Generalized Multivariate Analysis of Variance Models

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Abstract

In this paper, we consider a multivariate analysis of variance (MANOVA) model in which the $p \times N$ observation matrix $X = (x_1, \ldots, x_N)$ has $N$ independently distributed column vectors such that $X = \beta A + E$, where the $N$ columns of the $p \times N$ matrix $E$ are iid $N_p(0, \Sigma)$. We shall assume that $\Sigma$ is positive definite and the $m \times N$ matrix $A$ is of rank $m \leq N$. The $p \times m$ matrix $\beta$ is a matrix of unknown parameters, called regression parameters. For known orthogonal matrices $L = (L_1', L_2')'$ and $M = (M_1, M_2)$, the matrix $\beta$ can be rewritten as $\beta = L_1'(L_1\beta M_1)M_1' + L_2'(L_2\beta M_2)M_2'$. The general linear hypothesis of regression parameters corresponds to testing the hypothesis that $L_1\beta M_2 = 0$ and $L_2\beta M_2 = 0$. However, if we test only the hypothesis that $L_2\beta M_2 = 0$, we get a mixture of MANOVA and growth curve models, namely $\beta A = \tilde{\beta}_1 \tilde{A}_1 + L_1' \xi_3 \tilde{A}_2$, where $\tilde{\beta}_1 = \beta M_1$, $\xi_3 = L_1\beta M_2$, $\tilde{A}_1 = M_1' A$ and $\tilde{A}_2 = M_2' A$. Similarly, if we test the hypothesis that $L_2\beta M_2 = 0$ and $L_2\beta M_1 = 0$, we get the growth curve model, namely $\beta A = L_1'(L_1\beta) A = B\xi A$, by choosing $L_1' = B(B' B)^{-\frac{1}{2}}$ and $\xi = (B' B)^{-1} B' \beta$. Thus, all the models considered in the literature are special cases of MANOVA models. If we write $\xi = (\eta_1, \eta_2)$, where $\eta_2 = (\delta', \theta')'$, then we get the nested growth curve model first considered by Srivastava and Khatri (1979). In this paper, several aspects of testing and estimation problems that arise in these models are investigated.

*Keywords and Phrases:* Growth curve models, Estimation, Multivariate normal, Nested Growth Curve Model, Testing of Models.
1. INTRODUCTION

Consider a multivariate analysis of variance (MANOVA) model in which $N$ independent observation vectors $x_1, \ldots, x_N$ on the $p$-dimensional vector $x$ has the following form

$$
(M) \quad X = (x_1, \ldots, x_N) = \beta A + E,
$$

where the $N$ columns of the matrix $E$ are independent and identically distributed as $N_p(0, \Sigma)$, $\Sigma$ assumed unknown. The $p \times m$ matrix $\beta$ is a matrix of unknown parameters. We assume that the $m \times N$ matrix $A$ is a known matrix of full rank. The least squares estimate of $\beta$ is given by

$$
\hat{\beta} = XA'(AA')^{-1}
$$

and an unbiased estimate of $\Sigma$ based on the residuals is given by

$$
n \sum \hat{\epsilon}^2 = (X - \hat{\beta}A)(X - \hat{\beta}A)' = X[I - A'(AA')^{-1}A]X' = S
$$

where $n = N - m$.

Often, the matrix $A$ is considered to be the observations (non-random) or fixed values of $m$ covariables on which the outcome or the response matrix $X$ depends. It is usually desirable to eliminate redundant covariables. To test for the redundancy, we test the hypothesis

$$
H_1 : \beta C = 0 \quad \text{vs} \quad A_1 \neq H_1,
$$

where $C$ is a known matrix of order $m \times m_2$ of rank $m_2$. The above hypothesis can equivalently be written as

$$
H_1 : \beta C(C'C)^{-\frac{1}{2}} \equiv \beta M_2 = 0
$$

where $M_2$ is an $m \times m_2$ matrix and $M = (M_1, M_2)$ and
\[ \tilde{A} = M'A = (\tilde{A}_1', \tilde{A}_2')'. \] Then, we can write

\[
\begin{align*}
\beta X &= \beta MM'A \\
&= (\beta M_1, \beta M_2)\bar{A} \\
&= (\bar{\tilde{\beta}}_1, \bar{\tilde{\beta}}_2) \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} \\
&= \bar{\tilde{\beta}}_1 \tilde{A}_1 + \bar{\tilde{\beta}}_2 \tilde{A}_2,
\end{align*}
\]

where \( \bar{\tilde{\beta}}_i = \tilde{\beta}_i M_i, \ i = 1, 2. \) Under the hypothesis \( H_1, \ \bar{\tilde{\beta}}_2 = 0, \) and the MANOVA model now becomes

\[ E(X) = \bar{\tilde{\beta}}_1 \tilde{A}_1. \]

which is still a MANOVA model but with reduced number of parameters, the number of parameters reduces from \( pm \) to \( pm_1, \ m_1 = m - m_2. \) Next, we consider the hypothesis of the form

\[ H_2 : F\beta C = 0 \ vs \ A_2 \neq H_2. \]

where \( F : (p - q) \times p \) of rank \( p - q. \) This hypothesis can again be written as

\[ H_2 : L_2 \beta M_2 = 0 \ vs \ A_2 \neq H_2, \quad (3) \]

where

\[ L_2 = (FF')^{-\frac{1}{2}} F : (p - q) \times p \ such \ that \ L_2 L_2' = I_{p - q}. \]

and

\[ M_2 = C(C'C)^{-\frac{1}{2}} : m \times m_2 \ such \ that \ M_2 M_2 = I_{m_2}. \]

Define a \( p \times p \) orthogonal matrix \( L = (L_1', L_2')' \) where \( L_1 : q \times p, \) and an \( m \times m \) orthogonal matrix \( M = (M_1, M_2) \), where \( M_1 : m \times m_1, \ m_1 + m_2 = m. \) Then

\[ L_1' L_1 + L_2' L_2 = I_p, \ M_1 M_1' + M_2 M_2' = I_m, \]
\[
\beta = L'LL\beta MM'
= L' \begin{pmatrix}
L_1\beta M_1 & L_1\beta M_2 \\
L_2\beta M_1 & L_2\beta M_2
\end{pmatrix} M'
= L_1' (L_1\beta M_1) M_1' + L_2' (L_2\beta M_1) M_1' + L_1' (L_1\beta M_2) M_2' + L_2' (L_2\beta M_2) M_2'
\]

In hypothesis \( H_1 \), we tested the hypothesis that \( L_1\beta M_2 = 0 \) and \( L_2\beta M_2 = 0 \) and we were still in the MANOVA model. However, if we test only the hypothesis \( H_2 : L_2\beta M_2 = 0 \), then,

\[
\beta = (\beta M_1) M_1' + L_1' (L_1\beta M_2) M_2'
= \tilde{\beta}_1 M_1' + L_1' \xi_2 M_2'
\]

where

\[
\tilde{\beta}_1 = \beta M_1 \quad \text{and} \quad \xi_2 = L_1\beta M_2 = L_1\tilde{\beta}_2 : q \times m_2
\]

Hence, under \( H_2 \)

\[
\beta A = \tilde{\beta}_1 \tilde{A}_1 + L_1' \xi_2 \tilde{A}_2 \quad , \quad \tilde{A}_1 = M_1' A, \quad \tilde{A}_2 = M_2' A \quad (4)
\]

This model is called a mixture of MANOVA and GMANOVA (generalized MANOVA) models, since the first term represents a MANOVA model and the second term represents the GMANOVA model, where a model is said to be GMANOVA model if the mean matrix of the matrix \( X \) is of the form

\[
E(X) = B\xi A
\]

\( B : p \times q, \ \xi : q \times m \) and \( A : m \times N \). The above terminology of mixed MANOVA & GMANOVA model was introduced by Chinchilli and Elswick (1985). But the
problem of testing $H_2$ vs the model $M$ has been considered earlier, see for example, Srivastava and Carter (1983, p.139). Next, we shall consider the problem of testing the hypothesis

$$H_3 : L_2 \beta M_2 = 0, \quad L_2 \beta M_1 = 0 \quad vs \quad A_3 \neq H_3.$$ 

In this case

$$\beta = L_1' (L_1 \beta M_1) M_1' + L_1' (L_1 \beta M_2) M_2' = L_1' (L_1 \beta),$$

and

$$\beta A = L_1' (L_1 \beta) A = B (B' B)^{-\frac{1}{2}} (L_1 \beta) A = (GCM) = B \xi A,$$

(5)

by choosing $L_1'$ as $B (B' B)^{-\frac{1}{2}}$ and $\xi = (B' B)^{-\frac{1}{2}} L_1 \beta = (B' B)^{-1} B' \beta$. This is a growth curve model of Potthoff and Roy (1964), a one sample version of this was considered by Rao (1959). For testing and estimation of parameters in this model, see Khatri (1966), Grizzle and Allen (1969), Gleser and Olkin (1970), and Srivastava and Khatri (1979). The likelihood ratio test for testing that the model (GCM) holds against the model (M), has been given by Srivastava and Carter (1983, p.184) and independently by Chinchilli and Elswick (1985). Finally, if we test the hypothesis

$$H : L_2 \beta M_2 = 0, \quad L_1 \beta M_1 = 0 \quad vs \quad A \neq H,$$

we get, under $H$

$$E(LX) = L \beta MM' A$$
\[ \begin{pmatrix} 0 & \delta_1 \\ \delta_2 & 0 \end{pmatrix} \tilde{A} , \]

where \( \delta_1 = L_1 \beta M_2 \), \( \delta_2 = L_2 \beta M_1 \) and \( \tilde{A} = M'A \). This is a seemingly unrelated MANOVA model. Similarly, if we test the hypothesis that \( L_1 \beta M_1 = 0 \) and \( L_2 \beta M_2 = 0 \), we again get seemingly unrelated regression model. We shall not consider these problems in this paper and refer to V.K.Srivastava and Giles (1987).

