Testing Equality of Means and Simultaneous Confidence Intervals in Repeated Measures with Missing Data

by

Takashi Seo*
Department of Statistics
University of Toronto

and

M.S. Srivastava
Department of Statistics
University of Toronto

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Department of Statistics

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Takashi Seo* and M. S. Srivastava

Department of Statistics
University of Toronto
100 St George Street, Toronto
Ontario M5S 3G3, Canada

ABSTRACT

In this paper, repeated measures with intraclass correlation model is considered when the observations are missing at random. An exact test for the equality of the mean components is presented. Simultaneous confidence intervals of Scheffé's type as well as the ones based on Bonferroni inequality are given for linear contrasts of the mean components when the missing observations are of monotone type. Two examples are given to illustrate the method.

KEY WORDS: Contrasts; Intraclass Correlation Model; Missing Observation; Scheffé's Confidence Intervals; Testing Equality of Means

*On leave from Department of Information Sciences, Faculty of Science & Technology, Science University of Tokyo, Noda City, Chiba 278, Japan.

1. INTRODUCTION

Consider the two-components mixed linear model in which the random variable $x_{ij} = \mu_j + \alpha_i + \varepsilon_{ij}$, $i = 1, \cdots, n_j$, $j = 1, \cdots p$, where α_i and ε_{ij} are independently normally distributed, α_i's are i.i.d. $N(0, \sigma_\alpha^2)$ and ε_{ij}'s are i.i.d. $N(0, \sigma_\varepsilon^2)$. Thus, the means of x_{ij} is $\mathrm{E}[x_{ij}] = \mu_j$, $j = 1, \cdots, p$. Since $\mathrm{Var}(x_{ij}) = \sigma_\alpha^2 + \sigma_\varepsilon^2$, $\mathrm{Cov}(x_{ij}, x_{ij'}) = \sigma_\alpha^2$, $j \neq j'$ and $\mathrm{Cov}(x_{ij}, x_{i'j}) = 0, i \neq i'$, it follows that if $n_i \equiv n$ and if we define $\boldsymbol{x}_i = (x_{i1}, \cdots, x_{ip})'$, $i = 1, \cdots, n$, then $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n$ are i.i.d. $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_p)'$, $\boldsymbol{\Sigma} = \sigma^2[(1-\rho)\boldsymbol{I} + \rho\boldsymbol{1}\boldsymbol{1}']$, $\sigma^2 = \sigma_\alpha^2 + \sigma_\varepsilon^2$, $0 \leq \rho = \sigma_\alpha^2/\sigma^2 \leq 1$, $\boldsymbol{I}$ is a $p \times p$ identity matrix and $\boldsymbol{1}$ is a p-vector of ones, $\boldsymbol{1}' = (1, \cdots, 1)$. When the covariance matrix $\boldsymbol{\Sigma}$ is of the above structure, it is called an intraclass correlation model where ρ may lie between $[-(1/p-1),\ 1]$.

Simultaneous confidence intervals of Scheffé and Tukey type for linear contrasts $\boldsymbol{a}'\boldsymbol{\mu}$ for all non-null vector $\boldsymbol{a}$ such that $\boldsymbol{a}'\boldsymbol{1} = 0$, when ρ is unknown in the intraclass correlation model has been given by Bhargava and Srivastava [3]. When ρ is known, this problem was considered by Scheffé [9] and Miller [7]. However, when n_i's are not all equal, this problem has remained open, see Littell, Milliken, Stroup and Wolfinger [6] and Kleinbaum, Kupper, Muller, and Nizam [4].

In this paper, we shall assume that the observations are missing at random and the covariance matrix is of the intraclass correlation form. We first consider the case when the missing observations are of the monotone-type, considered by Rao [8], Anderson [1] and Bhargava [2] among others. Thus, our observations $\{x_{ij}\}$ can be written, without any loss of generality, in the following form:

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & \cdots & x_{1p} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & x_{n_p p} \\ x_{n_2 1} & x_{n_2 2} & & & \\ \vdots & & & & \\ x_{n_1 1} & & & & \end{bmatrix},$$

where $n_1 \geq n_2 \geq \cdots \geq n_p$. Since our interest is in the contrasts of the means, the observations $x_{n_2+1\,1}, \cdots, x_{n_1 1}$ do not provide any information and may be deleted. Thus, we shall consider the case when $n_1 = n_2$, omitting the observations $x_{n_2+1\,1}, \cdots, x_{n_1 1}$ if required. We consider the problem of testing the hypothesis $H : \mu_1 = \mu_2 = \cdots = \mu_p$ against the alternative $A \neq H$. We also provide simultaneous confidence intervals for contrasts $\boldsymbol{a}'\boldsymbol{\mu}$ for all non-null vector $\boldsymbol{a}$ such that $\boldsymbol{a}'\mathbf{1} = 0$.

