\tilde{f} = q(p-q) \qquad (4.7\ )$$

and

$$\tilde{\gamma}_2 = \tilde{f}[(p-q)^2 + q - 5]/48. \qquad (4.8\ )$$

Thus, we have the following

Theorem 4.1 *The growth curve model (2.1) is a MANOVA model if Σ is of the form (4.6). The likelihood ratio test for testing the hypothesis that Σ is of the form (4.6) is based on the statistic $\tilde{\lambda}$. The MANOVA model is given by*

$$Y_1 \equiv B_1'X = \xi A + B_1'E \ \ ,$$

where the columns of $B_1'E$ are iid $N_p(\mathbf{O}, \Lambda_{11})$, $\Lambda_{11} = B_1'\Sigma B_1$.

5 Testing General Hypotheses in GCM.

In sections 3 and 4 we tested respectively the adequacy of the growth curve model (GCM) and the independence of the two transformed matrices. If the first hypothesis is accepted and the second hypothesis is rejected, we are in the growth curve model. But there is still a possibility of redundancy of parameters. Thus, we may wish to test the hypothesis

$$H_2 : C\xi D = 0 \;\; vs \;\; A_2 \neq H_2 \;\; ,$$

where $C : c \times q$ and $D : m \times d$ are known matrices of ranks $c \leq q$ and $d \leq m$ respectively. Consider nonsingular matrices C_0 and D_0 defined by

$$C_0' = (C_1', C') : q \times q \;\; , \;\; CC_1' = 0 \;\; ,$$

and

$$D_0 = (D_1, D) : m \times m \qquad D_1'D = 0 \;\; .$$

Then

$$\begin{aligned} E(X) &= BC_0^{-1}C_0\xi D_0 D_0^{-1}A \\ &\equiv B_1\eta A_1 \;\; , \end{aligned} \tag{5.1 }$$

where $B_1 = BC_0^{-1}$, $A_1 = D_0^{-1}A$ and

$$\eta = C_0\xi D_0 = \begin{pmatrix} C_1\xi D_1 & C_1\xi D \\ C\xi D_1 & C\xi D \end{pmatrix} \;\; .$$

Under $H_5 : C\xi D = 0$. We have assumed that N columns of X are independently normally distributed with covariance Σ.

Let

$$A_0' = [A_1'(A_3')^{-1}, \;\; A_2']$$

be an orthogonal matrix such that $A_1A_1' = D_0^{-1}AA'D_0^{-1} = A_3A_3'$; $A_1A_2' = 0$ where A_3 is a lower triangular matrix.

Let

$$B_0 = [B_1(B_1'B_1)^{-1}, \quad B_2]$$

be a nonsingular matrix with $B_2'B_1 = 0$

Note that

$$B_1'B_0 = (I_q, 0) \quad , \quad A_1A_0' = (A_3, 0) \quad .$$

Hence, with $Y = B_0'XA_0'$,

$$\begin{aligned} E(Y) &= B_0'B_1\eta A_1A_0' \\ &= \begin{pmatrix} I_q \\ 0 \end{pmatrix} \eta(A_3, 0) \\ &= \begin{matrix} q \\ p-q \end{matrix} \overset{\begin{matrix} m & N-m \end{matrix}}{\begin{pmatrix} \eta A_3 & 0 \\ 0 & 0 \end{pmatrix}} \equiv \begin{matrix} q \\ p-q \end{matrix} \overset{\begin{matrix} m & N-m \end{matrix}}{\begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}} \quad , \end{aligned}$$

and under H

$$\delta = \eta A_3 = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & 0 \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{pmatrix} \equiv \begin{pmatrix} \delta_1 & \delta_3 \\ \delta_2 & 0 \end{pmatrix} \quad .$$

Let

$$Y = B_0'XA_0' = \begin{pmatrix} (B_1'B_1)^{-1}B_1'XA_0 \\ B_2'XA_0 \end{pmatrix} \quad .$$

Then

$$E(Y) = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix} \quad ; \text{ under } \ H, \quad E(Y) = \begin{pmatrix} \delta_1 & \delta_3 & 0 \\ \delta_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad .$$