The organization of the paper is as follows. In Sections 2 - 5, we review these problems and connect them with the existing literature that has been missed by many researchers in this area. The proofs and derivations are, however, different and appears simpler. In Section 6 and 7, we consider the nested model, first considered by Srivastava and Khatri (1979), and later generalized by von Rosen (1989). The testing of hypotheses problems that arise in these models are also considered in these sections.

2. Testing For a Mixture of MANOVA and GMANOVA Model vs MANOVA model

Consider the MANOVA model

\[ (M) \quad X = \beta A + E \]

\[
\begin{pmatrix}
M \\
p \times m
\end{pmatrix}
\begin{pmatrix}
\beta \\
m \times N
\end{pmatrix}
\begin{pmatrix}
E \\
p \times N
\end{pmatrix}
\]

where the \( N \) columns of \( E \) are iid \( N_p(0, \Sigma) \). We shall assume that \( \Sigma \) is unknown and positive definite and the \( m \times N \) matrix \( A \) of known constants is of rank \( m < N \). The mixture model arises when

\[
H_2 : \quad L_2 \beta M_2 = 0
\]

\[
\begin{pmatrix}
p - q \\
(p - q) \times p
\end{pmatrix}
\begin{pmatrix}
\beta \\
m \times m_2
\end{pmatrix}
\]

or equivalently \( F \beta C = 0 \) with \( L_2 = (FF')^{-1}F : (p - q) \times p \), \( M_2 = C(C'C)^{-\frac{1}{2}} : \)
\[ m \times m_1, L_2' L_2 = I_{p-q} \text{ and } M_2' M_2 = I_{m_2}. \text{ Under } H_2, \]

\[ E(X) = \hat{\beta}_1 \tilde{A}_1 + L'_1 \xi_2 \tilde{A}_2, \quad (6) \]

in the notation of the previous section, where \( \tilde{A}_1 = M_1' A, \quad \tilde{A}_2 = M_2' A, \quad \hat{\beta}_1 = \beta M_1 \text{ and } \xi_2 = L_1 \beta M_2. \text{ Letting} \]

\[ Z = L X G \quad \text{and} \quad \eta = L \hat{\beta}_1 = \left( \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right) \]

where \( G \) is an \( N \times N \) orthogonal matrix to be chosen later and \( L' = (L'_1, L'_2) \), we find that

\[ E(Z) = \begin{bmatrix} \eta \tilde{A}_1 + \left( \begin{array}{c} I_q \\ 0 \end{array} \right) \xi_2 \tilde{A}_2 \end{bmatrix} G \]

\[ = \begin{pmatrix} \eta_1 & \xi_2 \\ \eta_2 & 0 \end{pmatrix} \tilde{A} G, \]

Let \( \tilde{A} \tilde{A}' = TT' \), where \( T \) is a lower triangular matrix, given by

\[ T = \begin{pmatrix} m_1 & m_2 \\ m_1 & 0 \\ m_2 & T_{21} & T_{22} \end{pmatrix}. \]

Then for a semi-orthogonal matrix \( G_1 : m \times N, \quad G_1 G_1' = I_m \), \( \tilde{A} = TG_1 \). Let the orthogonal \( G \) be given by \( G = (G_1', G_0') \). Then

\[ E(Z) = \begin{pmatrix} \eta_1 & \xi_2 \\ \eta_2 & 0 \end{pmatrix} \begin{pmatrix} m & n \\ I_m & 0 \end{pmatrix}, \quad n = N - m, \]
\[
\begin{align*}
&= \left[
\begin{array}{c}
(\eta_1 T_{11} + \xi_2 T_{21} , \xi_2 T_{22}) \\
\eta_2 T_{11} , 0
\end{array}
\right] , \\
&= \left[
\begin{array}{c}
m_1 \\
m_2
\end{array}
\right], \\
&= \left[
\begin{array}{c}
\delta_1 \\
\delta_2 \\
\delta_3
\end{array}
\right] , 0
\right] , \\
&= M_x.
\end{align*}
\]

and the MLE of \( \Sigma \) is given by

\[
N \hat{\Sigma} = (Z - \hat{M}_2)(Z - \hat{M}_2)' ,
\]

where \( \hat{M}_2 \) is the MLE of \( M_2 \) obtained by minimizing

\[
d = |(Z - M_2)(Z - M_2)'|
\]

with respect to \( \delta_1, \delta_2, \) and \( \delta_3 \). Writing

\[
Z \equiv \begin{pmatrix} m_1 & m_2 & n \end{pmatrix}^{T} \begin{pmatrix} Z_1 & Z_3 & U \\ Z_2 & Z_4 \end{pmatrix} \text{ and } W = UU',
\]

we get

\[
d = \begin{vmatrix}
Z_1 - \delta_1 & Z_3 - \delta_3 & U \\
Z_2 - \delta_2 & Z_4
\end{vmatrix} (\cdot)' \\
= \begin{vmatrix}
W + \begin{pmatrix} Z_1 - \delta_1 \\
Z_2 - \delta_2
\end{pmatrix} (\cdot)' + \begin{pmatrix} Z_3 - \delta_3 \\
Z_4
\end{pmatrix} (\cdot)'
\end{vmatrix},
\]

where (\cdot)' denotes the transpose of the matrix on its left side. We shall follow this notation throughout this paper. In addition we shall also write for \((X - ABC)F(X - ABC)'\) as simply \((X - ABC)F(\cdot)'\).
Clearly, the MLE of $\delta_1$ and $\delta_2$ are given by

$$\hat{\delta}_1 = Z_1 \quad \text{and} \quad \hat{\delta}_2 = Z_2$$

respectively.

Thus, it remains to minimize with respect to $\delta_3$, the determinant

$$\left| W + \begin{pmatrix} Z_3 - \delta_3 \\ Z_4 \end{pmatrix} \right| = |W| \left| I + (Z_3' - \delta_3', \ Z_4') W^{-1} \begin{pmatrix} Z_3 - \delta_3 \\ Z_4 \end{pmatrix} \right|$$

$$= |W| \left| I + (Z_3 - \delta_3 - W_{12} W_{22}^{-1} Z_4) W_{1,2}^{-1} (Z_3 - \delta_3 - W_{12} W_{22}^{-1} Z_4) + Z_4' W_{22}^{-1} Z_4 \right|$$

$$\geq |W| \left| I + Z_4' W_{22}^{-1} Z_4 \right| ,$$

equality holding at

$$\hat{\delta}_3 = Z_3 - W_{12} W_{22}^{-1} Z_4 ,$$

where

$$W = \begin{pmatrix} q \quad p-q \\ p-q \quad W_{11} \quad W_{12} \\ & W'_{12} \quad W_{22} \end{pmatrix} ,$$

$$W_{1,2} = W_{11} - W_{12} W_{22}^{-1} W'_{12} .$$

Thus, under the hypothesis $H_2$, the MLE of $\Sigma$ is given by

$$N \hat{\Sigma} = W + \begin{pmatrix} W_{12} W_{22}^{-1} \\ I_{p-q} \end{pmatrix} Z_4 Z_4' (W_{22}^{-1} W'_{12}, \ I_{p-q})$$

The MLE of $\Sigma$ under the alternative hypothesis $A_2$ (i.e. MANOVA model) is given by

$$N \hat{\Sigma} = W .$$

Hence, the likelihood ratio test rejects the hypothesis $H_2$ for smaller values of the
\[ \lambda_2 = \frac{|W_{22}|}{|W_{22} + Z_4 Z_4'|} \equiv U_{p-q,m_2,n} \]

The asymptotic distribution of \( \lambda_2 \) is given by

\[
P\{-\frac{n - 1}{2} (p - q - m_2 + 1) \log U_{p-q,m_2,n} \geq Z\}
= P\{X_2^2 \geq Z\} + \gamma_2 P\{X_{2+4}^2 \geq Z\} - P(X_2^2 \geq Z) + O(n^{-3})
\]

where \( f = (p - q)m_2 \) and \( \gamma_2 = f[(p - q)^2 + m_2 - 5]/48 \). In terms of the original variables

\[
W = UU' = LXG_0X' \quad L^T
= LX[I - \tilde{A}'T^{-1}T^{-1}\tilde{A}]X' \quad L^T
= LX[I - A'(AA')^{-1}A]X' \quad L^T
= LSL'
\]

Hence

\[ W_{22} = L_2 S L_2' \quad \]

Recalling that \( G_1' = \tilde{A}'T^{-1} \), and

\[
\begin{pmatrix}
Z_1 & Z_3 \\
Z_2 & Z_4
\end{pmatrix} = \tilde{L}X \tilde{A}'T^{-1} \quad ,
\]

we get

\[ (Z_2, Z_4) = L_2 X \tilde{A}'T^{-1} \quad , \]

\[ Z_1 = L_1 X \tilde{A}_1 T_{11}^{-1} \quad , \]

and

\[ Z_2 = L_2 X \tilde{A}_1' T_{11}^{-1} \quad . \]
Hence,

\[ Z_4 Z'_4 = (Z_2, Z_4) \begin{pmatrix} Z'_2 \\ Z'_4 \end{pmatrix} - Z_2 Z'_2 \]
\[ = L_2 X X'(AA')^{-1}AX'L_2 - L_2 X A'M_1(M_1'AA'M_1)^{-1}M_1'AX'L_2 \]
\[ = L_2 X A'[AA']^{-1} - M_1(M_1'AA'M_1)^{-1}M_1'AX'L_2 \]
\[ = L_2 X A'[AA']^{-1}M_2[M_2'(AA')^{-1}M_2]^{-1}M_2'(AA')^{-1}AX'L_2' \]
\[ = L_2 \beta C[C'(AA')^{-1}C]^{-1}C'\beta' L'_2 \]
\[ = (FF')^{-\frac{1}{2}} F \beta C[C'(AA')^{-1}C]^{-1}C'\beta' F'(FF')^{-\frac{1}{2}} \]

and

\[ \lambda_2 = \frac{|FSF'|}{|FSF' + F\beta C[C'(AA')^{-1}C]^{-1}C'\beta' F'|} . \]