Finally, we consider the general case of missing observations, that is, when the missing observations are not necessarily of monotone type. However, in this case we provide only an exact test procedure but no confidence intervals. The organization of the paper is as follows.

In Section 2, we consider the case when the missing observations are of the monotone type and provide an exact test procedure for testing the equality of mean components. In Section 3, Bonferroni type and Scheffé's type of simultaneous confidence intervals for linear contrasts of the means are given. The general case of missing observations is considered in Section 4. The paper concludes in Section 5 with two examples.

2. TESTING THE EQUALITY OF MEAN COMPONENTS

In this section, we provide an exact test for testing the hypothesis $H : \mu_1 = \mu_2 = \cdots = \mu_p$ against the alternative $A \neq H$. To provide a test or simultaneous confidence intervals, we rewrite the observations in the following form:

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & \cdots & \cdots & x_{1p_1} \\ x_{21} & x_{22} & \cdots & \cdots & x_{2p_2} & \\ \vdots & \vdots & \cdots & & & \\ x_{n1} & \cdots & x_{np_n} & & & \end{bmatrix},$$

where $n = n_1 = n_2$ and $p = p_1 \geq p_2 \geq \cdots \geq p_n$. Since $n_1 = n_2$, all p_i's are greater than or equal to 2. Writing $\boldsymbol{x}_i = (x_{i1}, \cdots, x_{ip_i})'$, we find that $\boldsymbol{x}_i$'s are independently distributed as $N_{p_i}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$, $i = 1, \cdots n$, where $\boldsymbol{\mu}_i = (\mu_1, \cdots, \mu_{p_i})'$ and $\boldsymbol{\Sigma}_i = \sigma^2[(1-\rho)\boldsymbol{I}_{p_i} + \rho \mathbf{1}_i \mathbf{1}_i']$. $\boldsymbol{I}_{p_i}$ is a $p_i \times p_i$ identity matrix and $\mathbf{1}_i$ is a p_i-vector of ones, $\mathbf{1}_i' = (1, \cdots, 1) : 1 \times p_i$.

Let $\boldsymbol{C}_i$ be a $(p_i - 1) \times p_i$ matrix such that $\boldsymbol{C}_i \mathbf{1} = \mathbf{0}$ and $\boldsymbol{C}_i \boldsymbol{C}_i' = \boldsymbol{I}_{p_i - 1}$. Then, as in Srivastava [10], consider the transformation $\boldsymbol{y}_i = \boldsymbol{C}_i \boldsymbol{x}_i$. Clearly, then

$$\boldsymbol{y}_i = \boldsymbol{C}_i \boldsymbol{x}_i \sim N_{p_i - 1}(\boldsymbol{C}_i \boldsymbol{\mu}_i, \gamma^2 \boldsymbol{I}_{p_i - 1}),$$

where $\gamma^2 = \sigma^2(1-\rho)$. For illustrative purpose we shall assume that

$$\boldsymbol{C}_i = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{\sqrt{p_i(p_i-1)}} & \cdots & \cdots & \cdots & -\frac{1}{\sqrt{p_i(p_i-1)}} & \frac{p_i - 1}{\sqrt{p_i(p_i-1)}} \end{bmatrix},$$

where $i = 1, \cdots n$. We shall write $\boldsymbol{C}$: $(p-1) \times p$ for $\boldsymbol{C}_1$, since $p_1 \equiv p$.

The hypothesis H of testing the equality of means, $\mu_1 = \mu_2 = \cdots = \mu_p$ is equivalent to the hypothesis

$$H : \boldsymbol{C}_i \boldsymbol{\mu}_i = \mathbf{0}, \quad i = 1, \cdots, n.$$

Hence, under H, $\boldsymbol{y}_i$'s are i.i.d. $N_{p_i-1}(\mathbf{0}, \gamma^2 \boldsymbol{I}_{p_i - 1})$, $i = 1, \cdots, n$. Thus under H, y_{ij}'s are i.i.d. $N(0, \gamma^2)$, $i = 1, \cdots, n_{j+1}$, $j = 1, \cdots, p-1$, where $n_2 \equiv n$. Now, we have the following structure of the transformed observations:

$$\begin{bmatrix} y_{11} & y_{12} & & y_{1p-1} \\ y_{21} & y_{22} & \vdots & \vdots \\ \vdots & \vdots & & y_{n_p p-1} \\ \vdots & y_{n_3 2} & & \\ y_{n_2 1} & & & \end{bmatrix}.$$