Let

$$Y \quad = \quad \begin{matrix} q \\ p-q \end{matrix} \overset{\begin{matrix} m & N-m \end{matrix}}{\underset{\begin{matrix} m-d & d & N-m \end{matrix}}{\begin{pmatrix} Y_1 & \\ & Y_4 \\ Y_3 & \end{pmatrix}}}$$

$$= \begin{array}{c} {\scriptstyle q-c} \\ {\scriptstyle c} \\ {\scriptstyle p-q} \end{array} \begin{pmatrix} Y_{11} & Y_{12} & \\ Y_{21} & Y_{22} & Y_4 \\ Y_{31} & Y_{32} & \end{pmatrix}$$

$$V = Y_4 Y_4' = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}' & V_{33} \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{12}' & V_{22} & V_{23} \\ V_{13}' & V_{23}' & V_{33} \end{pmatrix}$$

Then, we need to minimize the following determinant d to get the maximum likelihood estimate of δ under A.

$$\begin{aligned} d = |R| &= |V + \begin{pmatrix} Y_1 - \delta \\ Y_3 \end{pmatrix} \begin{pmatrix} Y_1 - \delta \\ Y_3 \end{pmatrix}'| \\ &= |V||I + (Y_1' - \delta', \;\; Y_3')V^{-1} \begin{pmatrix} Y_1 - \delta \\ Y_3 \end{pmatrix}| \\ &= |V||I + Y_3' V_{33}^{-1} Y_3 + (Y_1 - \delta - P_{12} V_{33}^{-1} Y_3)' V_{12} (Y_1 - \delta - P_{12} V_{33}^{-1} Y_3)| \\ &\geq |V||I + Y_3' V_{33}^{-1} Y_3| \\ &= \frac{|V|}{|V_{33}|} |V_{33} + Y_3 Y_3'|. \end{aligned} \tag{5.2 }$$

The minimum occurs at

$$\hat{\eta} A_3 = \hat{\delta} = Y_1 - P_{12}' V_{33}^{-1} Y_3 \quad .$$

This gives

$$\begin{aligned} C_0 \hat{\xi} D_0 A_3 &= \hat{\delta} \\ &= (B_1' B_1)^{-1} B_1' X A_0' - P_{12}' V_{33}' B_2' X A_0' \\ &= [(B_1' B_1)^{-1} B_1' - P_{12}' V_{33}^{-1} B_2'] X A_0' \end{aligned}$$

Let

$$Q' = (Y_{12}' - \delta_3', \;\; Y_{22}', \;\; Y_{32}') \quad .$$

To get MLE of $\delta_1, \delta_2, \delta_3$,we need to minimize

$$
\begin{aligned}
&\left| V + \begin{pmatrix} Y_{11}-\delta_1 \\ Y_{21}-\delta_2 \\ Y_{31} \end{pmatrix} (\ )' + \begin{pmatrix} Y_{12}-\delta_3 \\ Y_{22} \\ Y_{32} \end{pmatrix} (\ )' \right| \\
\equiv\ & |V+QQ'| \left| I + \begin{pmatrix} Y_{11}-\delta_1 \\ Y_{21}-\delta_2 \\ Y_{31} \end{pmatrix}' (V+QQ')^{-1} \begin{pmatrix} Y_{11}-\delta_1 \\ Y_{21}-\delta_2 \\ Y_{31} \end{pmatrix} \right| \\
\geq\ & |V+QQ'||I + Y_{31}'(V_{33}+Y_{32}Y_{32}')^{-1}Y_{31}| \ , \\
=\ & \frac{|V+QQ'|}{|V_{33}+Y_{32}Y_{32}'|}|V_{33}+Y_{32}Y_{32}'+Y_{31}Y_{31}'| \\
=\ & \left| V + \begin{pmatrix} Y_{12}-\delta_3 \\ Y_{22} \\ Y_{33} \end{pmatrix} (\ )' \right| \frac{|V_{33}+Y_{32}Y_{32}'+Y_{31}Y_{31}'|}{|V_{33}+Y_{32}Y_{32}'|} \\
=\ & |V| \left| I + \begin{pmatrix} Y_{22} \\ Y_{32} \end{pmatrix}' \begin{pmatrix} V_{22} & V_{23} \\ V_{23}' & V_{33} \end{pmatrix}^{-1} \begin{pmatrix} Y_{22} \\ Y_{32} \end{pmatrix} \right| \times \frac{|V_{33}+Y_{32}Y_{32}'+Y_{31}Y_{31}'|}{|V_{33}+Y_{32}Y_{32}'|} \\
=\ & \frac{|V| \left| \begin{pmatrix} V_{22} & V_{23} \\ V_{23}' & V_{33} \end{pmatrix} + \begin{pmatrix} Y_{22} \\ Y_{32} \end{pmatrix} \begin{pmatrix} Y_{22} \\ Y_{32} \end{pmatrix}' \right|}{\begin{vmatrix} V_{22} & V_{23} \\ V_{23}' & V_{33} \end{vmatrix}} \times \frac{|V_{33}+Y_{32}Y_{32}'+Y_{31}Y_{31}'|}{|V_{33}+Y_{32}Y_{32}'|} \qquad (5.3)
\end{aligned}
$$