Similarly, the MLE of \( \xi_2 \) and \( \tilde{\beta}_1 \) are respectively,

\[ \hat{\xi}_2 = (L_1 S^{-1} L'_1)^{-1} L_1 S^{-1} X \hat{Q}_1 \hat{A}_2' (\hat{A}_2 Q_1 \hat{A}_2')^{-1} \]

and

\[ \tilde{\beta}_1 = (X - L'_1 \hat{\xi}_2 \hat{A}_2) \hat{A}_1' (\hat{A}_1 \hat{A}_1')^{-1} , \]

where

\[ Q_1 = I - \hat{A}_1' (\hat{A}_1 \hat{A}_1')^{-1} \hat{A}_1 \]
\[ = A'(AA')^{-1}A - A'(AA')^{-1}C[C'(AA')^{-1}C]^{-1}C'(AA')^{-1}A \]
\[ = A'(AA')^{-1}[(AA') - C(C'(AA')^{-1}C)C'(AA')^{-1}A \]

This problem was considered by Srivastava and Carter (1983) and later by Chinchilli and Elswick (1985).

3. Testing GMANOVA vs MANOVA

In this section, we consider the problem of testing that it is a GMANOVA model against the alternative that it is a MANOVA model. Under MANOVA model, we
have

\[(M) \quad E(X) = \beta A\]

and under GMANOVA model, we have

\[(GCM) \quad E(X) = B\xi A\]
\[= L_1'\eta A, \quad L_1 L_1' = I_q, \quad \eta = (B' B)^{\frac{1}{2}}\xi\]

Hence, under GCM model, with \(Z = L X G, \quad LL' = I_p, \quad GG' = I_N \) and \(G = (G_1', G_0')\), we get

\[E'(Z) = \begin{pmatrix} I_q \\ 0 \end{pmatrix} \eta A(G_1', G_0') \quad A = TG_1, \quad G_1 G_1' = I_m\]
\[= \begin{pmatrix} \eta \\ 0 \end{pmatrix} (T, 0)\]
\[= \begin{pmatrix} \gamma \\ 0 \\ 0 \end{pmatrix}, \quad \gamma = \eta T.\]

Writing

\[Z = \begin{pmatrix} q \\ p-q \end{pmatrix} \begin{pmatrix} m \quad n \\ Z_{11} \quad U \\ Z_{12} \quad Z_{12} \end{pmatrix},\]

we find that \(E(U) = 0\) and \(E(Z_{11}', Z_{12}') = (\gamma', 0)\). Hence, the MLE of \(\gamma\) is given by

\[\hat{\gamma} = Z_{11} - W_{12} W_{12}^{-1} Z_{12},\]
where

\[ W = UU' = \begin{pmatrix} W_{11} & W_{12} \\ W_{12}' & W_{22} \end{pmatrix}, \]

and the MLE of \( \Sigma \) is given by

\[ N\hat{\Sigma} = W + \begin{pmatrix} W_{12}W_{22}^{-1} \\ I_{p-q} \end{pmatrix}Z_{12}Z_{12}' \begin{pmatrix} W_{12}W_{22}^{-1} \\ I_{p-q} \end{pmatrix}'. \]

Hence, the LET for testing GM model vs M model is given by

\[ \lambda_3 = \frac{|W_{22}|}{|W_{22} + Z_{12}Z_{12}'|}, \]

where

\[ W_{22} = L_2SL_2^l. \]

The distribution of \( \lambda_3 \) is obtained from \( \lambda_2 \) by replacing \( m_2 \) by \( m \). Since,

\[ \begin{pmatrix} Z_{11} \\ Z_{12} \end{pmatrix} = \begin{pmatrix} L_1X_A^TP^{-1}' \\ L_2X_A^TP^{-1}' \end{pmatrix}, \]

and

\[ W_{12} = L_1SL_2^l, \]

\[ \hat{\gamma} = L_1X_A^TP^{-1}' - (L_1SL_2^l)(L_2SL_2^l)^{-1}L_2X_A^TP^{-1}' \]

\[ = L_1[S - SL_2^l(L_2SL_2^l)^{-1}L_2S]S^{-1}X_A^TP^{-1}' \]

\[ = L_1L_1'(L_1S^{-1}L_1')^{-1}L_1S^{-1}X_A'(AA')^{-1}T, \]

and

\[ \hat{\eta} = (L_1S^{-1}L_1')^{-1}L_1S^{-1}X_A'(AA')^{-1}. \]
We also find that

\[ Z_{12} Z'_{12} = L_2 X A' T^{-1'} T^{-1} A X' L'_2 \]

\[ = L_2 X A' (A A')^{-1} A X' L'_2 , \]

and hence in terms of the original variables

\[ \lambda_3 = \frac{|L_2 S L'_2|}{|L_2 S L'_2 + L_2 X A' (A A')^{-1} A X' L'_2|} . \]

Note that

\[ \lambda_3 = \frac{|L_2 S L'_2|}{|L_2 S L'_2 + L_2 S_1 L'_2|} , \quad S_1 = X A' (A A')^{-1} A X' \]

\[ = |I_p + (L_2 S L'_2)^{-1} (L_2 S_1 L'_2)|^{-1} \]

\[ = |I_p + L'_2 (L_2 S L'_2)^{-1} L_2 S_1|^{-1} \]

\[ = |I_p + [S^{-1} - S^{-1} L'_1 (L_1 S^{-1} L'_1)^{-1} L_1 S^{-1}] S_1|^{-1} \]

\[ = |S| |S + S_1 - L'_1 (L_1 S^{-1} L'_1)^{-1} L_1 S^{-1} S_1|^{-1} \]

\[ = |S| |S + S_1|^{-1} |I - (S + S_1)^{-1} L'_1 (L_1 S^{-1} L'_1)^{-1} L_1 S^{-1} S_1|^{-1} \]

\[ = |S| |S + S_1|^2 |I - L_1 S^{-1} S_1 (S + S_1)^{-1} L'_1 (L_1 S^{-1} L'_1)^{-1}|^{-1} \]

\[ = |S| |S + S_1|^2 |I - L_1 S^{-1} S_1 (S + S_1 - S)(S + S_1)^{-1} L'_1 (L_1 S^{-1} L'_1)^{-1}|^{-1} \]

\[ = |S| |S + S_1|^2 |I - L_1 S^{-1} L'_1 (L_1 S^{-1} L_1)^{-1} + L_1 (S + S_1)^{-1} L'_1 (L_1 S^{-1} L'_1)^{-1}|^{-1} \]

\[ = |S| |S + S_1|^2 |L_1 (S + S_1)^{-1} L'_1 (L_1 S^{-1} L'_1)^{-1}|^{-1} \]

\[ = \frac{|S|}{|S + S_1|} \cdot \frac{|L_1 S^{-1} L'_1|}{|L_1 (S + S_1)^{-1} L'_1|} \]

\[ = \frac{|S|}{|S + S_1|} \cdot \frac{|B' S^{-1} B|}{|B' (S + S_1)^{-1} B|} . \]

The above testing procedure was considered by Srivastava and Carter (1983) and later by Chinchilli and Elswick (1985).

From above, it also follows that

\[ \hat{\eta} = (B' B)^{\frac{1}{2}} (B' S^{-1} B)^{-1} B' S^{-1} X A' (A A')^{-1} \]
and hence the MLE of $\xi$ is given by

$$
\hat{\xi} = (B' S^{-1} B)^{-1} B' S^{-1} X A' (A A')^{-1}.
$$

which is an unbiased estimator of $\xi$ with

$$
\text{Cov} (\text{vec} \; \hat{\xi}) = \left( \frac{n-1}{n-p+q-1} \right) (A A')^{-1} \otimes (B' \Sigma^{-1} B)^{-1},
$$

where $\text{vec}(c_1, \ldots, c_r) = (c_1', \ldots, c_r')'$ and $F \otimes G$ denotes the Kronecker product $(f_{ij} G)$ for $F = (f_{ij})$.