Define

$$\bar{y}_{\cdot j} = \frac{1}{n_{j+1}} \sum_{i=1}^{n_{j+1}} y_{ij},$$

$$f\hat{\gamma}^2 = \sum_{j=1}^{p-1} \sum_{i=1}^{n_{j+1}} (y_{ij} - \bar{y}_{\cdot j})^2,$$

$$f = \sum_{j=1}^{p-1} n_{j+1} - (p-1).$$

Then under the hypothesis $H : \mu_1 = \mu_2 = \cdots = \mu_p$, $\sum_{j=1}^{p-1} \bar{y}^2_{\cdot j}$ is distributed as $\gamma^2 \chi^2_{p-1}$ and is independently distributed of $f\hat{\gamma}^2$ which is distributed as $\gamma^2 \chi^2_f$. Thus, a test for the hypothesis $H : \mu_1 = \mu_2 = \cdots = \mu_p$ against the alternative $A \neq H$ is based on the statistic

$$\frac{\sum_{j=1}^{p-1} n_{j+1} \bar{y}^2_{\cdot j}/(p-1)}{\hat{\gamma}^2}$$

which has an F distribution with $(p-1)$ and f degrees of freedom. Large values of this statistic lead to the rejection of the hypothesis H. When $n_1 = n_2 = \cdots = n_p$, this test statistic reduces to the usual test statistic given, for example in Srivastava and Carter [11], p.199.

3. SIMULTANEOUS CONFIDENCE INTERVALS FOR LINEAR CONTRASTS

Let

$$\bar{\boldsymbol{y}} = (\bar{y}_{\cdot 1}, \cdots, \bar{y}_{\cdot p-1})'.$$

Then

$$\mathrm{E}(\bar{\boldsymbol{y}}) = \boldsymbol{C\mu},$$

where $\boldsymbol{CC}' = \boldsymbol{I}_{p-1}$ and $\boldsymbol{C1} = \boldsymbol{0}$, $\boldsymbol{1}$ is a p-vector of ones, and

$$\mathrm{Cov}(\bar{\boldsymbol{y}}) = \gamma^2 \begin{bmatrix} n_2^{-1} & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & n_p^{-1} \end{bmatrix} \equiv \gamma^2 \boldsymbol{V}.$$

We wish to find simultaneous confidence intervals of $\boldsymbol{a}'\boldsymbol{\mu}$ for all non-null vector $\boldsymbol{a}$ such that $\boldsymbol{a}'\boldsymbol{1} = 0$. Since $\boldsymbol{a}'\boldsymbol{1} = 0$, we can write it as $\boldsymbol{a}' = \boldsymbol{b}'\boldsymbol{C}$ for some $\boldsymbol{b} : (p-1) \times 1$,

$\boldsymbol{C}\mathbf{1} = \mathbf{0}$. Since, under the hypothesis H, $\boldsymbol{C}\boldsymbol{\mu} = \mathbf{0}$, we also have $\boldsymbol{a}'\boldsymbol{\mu} = 0$ for all non-null p-vector $\boldsymbol{a}$ satisfying the above conditions. Now, we have

$$\boldsymbol{b}'\overline{\boldsymbol{y}} \sim N(\boldsymbol{b}'\boldsymbol{C}\boldsymbol{\mu},\ \gamma^2\boldsymbol{b}'\boldsymbol{V}\boldsymbol{b}).$$

We also note that the set of vector $\boldsymbol{a}$ such that $\boldsymbol{a}'\mathbf{1} = 0$ is equivalent to the set of vector $\boldsymbol{a}$ such that $\boldsymbol{a}' = \boldsymbol{b}'\boldsymbol{C}$, where $\boldsymbol{b}$ is a $(p-1)$ vector and $\boldsymbol{C}$ is a $(p-1)\times p$ matrix such that $\boldsymbol{C}\mathbf{1} = \mathbf{0}$ and $\boldsymbol{C}\boldsymbol{C}' = \boldsymbol{I}_{p-1}$. Hence $(1-\alpha)\times 100\%$ Scheffé's simultaneous confidence intervals for $\boldsymbol{a}'\boldsymbol{\mu}$ for all non-null vector $\boldsymbol{a}$ such that $\boldsymbol{a}'\mathbf{1} = 0$ are given by

$$\boldsymbol{b}'\overline{\boldsymbol{y}} \pm [(p-1)F_{p-1,f,\alpha}]^{\frac{1}{2}}\,\widehat{\gamma}\left(\sum_{j=1}^{p-1}\frac{b_j^2}{n_{j+1}}\right)^{\frac{1}{2}},$$

where $F_{p-1,f,\alpha}$ is the upper $100\alpha\%$ point of an F-distribution with $(p-1)$ and f degrees of freedom.