While obtaining the maximum likelihood estimators of δ under the alternative and δ_1, δ_2, and δ_3 under the hypothesis, we also obtained the value of the determinant of the maximum likelihood estimates of the covariance matrix under the alternative and hypothesis. These are given by the expression on the right of (5.2) and (5.3) respectively. Using these expressions, we can easily obtain the likelihood ratio test for testing the hypothesis H_2 against A_2. This is based on the statistic

$$\lambda_2 = \frac{|\tilde{V}|}{|\tilde{V}+ZZ'|} \times \frac{|V_{33}+Z_2Z_2'|}{|V_{33}|} \ , \qquad (5.4)$$

where

$$\tilde{V} \equiv \begin{pmatrix} V_{22} & V_{23} \\ V_{23}' & V_{33} \end{pmatrix} \text{ and } Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \equiv \begin{pmatrix} Y_{22} \\ Y_{32} \end{pmatrix} \ . \qquad (5.5)$$

In terms of the original variables,

$$\lambda_2 = \frac{|Q|}{|P+Q|} , \tag{5.6}$$

where

$$\begin{aligned}
Q &= C(B'V^{-1}B)^{-1}C' , \\
P &= (C\hat{\xi}D)(D'RD)^{-1}(C\hat{\xi}D)' , \\
\hat{\xi} &= (B'V^{-1}B)^{-1}B'V^{-1}XA'(AA')^{-1} , \\
R &= (AA')^{-1} + (AA')^{-1}AX'[V^{-1} - V^{-1}B(B'V^{-1}B)^{-1}B'V^{-1}]XA'(AA')^{-1} , \\
V &= X[I - A(AA')^{-1}A']X' ,
\end{aligned} \tag{5.7}$$

see Khatri (1966).

It may be noted that the statistic λ_2 is invariant under non singular linear transformations. Thus, we may assume without any loss of generality that $\Sigma = I$. Under this assumption, the joint distribution of $\tilde{V}$ and Z, defined in (5.5), under the hypothesis is given by

$$Const.|\tilde{V}|^{(n-p+q-c-1)/2}[etr - \frac{1}{2}(\tilde{V} + ZZ')] \quad , \quad n = N - m \quad .$$

Consider the transformations

$$\tilde{V} + ZZ' = TT' = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} T_1' & 0' \\ T_{12}' & T_2' \end{pmatrix}$$

$$Z = TW \quad , \quad W' = (W_1', W_2') \quad .$$

We find that the Jacobian of the transformation $J(\tilde{V}, Z \rightarrow T, W) = 2^p \Pi t_{ii}^{i+d}$, since Z is a matrix of order $(p-q+c) \times d$. It can be seen that T and W are independently distributed. The pdf of W is given by

$$Const.|I - WW'|^{(n-p+q-c-1)/2}$$

and

$$\lambda_2 = \frac{|I - W_2W_2'|}{|I - WW'|} = \frac{|I - W_2'W_2|}{|I - W_1'W_1 - W_2'W_2|} = |I - P'P|^{-1} \quad ,$$

where $P' = (I - W_2'W_2)^{-\frac{1}{2}}W_1'$. The pdf of P' is given by

$$Const.|I - P'P|^{(n-p+q-c-1)/2} \quad .$$

Hence

$$\lambda_2 = U_{c,d,n-p+q} \quad .$$

Theorem 5.1 *For the growth curve model (2.1), the likelihood ratio test statistic for testing the hypothesis* $C\xi D = 0$, $C : c \times q$, $D : m \times d$ *is given by* λ_2 *given in (5.4) or (5.6).*