4. Testing GCM model vs Mixture Model.

In this section, we consider the problem of testing the hypothesis that the model is a growth curve model against the alternative that it is a mixture of MANOVA and GCM models. That is, we test the hypothesis

$$
H_4: E(X) = L_1'(\xi_1, \xi_2) \hat{A}
$$

against the alternative

$$
A_4: E(X) = L' \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & 0 \end{pmatrix} \hat{A},
$$

where $L = (L_1', L_2')'$ is an orthogonal matrix and $\hat{A}$ is an $m \times N$ matrix of constants of rank $m < N$. The matrices $\xi_1$, $\xi_2$ and $\xi_3$ are of orders $q \times m_1$, $q \times m_2$ and $(p-q) \times m_1$ respectively. Thus, for an orthogonal matrix $G = (\hat{A}' T^{-1}', G_0)$, where $\hat{A} \hat{A}' = T T'$ and $T$ is a lower triangular matrix, we get under $H_4$

$$
E(Z) = E(L X G) = \begin{pmatrix} \xi_1 & \xi_2 \\ 0 & 0 \end{pmatrix} (T, 0) = \begin{pmatrix} \xi_1 T_{11} + \xi_2 T_{21} & \xi_2 T_{22} \\ 0 & 0 \end{pmatrix}.
$$
and under the alternative $A_4$

\[
E(Z) = E(LXG) = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & 0 \end{pmatrix} (T, 0) = \begin{pmatrix} \xi_1 T_{11} + \xi_2 T_{21} & \xi_2 T_{22} \\ \xi_3 T_{11} & 0 \end{pmatrix}
\]

where

\[
T = \begin{pmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{pmatrix}
\]

Writing

\[
Z = \underbrace{q}_{p} \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} U
\]

and

\[
W = UU' = \underbrace{q}_{p-q} \begin{pmatrix} W_{11} & W_{12} \\ W_{12} & W_{22} \end{pmatrix}
\]

we find that the likelihood ratio test is based on the statistic

\[
\lambda_4 = \frac{|W_{22}|}{|W_{22} + Z_3 Z_3' + Z_4 Z_4'|} \cdot \frac{|W_{22} + Z_4 Z_4'|}{|W_{22}|} = \frac{|W_{22} + Z_4 Z_4'|}{|W_{22} + Z_3 Z_3' + Z_4 Z_4'|} = U_{p-q, m_1, n+m_2}
\]

5. Testing General Hypotheses in GCM model.

Consider the growth curve model in which

\[
E(X) = B\xi A
\]
where $B : p \times q$, $\xi : q \times m$ and $A : m \times N$. We wish to test the hypothesis

$$H_5 : C\xi D = 0 \text{ vs } A_5 \neq H,$$

where $C : c \times q$, $c \leq q$ and $D : m \times d$, $d \leq m$ are of ranks $c$ and $d$ respectively.

Consider nonsingular matrices $C_0$ and $D_0$ defined by

$$C_0' = (C_1', C_2') : q \times q, \quad CC_1' = 0$$

and

$$D_0 = (D_1, D) : m \times m, \quad D_1' D = 0$$

Then

$$E(X) = BC_0^{-1}C_0\xi D_0 D_0^{-1} A$$

$$= B_1 \eta A_1,$$

where $B_1 = BC_0^{-1}$, $A_1 = D_0^{-1} A$ and

$$\eta = C_0 \xi D_0 = \begin{pmatrix} C_1 \xi D_1 & C_1 \xi D \\ C_\xi D_1 & C_\xi D \end{pmatrix}.$$ 

Under $H_5 : C\xi D = 0$. We have assumed that $N$ columns of $X$ are independently normally distributed with covariance $\Sigma$.

Let

$$A_0' = [A_1' (A_3')^{-1}, \quad A_2']$$

be an orthogonal matrix such that $A_1 A_1' = D_0^{-1} A A' D_0^{-1} = A_3 A_3'$; $A_1 A_2' = 0$ where $A_3$ is a lower triangular matrix.

Let

$$B_0 = [B_1 (B_1' B_1)^{-1}, \quad B_2]$$

be a nonsingular matrix with $B_2' B_1 = 0$.
Note that

\[ B'_1 B_0 = (I_q, 0), \quad A_1 A'_0 = (A_3, 0). \]

Hence

\[
M_1 = B'_0 B_1 \eta A_1 A'_0 \\
= \begin{pmatrix} I_q \\ 0 \end{pmatrix} \eta (A_3, 0) \\
= \begin{pmatrix} m & N-m \\ m & N-m \end{pmatrix} \equiv \begin{pmatrix} q \\ p-q \end{pmatrix} \begin{pmatrix} \eta A_3 & 0 \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}
\]

and under H

\[
\delta = \eta A_3 = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & 0 \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{pmatrix} \equiv \begin{pmatrix} \delta_1 & \delta_3 \\ \delta_2 & 0 \end{pmatrix}
\]

Let

\[
Y = B'_0 X A'_0 = \begin{pmatrix} (B'_1 B_1)^{-1} B'_1 X A_0 \\ B'_2 X A_0 \end{pmatrix}
\]

Then

\[
E(Y) = M_1 = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}; \text{ under } H, \quad M_1 = \begin{pmatrix} \delta_1 & \delta_3 & 0 \\ \delta_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
Let

\[
    Y = \begin{pmatrix}
    Y_1 & Y_4 & & & & \\
    Y_3 & & & & & \\
    & & & & & \\
    & & & & & \\
    & & & & & \\
    \end{pmatrix}
\]

\[
= \begin{pmatrix}
    Y_{11} & Y_{12} & & & & \\
    Y_{21} & Y_{22} & Y_4 & & & \\
    Y_{31} & Y_{32} & & & & \\
    \\
\end{pmatrix}
\]

\[
V = Y_4 Y_4' = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}' & V_{33} \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{12}' & V_{22} & V_{23} \\ V_{13}' & V_{23}' & V_{33} \end{pmatrix}
\]

Then, we need to minimize the following determinant \(d\) to get the maximum likelihood estimate of \(\delta\) under \(A\).

\[
d = |R| = |V + \begin{pmatrix} Y_1 - \delta \\ Y_3 \end{pmatrix} \begin{pmatrix} Y_1 - \delta \\ Y_3 \end{pmatrix}'|
\]

\[
= |V||I + (Y_1' - \delta', Y_3'V_{33}^{-1}Y_3)(Y_1 - \delta)\|
\]

\[
= |V||I + Y_3'V_{33}^{-1}Y_3 + (Y_1 - \delta - P_{12}V_{33}^{-1}Y_3)'V_{12}(Y_1 - \delta - P_{12}V_{33}^{-1}Y_3)|
\]

\[
\geq |V||I + Y_3'V_{33}^{-1}Y_3|
\]

\[
= \frac{|V|}{|V_{33}|}|V_{33} + Y_3 Y_3'|
\]

The minimum occurs at

\[
\hat{\eta}A_3 = \delta = Y_1 - P_{12}'V_{33}^{-1}Y_3
\]

This gives

\[
C_0 \hat{\Delta}_0 A_3 = \delta
\]
\[ = (B_1'B_1)^{-1}B_1'X A_0 - P_{12}'V_{33}B_2'X A_0' \]
\[ = [(B_1'B_1)^{-1}B_1' - P_{12}'V_{33}^{-1}B_2']X A_0' \]

Let
\[ Q' = (Y_{12}', \delta_3', Y_{22}', Y_{32}') \cdot \]

To get MLE of \( \delta_1, \delta_2, \delta_3 \), we need to minimize

\[
\begin{vmatrix}
V + 
\begin{pmatrix}
Y_{11} - \delta_1 \\
Y_{21} - \delta_2 \\
Y_{31}
\end{pmatrix}
+ 
\begin{pmatrix}
Y_{12} - \delta_3 \\
Y_{22} \\
Y_{32}
\end{pmatrix}'
\end{vmatrix} 

\]

\[
\begin{vmatrix}
Y_{11} - \delta_1 \\
Y_{21} - \delta_2 \\
Y_{31}
\end{vmatrix}' 

(V + Q Q')^{-1} 

\begin{pmatrix}
Y_{11} - \delta_1 \\
Y_{21} - \delta_2 \\
Y_{31}
\end{pmatrix}
\]

\[
\geq |V + Q Q'||I + Y_{31}(V_{33} + Y_{32}Y_{32}')^{-1}Y_{31}| ,
\]

\[
= \frac{|V + Q Q'|}{|V_{33} + V_{32}Y_{32}'|} |V_{33} + V_{32}Y_{32} + Y_{31}Y_{31}'|
\]

\[
= \left| V + \begin{pmatrix}
Y_{12} - \delta_3 \\
Y_{22} \\
Y_{33}
\end{pmatrix}' \right| \frac{|V_{33} + V_{32}Y_{32} + Y_{31}Y_{31}'|}{|V_{33} + V_{32}Y_{32}'|}
\]

\[
= |V| \left| I + \begin{pmatrix}
Y_{22}' \\
Y_{32}'
\end{pmatrix}' \begin{pmatrix}
V_{22} & V_{23} \\
V_{23}' & V_{33}
\end{pmatrix}^{-1} \begin{pmatrix}
Y_{22} \\
Y_{32}
\end{pmatrix} \right| \times \frac{|V_{33} + V_{32}Y_{32} + Y_{31}Y_{31}'|}{|V_{33} + V_{32}Y_{32}'|}
\]

\[
= |V| \left| \begin{pmatrix}
V_{22} & V_{23} \\
V_{23}' & V_{33}
\end{pmatrix} + \begin{pmatrix}
Y_{22} \\
Y_{32}
\end{pmatrix}' \begin{pmatrix}
Y_{22} \\
Y_{32}
\end{pmatrix} \right| \times \frac{|V_{33} + V_{32}Y_{32} + Y_{31}Y_{31}'|}{|V_{33} + V_{32}Y_{32}'|}
\]

\[
= \frac{|\bar{V}|}{|V + ZZ'|} \times \frac{|V_{33} + ZZ'|}{|V_{33}|} ,
\]

Thus, the likelihood ratio test is based on the statistic
where

\[ \tilde{V} \equiv \begin{pmatrix} V_{22} & V_{23} \\ V_{23} & V_{33} \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} Y_{22} \\ Y_{32} \end{pmatrix}. \]

It may be noted that the statistic \( \lambda_5 \) is invariant under non singular linear transformations. Thus, we may assume without any loss of generality that \( \Sigma = I \). Under this assumption, the joint distribution of \( \tilde{V} \) and \( Z \) under the hypothesis if given by

\[ \text{Const.} |\tilde{V}|^{(n-p+q-c-1)/2}[\text{etr} - \frac{1}{2}(\tilde{V} + ZZ')] , \quad n = N - m \]

Consider the transformations

\[ \tilde{V} + ZZ' = TT' = \begin{pmatrix} T_1 & T_1' \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} T_1' & 0' \\ T_1 & T_2' \end{pmatrix} \]

\[ Z = TW , \quad W' = (W'_1, W'_2). \]

we find that the Jacobian of the transformation \( J(\tilde{V}, Z \to T, W) = 2^p \sqrt{\Pi_{ii}^d} \), since \( Z \) is a matrix of order \((p-q+c) \times d \). It can be seen that \( T \) and \( W \) are independently distributed. The pdf of \( W \) is given by

\[ \text{Const.} |I - WW'|^{(n-p+q-c-1)/2} \]

and

\[ \lambda_5 = \frac{|I - W_2W'_2|}{|I - WW'|} = \frac{|I - W'_2W_2|}{|I - W'_1W_1 - W'_2W_2|} = |I - P'P|^{-1} , \]

where \( P' = (I - W'_2W_2)^{-\frac{1}{2}}W'_1 \). The pdf of \( P' \) is given by

\[ \text{Const.} |I - P'P|^{(n-p+q-c-1)/2} . \]

Hence

\[ \lambda_5 = U_{c,d,n-p+q} . \]

With a slight change of notation, consider the GCM model in which

\[(GCM) \quad E(X) = B_1 \xi A\]

\[B_1 : p \times q \quad \xi : q \times m \quad A : m \times N\]

matrices \(A\) and \(B_1\) are of full rank. Consider the case when

\[
(H_6) \quad \xi = \begin{pmatrix}
q_1 \\
q_2
\end{pmatrix}
\begin{pmatrix}
\xi_1 & \xi_2 \\
\xi_3 & \xi_4
\end{pmatrix}
\]

with \(\xi_4 = 0, q_1 + q_2 = q\) and \(m_1 + m_2 = m\).

Then writing \(\eta'_1 = (\xi'_1, \xi'_3), \eta_2 = \xi_2, B_1 = (B_2, \tilde{B})\) and \(A' = (A'_1, A'_2)\), we get

\[B_1 \xi A = B_1 \eta_1 A_1 + B_2 \eta_2 A_2\]

where the columns of \(B_2\) are a subset of the columns of \(B_1\). This model was considered by Srivastava and Khatri (1979, p.197) and a conditional approach was proposed for the maximum likelihood estimation and for obtaining the likelihood ratio test for lasting \(H_4\) against the alternative (GCM). The conditional approach is useful in obtaining the estimates in terms of the original parameters. However, it is not necessary to convert all the results in the original variables especially when it gets considerably involved. For testing of hypotheses we shall reduce the problem to a canonical form and solutions will be provided in terms of canonical variables. Finally, we may note that under \(H_6\) we can also write (slight change of notation)

\[B \xi A_1 = B_1 \tilde{\xi}_1 A_1 + B_2 \tilde{\xi}_2 A_2\]

where \(B = (B_1, B_2), \tilde{\xi}_1 = (\xi_1, \xi_2), \tilde{\xi}_2 = \xi_3\) and \(A_1 = (A_2, \tilde{A})\) and rows of \(A_2\) are a subset of \(A_1\). Thus the given solution is applicable in both situations.
6.1 Maximum Likelihood Estimates of $\eta_1, \eta_2$ and $\Sigma$.

The MLE of $\Sigma$ is given by

$$\hat{\Sigma} = (X - B_1\hat{\eta}_1 A_1 - B_2\hat{\eta}_2 A_2)(X')'$$

where $\hat{\eta}_1$ and $\hat{\eta}_2$ are those values of $\eta_1$ and $\eta_2$ that minimizes the determinant

$$d_4 = |(X - B_1\eta_1 A_1 - B_2\eta_2 A_2)(X')'|.$$

We minimize $d_4$ first with respect to $\eta_1$ for fixed $\eta_2$. From theorem 1.10.3 of Srivastava and Khatrri (1979) we obtain

$$B_1\hat{\eta}A_1 = B_1(B_1'^{-1}S_\eta^{-1}B_1')^{-1}B_1'S_\eta^{-1}(X - B_2\eta_2 A_2)P_1,$$

where

$$P_1 = A_1'(A_1A_1')^{-1}A_1$$

$$S_\eta = (X - B_2\eta_2 A_2)(X' - S_1\eta),$$

$$S_1\eta = (X - B_2\eta_2 A_2)P_1(X - B_2\eta_2 A_2)' .$$

Let $B_0$ be a matrix such that $(B_1, B_0)$ is nonsingular and $B_0'B_1 = 0$. Then at $\hat{\eta}_1$

$$d_4 = |S_\eta + [I - B_1(B_1'^{-1}S_\eta^{-1}B_1)^{-1}B_1'S_\eta^{-1}]S_1\eta (X')|$$

$$= |S_\eta||I + S_1\eta([I - B_1(B_1'^{-1}S_\eta^{-1}B_1)^{-1}B_1'S_\eta^{-1}]S_\eta^{-1})||$$

$$= |S_\eta||I + S_1\eta[S_\eta^{-1} - S_\eta^{-1}B_1(B_1'^{-1}S_\eta^{-1}B_1)'B_1'S_\eta^{-1}]||$$

$$= |S_\eta||I + S_1\eta B_0(B_0'S_\eta B_0)^{-1}B_0'||$$

$$= |S_\eta|(B_0'S_\eta B_0)^{-1}||B_0'(S_\eta + S_1\eta)B_0||.$$

Since $B_0'B_1 = 0$, $B_0'B_0 = 0$ and $B_0'B_2 = 0$. Hence

$$B_0'S_1\eta B_0 = B_0'XP_1X'B_0.$$
and
\[ B_0' S_n B_0 = B_0' X X'B_0 - B_0' X P_1 X'B_0 = B_0' X Q_1 X'B_0 , \quad Q_1 = I - P_1 . \]

Thus at \( \hat{\eta}_1 \),
\[
d_4 = |S_n||B_0' X Q_1 X'B_0|^{-1}|B_0' X X'B_0|
\]
\[
= |(X - B_2 \hat{\eta}_2 A_2)Q_1(X - B_2 \hat{\eta}_2 A_2)'| |B_0' X Q_1 X'B_0|^{-1}|B_0' X X'B_0|
\]
\[
= |(\tilde{X} - B_2 \hat{\eta}_2 \tilde{A}_2)(\tilde{X} - B_2 \hat{\eta}_2 \tilde{A}_2)'| |B_0' X Q X'B_0|^{-1}|B_0' X X'B_0| ,
\]

where \( \tilde{X} = X Q_1 \) and \( \tilde{A}_2 = A_2 Q_1 \). Again, using Theorem 1.10.3 from Srivastava and Khatri (1979), we get
\[
B_2 \hat{\eta}_2 \tilde{A}_2 = B_2 (B' \hat{S}^{-1} B_2)^{-1} B' \hat{S}^{-1} \tilde{X} \hat{P}_2 ,
\]

where
\[
\hat{\tilde{S}} = \tilde{X} \tilde{X}' - \hat{S}_1 , \quad \hat{\tilde{S}}_1 = \tilde{X} \hat{P}_2 \tilde{X}' , \quad \hat{\tilde{P}}_2 = \hat{\tilde{A}}_2 (\hat{\tilde{A}}_2')^{-1} \tilde{A}_2 .
\]

The minimum value of \( d_4 \) at \( \hat{\eta}_1 \) and \( \hat{\eta}_2 \)
\[
= |\tilde{T}| \quad |B_0' X Q X'B_0|^{-1} \quad |B_0' X X'B_0| ,
\]

where
\[
\tilde{T} = \hat{\tilde{S}} + [I - B_2 (B' \hat{S}^{-1} B_2)^{-1} B' \hat{S}^{-1}] \hat{\tilde{S}}_1 [I]^2
\]

Note that
\[
\hat{\tilde{S}} = X Q_1 X' - X Q_1 A_2' (A_2 Q_1 A_2')^{-1} A_2 Q_1 X'
\]
\[
= X Q_1 [I - A_2' (A_2 Q_1 A_2')^{-1} A_2] Q_1 X'
\]
where

\[ Q_1 = I - A'_1 (A_1 A'_1)^{-1} A_1 = I - P_1 \]

and

\[ S_{\tilde{\eta}} = (X - B_2 \tilde{\eta}_2 A_2)^T Q_1 (X - B_2 \tilde{\eta}_2 A_2)^T = (X - B_2 \tilde{\eta}_2 A_2)(X' Q_1 A_2 A_1' A_1)^{-1} , \]

Hence, the MLE of \( \eta_2 \) is

\[ \hat{\eta}_2 = (B'_2 S_{\tilde{\eta}} B_2)^{-1} B'_2 S_{\tilde{\eta}} B_2 (X Q_1 A_2 A_1')^{-1} \]

and the MLE of \( \eta_1 \) is

\[ \hat{\eta}_1 = (B'_1 S_{\tilde{\eta}} B_1)^{-1} B'_1 S_{\tilde{\eta}} B_1 (X - B_2 \tilde{\eta}_2 A_2) A_1 (A_1 A'_1)^{-1} \]

The MLE of \( \Sigma \) is given by

\[ N \hat{\Sigma} = (X - B_1 \hat{\eta}_1 A_1 - B_2 \tilde{\eta}_2 A_2)(X' Q_1 A_2 A_1')^{-1} . \]

6.2. Canonical Reduction.

Let

\[ B_1 = L_1' P, \quad P = \begin{pmatrix} P_1 & P_{12} \\ 0 & P_2 \end{pmatrix} \]

where \( P \) is an upper triangular matrix and \( L_1 L_1' = I_q \). Then

\[ B_1 \xi = L_1' \begin{bmatrix} \delta \\ \delta_1 \end{bmatrix} = P \eta_1, \quad \delta_1 = P_1 \eta_2 . \]
Let $L' = (L'_1, L'_2)$ be an orthogonal matrix. Then

$$LB_1\xi = \begin{pmatrix} I_q \\ 0 \end{pmatrix} \begin{pmatrix} \delta, \delta_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \delta, \delta_1 \\ 0, 0 \end{pmatrix}$$