6 Nested Models.

In section 5, we considered the problem of testing the hypothesis $H : C\xi D = 0$ in the GCM model $E(X) = B\xi A$, where the columns of X are independently normally distributed with common covariance Σ. This testing problem was rewritten in (5.1) as

$$E(X) = B_1\eta A_1$$

where

$$\eta = \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{pmatrix}$$

and testing the hypothesis was equivalent to testing the hypothesis that $\eta_4 = 0$. Writing $B_1 = (B_2, \tilde{B})$, $A_1' = (\tilde{A}_1', \tilde{A}_2')$, $\eta_{(1)} = (\eta_1', \eta_3')'$ and $\eta_{(2)} = \eta_2$, we can write $B_1\eta A_1$ as

$$B_1\eta A_1 = B_1\eta_{(1)}\tilde{A}_1' + B_2\eta_{(2)}\tilde{A}_2' \quad , \qquad (6.1\ )$$

where B_2 is a subset of B_1. Srivastava and Khatri (1979) proposed this model and gave an outline of the procedure to obtain tests and estimates in Problem 6.9 p.196. The model of the type (6.1) is also called sum of profiles. Banken

(1984), Kariya (1985), Verbyla and Venables (1988) and von Rosen (1989) also independently considered this model. Recently Andersson, Marden and Perlman (1993) have discussed this model. For the details of the estimation and testing of the hypothesis problems considered in Srivastava and Khatri (1979), see Srivastava (1997 b).

In the above we demonstrated that the testing of hypothesis problem of Section 5 can be written as a nested or sum of profiles models. We now give another example of nested model. Consider two treatments applied to N_1 and N_2 subjects respectively. Suppose the second treatment is a placebo. Then if we represent the two responses by y_{ti} and z_{ti} and if the response is linear in time for the first treatment (and constant for the second, placebo), we get

$$E(y_{ti}) = \beta_{01} + \beta_{11}t \quad , i = 1, \ldots, N_1$$

and

$$E(z_{ti}) = \beta_{02} \quad , i = 1, \ldots, N_2$$

t=1,2,3. Writing

$$\mathbf{y}_i = \begin{pmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \end{pmatrix} \quad , i = 1, \ldots, N_1$$

and

$$\mathbf{z}_i = \begin{pmatrix} z_{1i} \\ z_{2i} \\ z_{3i} \end{pmatrix} \quad , i = 1, \ldots, N_2$$

we find that

$$E(\mathbf{y}_i) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_{01} \\ \beta_{11} \end{pmatrix} \quad i = 1, \ldots, N_1$$

and

$$E(z_i) = \begin{pmatrix} \beta_{02} \\ \beta_{02} \\ \beta_{02} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_{02} \\ 0 \end{pmatrix} \quad , i = 1, \ldots, N_1 \quad .$$

Thus, if we let

$$B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

and

$$X = (\mathbf{y}_1, \ldots, \mathbf{y}_{N_1},\ \mathbf{z}_1, \ldots, \mathbf{z}_{N_2}) ,$$

we get

$$E(X) = B_1 \begin{pmatrix} \beta_{01} & \beta_{02} \\ \beta_{11} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}'_{N_1} & 0 \\ 0 & \mathbf{1}'_{N_2} \end{pmatrix} ,$$

where $\mathbf{1}'_k$ is a vector of ones of length k, $\mathbf{1}'_k = (1, \ldots, 1) : 1 \times k$. This can be written as

$$\begin{aligned} E(X) &\equiv B_1 \begin{pmatrix} \beta_{01} & \beta_{02} \\ \beta_{11} & 0 \end{pmatrix} A \\ &= B_1 \xi_1 A_1 + B_2 \xi_2 A_2 , \end{aligned}$$

where

$$B_1 = (B_2, \tilde{B}) \text{ and } A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} .$$

6.1 Maximum Likelihood Estimates.

In this section, we consider the nested growth curve model in which

$$E(X) = B_1 \xi A ,$$

where

$$\xi = \begin{matrix} & \begin{matrix} m_1 & m_2 \end{matrix} \\ \begin{matrix} q_1 \\ q_2 \end{matrix} & \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & 0 \end{pmatrix} \end{matrix} , m_1 + m_2 = m, \quad q_1 + q_2 = q \quad .$$

Thus, writing

$$\eta_1 = \begin{pmatrix} \xi_1 \\ \xi_3 \end{pmatrix}, \quad \eta_2 \equiv \xi_2, \quad B_1 = (B_2, \tilde{B}), \quad A' = \begin{pmatrix} A_1' \\ A_2' \end{pmatrix},$$

we get

$$B_1 \xi A = B_1 \eta_1 A_1 + B_2 \eta_2 A_2 \quad . \tag{6.2 }$$

We shall assume that the columns of X are independently normally distributed with covariance matrix Σ. It is also known that B_2 is a subset of B_1, we shall assume that they are of full rank. To obtain the maximum likelihood estimates, we first note that for a given η_1 and η_2, the MLE of Σ is given by

$$N\hat{\Sigma} = (X - B_1\eta_1 A_1 - B_2\eta_2 A_2)(\quad)'$$

where we shall write $(X-\theta)(X-\theta)'$ as simply $(X-\theta)(\quad)'$. To obtain the values of η_1 and η_2, we minimize the determinant

$$d = |(X - B_1\eta_1 A_1 - B_2\eta_2 A_2)(\quad)'|$$

with respect to η_1 and η_2. We minimize d first with respect to η_1 for fixed η_2. From Lemma 8.2 we get

$$B_1\hat{\eta}_1 A_1 = B_1(B_1' S_\eta^{-1} B_1)^{-1} B_1' S_\eta^{-1}(X - B_2\eta_2 A_2)P_1 \quad ,$$

where

$$\begin{aligned} P_1 &= A_1'(A_1 A_1')^{-1} A_1 \quad . \\ S_\eta &= (X - B_2\eta_2 A_2)(\quad)' - S_{1\eta} \quad , \\ S_{1\eta} &= (X - B_2\eta_2 A_2)P_1(X - B_2\eta_2 A_2)' \quad . \end{aligned}$$

Let B_0 be a matrix such that (B_1, B_0) is nonsingular and $B_0' B_1 = 0$. Then at $\hat{\eta}_1$

$$d_4 = |S_\eta + [I - B_1(B_1' S_\eta^{-1} B_1)^{-1} B_1' S_\eta^{-1}] S_{1\eta}[\quad]'|$$

$$\begin{aligned}
&= |S_\eta||I + S_{1\eta}[I - B_1(B_1'S_\eta^{-1}B_1)^{-1}B_1'S_\eta^{-1}]'S_\eta^{-1}[\ \ ]| \\
&= |S_\eta||I + S_{1\eta}[S_\eta^{-1} - S_\eta^{-1}B_1(B_1'S_\eta^{-1}B_1)'B_1'S_\eta^{-1}]| \\
&= |S_\eta||I + S_{1\eta}B_0(B_0'S_\eta B_0)^{-1}B_0'| \\
&= |S_\eta||(B_0'S_\eta B_0)^{-1}||B_0'(S_\eta + S_{1\eta})B_0| \quad ,
\end{aligned}$$

since $B_0'B_1 = 0$, $B_0'\tilde{B} = 0$ and $B_0'B_2 = 0$. Hence

$$B_0'S_{1\eta}B_0 = B_0'XP_1X'B_0$$

and

$$\begin{aligned}
B_0'S_\eta B_0 &= B_0'XX'B_0 - B_0'XP_1X'B_0 \\
&= B_0'XQ_1X'B_0 \quad , \quad Q_1 = I - P_1 \quad .
\end{aligned}$$