Define an orthogonal matrix $G = (A'T^{-1}', G_0)$ where $AA' = TT'$ and $T$ is a lower triangular matrix. Then

$$LB_1\xi AG_1 = \begin{pmatrix} \delta, \delta_1 \\ 0, 0 \end{pmatrix} [T, 0] = \begin{pmatrix} \delta, \delta_1 \\ 0, 0 \end{pmatrix} T, 0$$

$$= \begin{pmatrix} \mu, \mu_1 \\ 0, 0 \end{pmatrix} 0$$

$$= [M_1, 0]$$

Consider the transformation

$$Z = LXG$$

$$= \begin{pmatrix} m_1 & m_2 \\ q_1 & Y_{11} & Y_{21} \\ q_2 & Y_{12} & Y_{22} & V \\ p-q & Y_{13} & Y_{23} \end{pmatrix} = (Y, V).$$
and define \( Y_{(1)} = (Y_{11}, Y_{12}), \ Y_{(2)} = (Y_{22}, Y_{23}) \), and

\[
W = VV' = \begin{pmatrix}
    W_{11} & W_{12} & W_{13} \\
    W_{12} & W_{22} & W_{23} \\
    W_{13} & W_{23} & W_{33}
\end{pmatrix}
= \begin{pmatrix}
    q_1 & q_2 & p-q \\
    q_1 & p-q_1 & q \\
    q & p-q
\end{pmatrix}

Then the MLE of \( \Sigma \) is given by

\[
N \hat{\Sigma} = W + \begin{pmatrix}
    Y_{11} - \hat{\mu} \\
    Y_{13}
\end{pmatrix}
\begin{pmatrix}
    Y_{21} - \mu_1 \\
    Y_{(2)}
\end{pmatrix}
\begin{pmatrix}
    Y_{(1)} - \mu \\
    Y_{13}
\end{pmatrix}
\begin{pmatrix}
    Y_{21} - \mu_1 \\
    Y_{(2)}
\end{pmatrix}^T
\]

where \( \hat{\mu} \) and \( \hat{\mu}_1 \) are the values of the matrices \( \mu \) and \( \mu_1 \) that minimize the determinant

\[
d = \left| W + \begin{pmatrix}
    Y_{11} - \mu \\
    Y_{13}
\end{pmatrix}
\begin{pmatrix}
    Y_{21} - \mu_1 \\
    Y_{(2)}
\end{pmatrix}^T\right|
\]

\[
= |Q| \left| I + (Y_{11}' - \mu_0 Y_{13}')Q^{-1} \begin{pmatrix}
    Y_{11} - \mu \\
    Y_{13}
\end{pmatrix}\right|
\geq |Q| |I + Y_{13}'Q_{33}^{-1}Y_{13}| ,
\]

where

\[
Q = W + \begin{pmatrix}
    Y_{21} - \mu_1 \\
    Y_{(2)}
\end{pmatrix}^T
\begin{pmatrix}
    Q_{11} & Q_{12} & Q_{13} \\
    Q_{22} & Q_{23} \\
    Q_{33}
\end{pmatrix}
\begin{pmatrix}
    Q_{11} & Q_{12} \\
    Q_{13} & Q_{33}
\end{pmatrix}
\]
and $Q_{33} = W_{33} + Y_{23}Y'_{23}$, $Y'_{(2)} = (Y'_{22}, Y'_{23})$. The equality holds at

$$
\hat{\mu} = Y_{(1)} - Q_{(13)}Q_{33}^{-1} Y_{13}
$$

Next, we find that

$$|Q| \geq |W||I + Y'_{(2)}W_{(22)}^{-1}Y_{(2)}|
$$

and the equality holds at

$$
\hat{\mu}_1 = Y_{21} - W_{(12)}W_{(22)}^{-1}Y_{(2)}.
$$

Thus

$$
\hat{\mu} = Y_{(1)} - \hat{Q}_{(13)}\hat{Q}_{33}^{-1} Y_{13},
$$

where $\hat{Q}$ is $Q$ with $\hat{\mu}_1$ in place of $\mu_1$.

The minimum value of

$$
d = |I + Y'_{13}Q_{33}^{-1}Y_{13}||I + Y'_{(2)}W_{(22)}^{-1}Y_{(2)}||W|
$$

$$
= \frac{|W_{(22)} + Y_{(2)}Y'_{(2)}|}{|W_{(22)}|} \cdot \frac{|Q_{33} + Y_{13}Y'_{13}|}{|Q_{33}|} \cdot |W|
$$

$$
= \frac{|W_{(22)} + Y_{(2)}Y'_{(2)}|}{|W_{(22)}|} \cdot \frac{|W_{33} + Y_{23}Y'_{23} + Y_{13}Y'_{13}|}{|W_{33} + Y_{23}Y'_{23}|} \cdot |W|
$$

6.3 Testing for Nested model vs GCM model.

To test the hypothesis that it is a nested model against the alternative model it is a GCM model, we need to obtain the estimates of the parameters under GCM model, that is, when $\xi_4 \neq 0$. In this case

$$
LB\xi = \begin{pmatrix} I_q \\ 0 \end{pmatrix} P\xi = \begin{pmatrix} \theta \\ 0 \end{pmatrix}, \quad \theta = P\xi
$$
Hence,

\[
E(Z) = E(LXG) = E(Y, V)
\]

\[
= E\left[\begin{array}{c}
Y_{(12)} \\
Y_{(22)}
\end{array}\right],
\]

where

\[
Y_{(12)} = \begin{bmatrix}
Y_{11} & Y_{21} \\
Y_{12} & Y_{22}
\end{bmatrix}, \quad Y_{(22)} = (Y_{13}, Y_{23})
\]

Thus, the MLE of \( \Sigma \) is given by

\[
N\hat{\Sigma} = W + \left(\begin{array}{c}
Y_{(12)} - \theta \\
Y_{(22)}
\end{array}\right) (\cdot)'\]

The MLE of \( \theta \) is obtained by minimizing the determinant of \( N\hat{\Sigma} \). This gives

\[
\hat{\theta} = Y_{(12)} - W_{(13)}W_{33}^{-1}Y_{(22)}
\]

and the minimum value of the determinant of \( N\Sigma \) is

\[
|N\hat{\Sigma}| = |W||I + Y'_{(22)}W_{33}^{-1}Y_{(22)}|
\]

\[
= |W||W_{35}^{-1}||W_{33} + Y_{13}Y'_{13} + Y_{23}Y'_{23}|.
\]

Hence the likelihood ratio test statistic is based on the statistic

\[
\lambda_6 = \frac{|W_{33} + Y_{23}Y'_{23}|}{|W_{33}|}, \quad \frac{|W_{(22)}|}{|W_{(22)} + Y_{(22)}Y'_{(22)}}.
\]

To obtain the distribution of \( \lambda_6 \), we first note that the distribution of \( \lambda_6 \) does not depend on \( \Sigma \). Thus, without any loss of generality, we shall assume that \( \Sigma = I \).

Next, we make the following transformations.

\[
W_{(22)} + Y_{(2)}Y'_{(2)} = \tilde{K}\tilde{K}',
\]
and

\[ Y_{(2)} = \tilde{K}U \]

where \( \tilde{K} \) is an upper triangular matrix. Then \( J(W_{(2)}, Y_{(2)} \to \tilde{K}, U) = 2^p \Pi k_{ii}^{i+m} \).

The joint density of \( W_{(22)} \) and \( Y_{(2)} \) is given by

\[ C_{1}|W_{(22)}|^{\frac{n-p+q_1-1}{2}}etr - \frac{1}{2}(W_{(22)} + Y_{(2)}Y'_{(2)}) \]

Hence, the joint pdf of \( \tilde{K} \) and \( U \) is given by

\[ C_{2}p|\tilde{K}|^{(n-p+q_1-1)} \prod_{i=1}^{q_1} k_{ii}^{i+m} |I - UU'|^{\frac{n-p+q_1-1}{2}}etr - \frac{1}{2} \tilde{K} \tilde{K}' \]

Thus \( \tilde{K} \) and \( U \) are independently distributed. The pdf of \( U \) is given by

\[ C_{1}|I - UU'|^{\frac{n-p+q_1-1}{2}} \]

Writing

\[ \tilde{K} = \begin{pmatrix} \tilde{K} & \tilde{K}_{12} \\ 0 & \tilde{K}_2 \end{pmatrix} \text{ and } U = \begin{pmatrix} U_1 \\ p \end{pmatrix} \begin{pmatrix} U_2 \end{pmatrix} \]

we find that

\[ W_{33} + Y_{23}Y'_{23} = \tilde{K}_2 \tilde{K}'_2 \]

\[ W_{33} = \tilde{K}_2(I - U_2U_2')\tilde{K}'_2 \]

Hence

\[ \lambda_8 = \frac{|I - UU'|}{|I - U_2U_2'|} = \frac{|I - U_1'U_1 - U_2'U_2|}{|I - U_2'U_2|} \]
Let

\[ R'_1 = (I - U'_1 U_2) \frac{1}{2} U'_1 \]

Then

\[ \lambda_6 = |I - R'_1 R_1| \]

where the pdf of \( R_1 \) is given by

\[ \text{Const. } |I - R'_1 R_1| \frac{n - p + q - q_2 - 1}{2} \]

Hence \( \lambda_6 \) is distributed as \( U_{q_2}, m_2, n - p + q \)

7. Generalized Nested Models

With a slight change of notation, consider the GCM model in which

\[ E(X) = B_1 \xi A \]

\( B_1 : p \times q, \xi : q \times m, A : m \times N \); matrices \( A \) and \( B_1 \) are full rank. Consider the case when

\[ (H_5) \quad \xi = \begin{pmatrix} \xi_1 & \xi_4 & \xi_7 \\ \xi_2 & \xi_5 & \xi_8 \\ \xi_3 & \xi_6 & \xi_9 \end{pmatrix} \]

where \( \xi_6 = 0, \xi_8 = 0, \xi_9 = 0 \). Let

\[ \eta_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} \xi_4 \\ \xi_5 \end{pmatrix}, \quad \eta_3 = (\xi_7) \]
\[ B_1 = \begin{pmatrix} q_1 + q_2 & q_3 \\ B_2, & \hat{B} \end{pmatrix} = \begin{pmatrix} q_1 & q_2 & q_3 \\ B_3, & B_2, & \hat{B} \end{pmatrix}, \ A' = \begin{pmatrix} A_1', & A_2', & A_3' \end{pmatrix} \]
\[ q_1 + q_2 + q_3 = q \text{ and } m_1 + m_2 + m_3 = m. \]

Then
\[ B_1 \xi A = B_1 \eta_1 A_1 + B_2 \eta_2 A_2 + B_3 \eta_3 A_3 \]

where \( B_3 \subset B_2 \subset B_1 \).