Thus at $\hat{\eta}_1$,

$$\begin{aligned}
d_4 &= |S_\eta||B_0'XQ_1X'B_0|^{-1}|B_0'XX'B_0| \\
&= |(X - B_2\eta_2A_2)Q_1(X - B_2\eta_2A_2)'||B_0'XQ_1X'B_0|^{-1}|B_0'XX'B_0| \\
&= |(\tilde{X} - B_2\eta_2\tilde{A}_2)(\tilde{X} - B_2\eta_2\tilde{A}_2)'||B_0'XQ_1X'B_0|^{-1}|B_0'XX'B_0| \quad ,
\end{aligned}$$

where $\tilde{X} = XQ_1$ and $\tilde{A}_2 = A_2Q_1$. Again, using Theorem 8.2 to minimize d_4 with respect to η_2, we get

$$B_2\hat{\eta}_2\tilde{A}_2 = B_2(B_2'\tilde{S}^{-1}B_2)^{-1}B_2'\tilde{S}^{-1}\tilde{X}\tilde{P}_2 \quad ,$$

where

$$\tilde{S} = \tilde{X}\tilde{X}' - \tilde{S}_1 \quad , \quad \tilde{S}_1 = \tilde{X}\tilde{P}_2\tilde{X}' \quad , \tilde{P}_2 = \tilde{A}_2'(\tilde{A}_2\tilde{A}_2')^-\tilde{A}_2 \quad .$$

The minimum value of d_4 at $\hat{\eta}_1$ and $\hat{\eta}_2$

$$= |\tilde{T}| \ \ |B_0{}'XQ_1X'B_0|^{-1} \ \ |B_0{}'XX'B_0| \quad ,$$

where

$$\tilde{T} = \tilde{S} + [I - B_2(B_2'\tilde{S}^{-1}B_2)^{-1}B_2'\tilde{S}^{-1}]\tilde{S}_1[\ \ ]'$$

Note that

$$\tilde{S} = XQ_1X' - XQ_1A_2'(A_2Q_1A_2')^-A_2Q_1X'$$
$$= XQ_1[I - A_2'(A_2Q_1A_2')^-A_2]Q_1X'$$

and

$$S_{\hat{\eta}} = (X - B_2\hat{\eta}_2A_2)Q_1(X - B_2\hat{\eta}_2A_2)'$$
$$= (\tilde{X} - B_2\hat{\eta}_2\tilde{A}_2)(\ \ )' \ ,$$

Hence, the MLE of η_2 is

$$\hat{\eta}_2 = (B_2'\tilde{S}^{-1}B_2)^{-1}B_2'\tilde{S}^{-1}XQ_1A_2'(A_2Q_1A_2')^- \tag{6.3 }$$

and the MLE of η_1 is

$$\hat{\eta}_1 = (B_1'S_{\hat{\eta}}^{-1}B_1)^{-1}B_1'S_{\hat{\eta}}^{-1}(X - B_2\hat{\eta}_2A_2)A_1'(A_1A_1')^{-1} \tag{6.4 }$$

The MLE of Σ is given by

$$N\hat{\Sigma} = (X - B_1\hat{\eta}_1A_1 - B_2\hat{\eta}_2A_2)(\ \ )' \ . \tag{6.5 }$$

Thus, we get the following

Theorem 6.1 *For the nested growth curve model (6.1), the MLE of* η_1, η_2 *and* Σ *are given by (6.2), (6.3) and (6.4) respectively.*

6.2 Testing For Nested model vs GCM.

The problem of testing the hypothesis that it is a nested model (6.1) against the alternative that it is a GCM, we can use the results of Section 5 or obtain the Likelihood ratio procedure directly. It is based on the statistic

$$\lambda_3 = \frac{|S + (I - T_1S^{-1})S_1(I - S^{-1}T_1)|}{|\tilde{S} + (I - \tilde{T}_2\tilde{S}^{-1})\tilde{S}_1(I - \tilde{S}^{-1}\tilde{T}_2)|} \cdot \frac{|B_0'XQ_1X'B_0|}{|B_0'XX'B_0|} \ ,$$

where

$$\begin{aligned} T_1 &= B_1(B_1'\tilde{S}^{-1}B_1)^{-1}B_1' , \\ \tilde{T}_2 &= B_2(B_2'\tilde{S}^{-1}B_2)^{-1}B_2' , \\ S &= X[I - A'(AA')^{-1}A]X' , \\ S_1 &= XA'(AA')^{-1}AX' , \end{aligned}$$

$\tilde{S}_1$ and $\tilde{S}$ have been defined earlier. Following Srivastava (1997 b), it can be shown that

$$\frac{|B_0'XQ_1X'B_0|}{|B_0'XX'B_0|} = \frac{|B_1'(XQ_1X')^{-1}B_1|}{|B_1'(XX')^{-1}B_1|} \cdot \frac{|XQ_1X'|}{|XX'|} ,$$

since (B_1, B_0) is nonsingular and $B_1'B_0 = 0$.

The distribution of λ_3 can be shown to be $U_{q_2,\ m_2,\ n-p+q}$

7 Generalized Nested Models

With a slight change of notation, consider the GCM model in which

$$E(X) = B_1\xi A$$

$B_1 :\ p \times q,\ \xi :\ q \times m,\ A :\ m \times N$; matrices A and B_1 are full rank. Consider the case when

$$(H_7) \qquad \xi = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \end{array} \begin{array}{c} \begin{array}{ccc} m_1 & m_2 & m_3 \end{array} \\ \begin{pmatrix} \xi_1 & \xi_4 & \xi_7 \\ \xi_2 & \xi_5 & \xi_8 \\ \xi_3 & \xi_6 & \xi_9 \end{pmatrix} \end{array}$$

where $\xi_6 = 0,\ \xi_8 = 0,\ \xi_9 = 0$. Let

$$\eta_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} ,\ \eta_2 = \begin{pmatrix} \xi_4 \\ \xi_5 \end{pmatrix} ,\ \eta_3 = (\xi_7) ,$$

$$B_1 = \begin{pmatrix} \overset{q_1+q_2}{B_2}, & \overset{q_3}{\tilde{B}} \end{pmatrix} = \begin{pmatrix} \overset{q_1}{B_3}, & \overset{q_2}{B_{22}}, & \overset{q_3}{\tilde{B}} \end{pmatrix}, \quad A' = \begin{pmatrix} \overset{m_1}{A_1'}, & \overset{m_2}{A_2'}, & \overset{m_3}{A_3'} \end{pmatrix}$$

$q_1 + q_2 + q_3 = q$ and $m_1 + m_2 + m_3 = m$.

Then

$$B_1 \xi A = B_1 \eta_1 A_1 + B_2 \eta_2 A_2 + B_3 \eta_3 A_3$$

where $B_3 \subset B_2 \subset B_1$.

This generalized nested model was considered by von Rosen (1989) who gave the MLE of the parameters but no testing problem was considered. We obtain the MLE of the parameters by following the approach of Srivastava and Khatri (1979). For testing and estimation problems, see Srivastava (1997 b). The MLE can also be found in von Rosen (1989).

8 Appendix A

Here we present some results which are useful in obtaining maximum likelihood tests and estimates.

Lemma 8.1 *Let (B, B_0) be a $p \times p$ nonsingular matrix such that $B'B_0 = 0$. Then for any $p \times p$ symmetric positive definite matrix S,*

$$S^{-1} - S^{-1}B(B'S^{-1}B)^{-1}B'S^{-1} = B_0(B_0'SB_0)^{-1}B_0' \ .$$

For a proof, see Srivastava and Khatri (1979), p.19

Lemma 8.2 *Let $X : p \times N, B : p \times q$ and $A : m \times N$ be matrices such that $(X - B\xi A)(X - B\xi A)'$ is positive definite for every $q \times m$ matrix ξ. Then*

$$|(X - B\xi A)(X - B\xi A)'| \geq |T| \quad \text{for all } \xi,$$

where

$$\begin{aligned} T &= S + [I - KS^{-1}]S_1[I - KS^{-1}]' \ , \\ K &= B(B'S^{-1}B)^{-}B' \ , \\ S_1 &= XHX', \quad H = A'(AA')^{-}A \\ S &= X[I - H]X' = XX' - S_1 \ . \end{aligned}$$

Khatri, C.G. & Srivastava, M.S. (1975). "On the likelihood ratio test for covariance matrix in growth curve model." *In Applied Statistics,* Ed. R.P. Gupta, 187-198. New York : North-Holland.