This generalized nested model was considered by von Rosen (1989) who gave the MLE of the parameters but no testing problem was considered. In this paper, we obtain the MLE of the parameters by following the approach of Srivastava and Khatri (1979). To carry out testing of the hypothesis, however, we shall use the canonical reduction method.

### 7.1 Maximum Likelihood Estimate of \( \eta_1, \eta_2, \eta_3 \) and \( \Sigma \).

The MLE of \( \Sigma \) is given by

\[ N \hat{\Sigma} = (X - B_1 \hat{\eta} A_1 - B_2 \hat{\eta} A_2 - B_3 \hat{\eta} A_3)( \cdot)' \]

where \( \hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3 \) are those values of \( \eta_1, \eta_2 \) and \( \eta_3 \) that minimize the determinant

\[ d_5 = |(X - B_1 \eta_1 A_1 - B_2 \eta_2 A_2 - B_3 \eta_3 A_3)( \cdot)'| \]

We minimize \( d_5 \) first with respect to \( \eta_1 \) for fixed \( \eta_2 \) and \( \eta_3 \). From Srivastava and Khatri (1979), Theorem 1.10.3, we obtain

\[ B_1 \hat{\eta} A_1 = B_1 (B_1' S_{\eta_{23}}^{-1} B_1)^{-1} B_1' S_{\eta_{23}}^{-1} (X - B_2 \eta_2 A_2 - B_3 \eta_3 A_3) P_1, \]

where

\[ P_1 = A_1'(A_1 A_1')^{-1} A_1 \]

\[ S_{\eta_{23}} = (X - B_2 \eta_2 A_2 - B_3 \eta_3 A_3)( \cdot)' - S_{1\eta_{23}} \]
\[ S_{1\eta_2} = (X - B_2 \eta_2 A_2 - B_3 \eta_3 A_3) P_1(\ )'. \]

Let \( B_0 \) be a matrix such that \((B_1, B_0)\) is non-singular and \( B_0' B_1 = 0 \). Then at \( \eta_1 \)

\[
d_4 = \left| S_{\eta_2} + [I - B_1 (B_1' S^{-1}_{\eta_2} B_1)^{-1} B_1' S^{-1}_{\eta_2} S_{1\eta_2} B_1]' \right| \\
= \left| S_{\eta_2} ||I + S_{1\eta_2} [S^{-1}_{\eta_2} - S^{-1}_{\eta_2} B_1 (B_1' B_1' S^{-1}_{\eta_2} B_1)^{-1} B_1' S^{-1}_{\eta_2} B_1]|\right| \\
- \left| S_{\eta_2} ||I + S_{1\eta_2} B_0 (B_0' S^{-1}_{\eta_2} B_0)^{-1} B_0' \right| \\
= \left| S_{\eta_2} ||B_0' S^{-1}_{\eta_2} B_0|^{-1} |B_0' S^{-1}_{\eta_2} B_0 + B_0' S_{1\eta_2} B_0| \\
= \left| S_{\eta_2} ||B_0' X Q_1 X' B_0|^{-1} |B_0' X X' B_0| , \right|
\]

The last two expressions do not depend on \( \eta_2 \) and \( \eta_3 \).

Next we minimize

\[
|S_{\eta_2}| = |(X - B_2 \eta_2 A_2 - B_3 \eta_3 A_3) Q_1(\ )'| , \quad Q_1 = I - P_1 \\
= |(\tilde{X} - B_2 \eta_2 \tilde{A}_2 - B_3 \eta_3 \tilde{A}_3)(\ )'| ,
\]

where

\[
\tilde{X} = X Q_1, \quad \tilde{A}_2 = A_2 Q_1, \quad \tilde{A}_3 = A_3 Q_1
\]

Again, using Theorem 1.10.3 from Srivastava and Khatri (1979), we get

\[
B_2 \eta_2 \tilde{A}_2 = B_2 (B_2' S_{\eta_2} B_2)^{-1} B_2' S_{\eta_2}^{-1} (\tilde{X} - B_3 \eta_3 \tilde{A}_3) \tilde{P}_2 ,
\]

where

\[
\tilde{P}_2 = \tilde{A}_2' (\tilde{A}_2 \tilde{A}_2')^{-1} \tilde{A}_2 , \quad \tilde{Q}_2 = I - \tilde{P}_2 , \\
S_{\eta_3} = (\tilde{X} - B_3 \eta_3 \tilde{A}_3)(\ )' S_{1\eta_3} , \\
S_{1\eta_3} = (\tilde{X} - B_3 \eta_3 \tilde{A}_3) \tilde{P}_2(\ )' .
\]
Hence, at $\eta_2$

\[
|S_{\eta_3}| = |S_{\eta_3} + [I - B_2(B_2'S_{\eta_3}B_2)^{-1}B_2'S_{\eta_3}^{-1}]S_{1m}[^{\prime}[^{\prime}]]|
\]
\[
= |S_{\eta_3}||I + S_{1m}([I - B_2(B_2'S_{\eta_3}B_2)^{-1}B_2'S_{\eta_3}^{-1}]S_{\eta_3}[^{\prime}[^{\prime}]]|
\]
\[
= |S_{\eta_3}||I + S_{1m}B_0(B_0'S_{\eta_3}B_0)^{-1}B_0[^{\prime}[^{\prime}]]|
\]
\[
= |S_{\eta_3}||B_0'S_{\eta_3}B_0[^{\prime}[^{\prime}]]|B_0'(S_{\eta_3} + S_{1m})B_0[^{\prime}[^{\prime}]],
\]

where $(B_2, B_0)$ is non singular and $B_0'B_2 = 0$.

Now to minimize,

\[
|S_{\eta_3}| = |(\tilde{X} - B_3\eta_3\tilde{A}_3)\tilde{Q}_2(\tilde{\gamma}')|
\]
\[
= |(\tilde{X} - B_3\eta_3\tilde{A}_3)(\tilde{\gamma}')|
\]

where $\tilde{A}_3 = \tilde{A}_3\tilde{Q}_2$ and $\tilde{X} = \tilde{X}\tilde{Q}_2$, we use Theorem 1.10.3 of Srivastava and Khatri (1979). This gives

\[
B_3\tilde{\eta}_3\tilde{A}_3 = B_3(B_2'b_{\eta_3}^{-1}B_3)^{-1}B_2'b_{\eta_3}^{-1}\tilde{X}\tilde{P}_3
\]
\[
\tilde{P}_3 = \tilde{A}_3'(\tilde{A}_3\tilde{A}_3')^{-1}\tilde{A}_3
\]
\[
\tilde{s} = \tilde{X}[I - \tilde{P}_3]\tilde{X}'.
\]

Thus, we get the MLE of $\eta_1$, $\eta_2$, and $\eta_3$. These results are stated in the following Theorem. The MLE of $\eta_3$, $\eta_2$ and $\eta_1$ are respectively given by

1. $B_3\tilde{\eta}_3\tilde{A}_3 = B_3(B_2'b_{\eta_3}^{-1}B_3)^{-1}B_2'b_{\eta_3}^{-1}\tilde{X}\tilde{P}_3$
2. $B_2\tilde{\eta}_2\tilde{A}_2 = B_2(B_2'b_{\eta_3}^{-1}B_2)^{-1}B_2'b_{\eta_3}^{-1}(\tilde{X} - B_3\tilde{\eta}_3\tilde{A}_3)\tilde{P}_2$
3. $B_1\tilde{\eta}_1A_1 = B_1(B_1'S_{\eta_2}^{-1}B_1)^{-1}B_1'S_{\eta_2}^{-1}(\tilde{X} - B_2\tilde{\eta}_2A_2 - B_3\tilde{\eta}_3A_3)P_1$.
7.2 Canonical Reduction of the Model

Consider the growth curve model

\[ E(X) = B_1 \xi A, \]

where \( B_1 : p \times q, \) \( \xi : q \times m, \) \( A : m \times N. \) We shall assume that all matrices \( B_1 \) and \( A \) are of full rank.