Khatri, C.G. & Srivastava, M.S. (1976). "Asymptotic expansions of the non-null distributions of the likelihood ratio critera for covariance matrices II" *Metron* **34**, 55-71.

Kleinbaum, D.G. (1973). "A generalization of the growth curve model which allows missing data." *J. Multivariate Anal.* **3**, 117-124.

Kshirsager, A.M. and Smith, W.B. (1995). "Growth Curves." *Marcel Dekker* , New York.

Lee, J.C. & Geisser, S. (1972). "Growth curve prediction." *Sankyā Ser. A* **34**, 393-412.

Lee, J.C. & Geisser, S (1975). "Applications of growth curve prediction." *Sankyā Ser. A* **37**, 239-256.

Liski, E.P. (1985). "Estimation from incomplete data in growth curve models." *Comm. Statist. Comput.* **14**, 13-27.

Marden, J.I. (1983). "Admissibility of invariant tests in the general multivariate analysis of variance problem." *Ann. Statist.* **11**, 1086-1099.

Potthoff, R.F. & Roy, S.N. (1964). "A generalized multivariate analysis of variance model useful especially for growth curve problems." *Biometrika.* **51**, 313-326.

Rao, C. Radhakrishna. (1948). "Tests of significance in multivariate analysis." *Biometrika* **35**, 58-79.

Rao, C. Radhakrishna (1959). "Some problems involving linear hypotheses in multivariate analysis." *Biometrika* **46**, 49-58.

Rao, C. Radhakrishna (1961). "Some observations on multivariate statistical methods in anthropological research." *Bull. Inst. Internat. Statist.* **38**, 99-109.

Rao, C. Radhakrishna (1965). "The theory of least squares when the parameters are stochastic and its application to the analysis of growth curves." *Biometrika* **52**, 447-458.

Rao, C. Radhakrishna (1966). "Covariance adjustment and related problems in multivariate analysis." *In Multivariate Analysis*, Ed. P.R. Krishnaiah, pp. 87-103. New York : Academic Press.

Rao, C. Radhakrishna (1967). "Least squares theory using an estimated dispersion matrix and its application to measurement of signals. " *In Proc. Fifth Berkeley Symp. Math. Statist. Prob.*, **1**, Ed. L.M. LeCam and J. Neuman, pp. 355-372. Berkeley and Los Angeles, California : University of California Press.

von Rosen, D. (1989). "Maximum likelihood estimators in multivariate linear normal models." *J. Multivar. Analysis* **31**, 187-200.

von Rosen, D. (1991). "The growth curve model : A review." *Comm. in Statist. Theory & Methods* **20**, 2791-2822.

Srivastava, M.S. (1997 a). "Reduced rank discrimination." *Sandinavial Journal of Statis.* **24**, 115-124.

Srivastava, M.S. (1997 b). "Generalized Multivariate analysis of variance Models. *Tech. report University of Toronto*

Srivastava, M.S. & Carter, E.M. (1977). "Asymptotic non-null distribution of a likelihood ratio criterion for sphericity in the growth curve model." *Sankyā Ser. B* **39**, 160-165.

Srivastava, M.S. & Carter, E.M. (1983). "An introduction to Applied Multivariate Statistics." New York : North-Holland.

Srivastava, M.S. and Khatri, C.G. (1979). "An introduction to Multivariate Statistics ." New York : North-Holland.

Tsai, K.T. & Koziol, J.A. (1988). "Score and Wald tests for the multivariate growth curve model with missing data." *Ann. Inst. Statist. Math.* **40**, 179-186.

Verbyla, A.P. & Venables, W.N. (1988). "An extension of the growth curve model." *Biometrika.* **75**, 129-138.

Ware, J.H. and Bowden, R.E. (1977). "Circadian rythm analysis when output is collected at intervals." *Biometrics.* **33**, 566-571.