Suppose

\[
\xi = \begin{bmatrix} m_1 & m_2 & m_3 \\ \eta_1 & \begin{pmatrix} \eta_2 \\ 0 \end{pmatrix} & \begin{pmatrix} \eta_3 \\ 0 \end{pmatrix} \\ q_1 & q_2 & q_3 \end{bmatrix}
\]

\( q_1 + q_2 + q_3 = q, \) \( m_1 + m_2 + m_3 = m, \) \( \eta_2 \) is a \((q_1 + q_2) \times m_2\) matrix and \( \eta_3 \) is a \(q \times m_3\) matrix. We can write

\[ B_1 = L_1 P \]

where \( P \) is an upper triangular matrix and \( L_1 L_1' = I_q. \) Thus, if we write

\[
P = \begin{pmatrix} P_1 & P_{12} & P_{13} \\ 0 & P_{22} & P_{23} \\ 0 & 0 & P_{33} \end{pmatrix} = \begin{pmatrix} P_2 & P_{(13)} \\ 0 & P_{33} \end{pmatrix},
\]

then

\[
B_1 \xi = L_1' \begin{bmatrix} P \eta_1, & \begin{pmatrix} P_2 \eta_2 \\ 0 \end{pmatrix}, & \begin{pmatrix} P_1 \eta_3 \\ 0 \end{pmatrix} \\ \delta_1, & \begin{pmatrix} \delta_2 \\ 0 \end{pmatrix}, & \begin{pmatrix} \delta_3 \\ 0 \end{pmatrix} \end{bmatrix}
\]

\[ \equiv L_1' \begin{bmatrix} \delta_1, & \begin{pmatrix} \delta_2 \\ 0 \end{pmatrix}, & \begin{pmatrix} \delta_3 \\ 0 \end{pmatrix} \end{bmatrix} \]
Define orthogonal matrices $L' = (L'_1, L'_4)$ and $G = (A'T^{-1'}, G_0)$ where $AA' = TT'$ and $T$ is a lower triangular matrix. Then for

$$Z = LXG = \begin{bmatrix}
q_1 & m_1 & m_2 & m_3 \\
q_2 & Y_{11} & Y_{21} & Y_{31} \\
q_3 & Y_{12} & Y_{22} & Y_{32} \\
p-q & Y_{13} & Y_{23} & Y_{33} \\
p-q & Y_{14} & Y_{24} & Y_{34}
\end{bmatrix}, \quad q_1 + q_2 + q_3 = q,$$

and $m_1 + m_2 + m_3 = m$, we get

$$E(Z) = LB_1\xi AG$$
$$= LL'_1 P\xi AG$$
$$= \begin{bmatrix} I_q \\ 0 \end{bmatrix} P\xi(T, 0)$$
$$\equiv \begin{bmatrix}
0 & m_1 & m_2 & m_3 \\
\theta_1 & q & q_1 + q_2 & q_1 \\
0 & q_2 & 0 & q_2 \\
0 & q_3 & 0 & q_3 \\
p-q & 0 & p-q & 0
\end{bmatrix},$$

where $\theta_1 : q \times m_1$, $\theta_2 : (q_1 + q_2) \times m_2$ and $\theta_3 : q_1 \times m_3$.

Let

$$W = VV' = \begin{bmatrix}
W_{11} & W_{12} & W_{13} & W_{14} \\
W_{22} & W_{23} & W_{24} \\
W_{33} & W_{34} \\
W_{44}
\end{bmatrix} = \begin{bmatrix} \theta_1 & q & p-q & q_1 + q_2 \\
p-q & q_2 & p-q & q_2 - q_3 \\
\theta_3 & q_3 & 0 & p-q \\
W'_{14} & W_{14} & W_{44}
\end{bmatrix} = \begin{bmatrix} W_{11} & W_{14} \\
W'_{14} & W_{44}
\end{bmatrix} = \begin{bmatrix} W_{(11)} & W_{(14)} \\
W'_{(14)} & W_{(44)}
\end{bmatrix} = \begin{bmatrix} W_{(22)} & W_{(12)} \\
W'_{(12)} & W_{(44)}
\end{bmatrix}.$$
\[
\begin{pmatrix}
q_1 & p-q_1 \\
W_{11} & W_{(13)} \\
W'_{(13)} & W_{(44)}
\end{pmatrix}
\]

\[
Y_{(1)} = (Y_{11}', Y_{12}', Y_{13}'), \ Y_{(21)} = (Y_{21}', Y_{22}'), \ Y_{(23)} = (Y_{23}', Y_{24}'), \ Y_{(32)} = (Y_{32}', Y_{33}', Y_{34}').
\]

The maximum likelihood estimate of \(\Sigma\) is given by

\[
N\hat{\Sigma} = W + \begin{pmatrix} Y_1 - \hat{\theta}_1 \\ Y_{14} \end{pmatrix} (') + \begin{pmatrix} Y_{21} - \hat{\theta}_2 \\ Y_{23} \end{pmatrix} (') + \begin{pmatrix} Y_{31} - \hat{\theta}_3 \\ Y_{32} \end{pmatrix} (')
\]

where \(\hat{\theta}_1, \hat{\theta}_2\) and \(\hat{\theta}_3\) are those values of \(\theta_1, \theta_2\) and \(\theta_3\) that minimizes the determinant of \(N\hat{\Sigma}\). Writing

\[
Q = W + \begin{pmatrix} Y_{(21)} - \theta_2 \\ Y_{(23)} \end{pmatrix} (') + \begin{pmatrix} Y_{31} - \theta_3 \\ Y_{(32)} \end{pmatrix} (')
\]

this is equivalent to minimizing

\[
d_5 = |Q| |I + (Y_{(1)}' - \theta_1', Y_{14}')Q^{-1}(Y_{11}' - \theta_1', Y_{(1)})|
\geq |Q| |I + Y_{14}'Q_{44}Y_{14}|,
\]

the equality holds at

\[
\hat{\theta}_1 = Y_{(1)} - Q_{(14)}Q_{44}^{-1}Y_{14} ,
\]

where

\[
Q_{44} = W_{44} + Y_{24}'Y_{24} + Y_{34}'Y_{34}',
\]

and

\[
Q = \begin{pmatrix} Q_{(11)} & Q_{(14)} \\ Q_{(14)}' & Q_{(44)} \end{pmatrix}.
\]
Letting

\[
\dot{Q} = W + \begin{pmatrix} Y_{31} - \theta_3 \\ Y_{32} \end{pmatrix} (\gamma)'
= \begin{pmatrix} Q_{(22)} & \dot{Q}_{(12)} \\ \dot{Q}_{(12)}' & Q_{(44)} \end{pmatrix}
\]

we get

\[
\dot{Q}_{(44)} = W_{(44)} + \begin{pmatrix} Y_{33} \\ Y_{34} \end{pmatrix} (\gamma)'
\]

and

\[|Q| \geq |\dot{Q}| |I + Y_{(23)}'\dot{Q}_{(44)}^{-1} Y_{(23)}|,
\]

the equality holding at

\[
\dot{\theta}_2 = Y_{(21)} = \dot{Q}_{(12)} Q_{(44)}^{-1} Y_{(23)}
\]

Similarly,

\[|\dot{Q}| \geq |W| |I + Y_{(32)} W_{(33)}^{-1} Y_{(32)}|
\]

and the equality holding at

\[
\dot{\theta}_3 = Y_{31} = W_{(13)} W_{(33)}^{-1} Y_{(32)}.
\]

Thus, the MLE of \(\theta_1\) and \(\theta_2\) can be obtained in succession. Thus the minimum value of the determinant of \(\hat{N}\Sigma\) under the model is given by

\[
|\hat{N}\hat{\Sigma}| = |I + Y_{14}' Q_{44}^{-1} Y_{14}||I + Y_{(23)}' \dot{Q}_{(44)}^{-1} Y_{(23)}||I + Y_{(32)}' W_{(33)}^{-1} Y_{(32)}||W|
= |W||Q_{44} + Y_{14} Y_{14}'||\dot{Q}_{(44)} + Y_{(23)} Y_{(23)}'||W_{(33)} + Y_{(32)} Y_{(32)}'||W||
\]

\[
= |W| |Q_{44} + Y_{24} Y_{24}' + Y_{34} Y_{34}' + Y_{14} Y_{14}'| \\
\]

\[
|Q_{44} + Y_{24} Y_{24}' + Y_{34} Y_{34}'| \\
\]

\[
= |W| \left| W_{44} + Y_{24} Y_{24}' + Y_{34} Y_{34}' \right| \times
\]

\
\[
\frac{|W_{(44)} + \begin{pmatrix} Y_{33} \\ Y_{34} \end{pmatrix} (\cdot)' + Y_{(23)}Y_{(23)\prime}|}{|W_{(44)} + \begin{pmatrix} Y_{33} \\ Y_{34} \end{pmatrix} (\cdot)'|} \times \frac{|W_{(33)} + Y_{(32)}Y_{(32)\prime}|}{|W_{(33)}|}.
\]

Suppose, under the alternative model

\[
E(Z) = \begin{pmatrix}
\begin{bmatrix}
m_1 + m_2 \\
p - q
\end{bmatrix}
& m_3 \\
q_1 & \begin{bmatrix}
\delta \\
0
\end{bmatrix}
\end{bmatrix},
\begin{bmatrix}
q_1 & \begin{bmatrix}
\delta_1 \\
0
\end{bmatrix}
\end{bmatrix},
\begin{bmatrix}
0 \\
p - q
\end{bmatrix}
\end{pmatrix},
\]

that, it is a sum of only two profiles (instead of three profiles in the hypothesis).

Then, if \( \hat{\Sigma}_A \) denotes the estimate of \( \Sigma \) under the alternative,

\[
[N\hat{\Sigma}_A] = \frac{|W_{(33)} + Y_{(32)}Y_{(32)\prime}|}{|W_{(33)}|}, \frac{|W_{44} + Y_{24}Y_{24}' + Y_{34}Y_{34}' + Y_{44}Y_{44}|}{|W_{44} + Y_{24}Y_{24}' + Y_{34}Y_{34}'|}.
\]

Hence, the likelihood ratio test is based on the statistic

\[
\lambda_7 = \frac{|W_{(44)} + \begin{pmatrix} Y_{33} \\ Y_{34} \end{pmatrix} (\cdot)'|}{|W_{(44)} + \begin{pmatrix} Y_{33} \\ Y_{34} \end{pmatrix} (\cdot)'|} \times \frac{|Y_{33}' , Y_{34}' + Y_{(23)}Y_{(23)\prime}|}{U_{p - q_1 - q_2, m_2, n + m_3}}.
\]
REFERENCES