We can also use Bonferroni inequality to obtain simultaneous confidence intervals for ℓ linear contrasts $\boldsymbol{a}_1'\boldsymbol{\mu}, \cdots, \boldsymbol{a}_\ell'\boldsymbol{\mu}$, where $\boldsymbol{a}_i' = \boldsymbol{b}_i'\boldsymbol{C}$, $i = 1, \cdots, \ell$. This is given by

$$\boldsymbol{b}_i'\overline{\boldsymbol{y}} \pm t_{f,\frac{\alpha}{2\ell}}\widehat{\gamma}\left(\sum_{j=1}^{p-1}\frac{b_j^{(i)2}}{n_{j+1}}\right)^{\frac{1}{2}}, \quad i = 1, \cdots, \ell,$$

where $\boldsymbol{b}_i = (b_1^{(i)}, \cdots, b_{p-1}^{(i)})'$. The simultaneous confidence intervals by Bonferroni inequality should be used only if

$$(p-1)F_{p-1,f,\alpha} \geq t^2_{f,\frac{\alpha}{2\ell}}\ ;$$

otherwise Scheffé's simultaneous confidence intervals should be used. According to the tables computed by Dunn [5], it holds that $(p-1)F_{p-1,f,\alpha} < t^2_{f,\frac{\alpha}{2\ell}}$ if ℓ is considerably bigger than $p-1$(see, Miller [7], p.69).

4. MISSING OBSERVATIONS: GENERAL CASE

In this section, we shall not assume any pattern for the missing observations. However, we shall assume that the observations are missing at random and that observations are available on at least two components of the random vector; if there are random vectors with observations on only one component, they will be deleted from consideration.

The purpose in this section is to propose an exact test for the equality of mean components. To calculate $\{y_{ij}\}$ in this general case, we can obtain it by the transformation of the observations except for the missing components. As an example which is described in Section 5 later, suppose we have the observations $\boldsymbol{x}_i = (x_{i1}, x_{i2}, *, x_{i4}, *, x_{i6}, x_{i7})'$ for the ith subject. Then, since x_{ij}'s are assumed to have the intraclass correlation covariance structure, $\tilde{\boldsymbol{x}}_i \equiv (x_{i1}, x_{i2}, x_{i4}, x_{i6}, x_{i7})'$, which is obtained by deleting missing observations, are distributed as $N_5(\tilde{\boldsymbol{\mu}}, \boldsymbol{\Sigma})$, where $\tilde{\boldsymbol{\mu}} = (\mu_1, \mu_2, \mu_4, \mu_6, \mu_7)'$ and $\boldsymbol{\Sigma} = \sigma^2[(1-\rho)\boldsymbol{I}_5 + \rho\boldsymbol{1}\boldsymbol{1}']$. Hence, after the transformation by $\boldsymbol{C}$: 4×5 in section 2 such that

$$\tilde{\boldsymbol{y}}_i \equiv (\tilde{y}_{i1}, \tilde{y}_{i2}, \tilde{y}_{i3}, \tilde{y}_{i4})' = \boldsymbol{C}\tilde{\boldsymbol{x}}_i,$$

we can obtain $\boldsymbol{y}_i$ given by

$$\boldsymbol{y}_i = (\tilde{y}_{i1}, *, \tilde{y}_{i2}, *, \tilde{y}_{i3}, \tilde{y}_{i4})' \equiv (y_{i1}, 0, y_{i3}, 0, y_{i5}, y_{i6})'$$

The test statistic in this case can be given by

$$\frac{\sum_{j=1}^{p-1} n_{j+1}^* \overline{y}_{\cdot j}^2/(p-1)}{\hat{\gamma}^2},$$

where

$$\overline{y}_{\cdot j} = \frac{1}{n_{j+1}^*}\sum_{i=1}^{n_{j+1}} y_{ij},$$

$$f\hat{\gamma}^2 = \sum_{j=1}^{p-1}\sum_{i=1}^{n_{j+1}^*}(y_{ij} - \overline{y}_{\cdot j})^2,$$

$$f = \sum_{j=1}^{p-1} n_{j+1}^* - (p-1),$$

and n_{j+1}^* is the total number of observations after excluding the missing observations for $(j+1)$th component of all subjects in $\{x_{ij}\}$. In this calculation, we note that the values of missing observations are zero.

As an alternative, we can obtain an exact test statistic, which appear simpler than the above scheme, as follows.

Since, under the hypothesis the components in each random vectors are exchangeable in the sense of distribution, we shall ignore the location of the missing values. For example, the random vectors $(x_{i1},\ *, x_{i\ p-1}, \cdots, x_{ip})$ and $(x_{j1}, x_{j2}, \cdots, x_{j\ p-1}, *)$ where the observation is missing on the second component of the ith observation and on the pth component of the jth observation, $j \neq i$, will all be considered as $(p-1)$ vectors. In this general case, we shall divide the data into $(p-1)$ groups. The first group will consist of the data where observations on all the components are available. The second group will consist of random vectors where observations on only one component is missing, the third group will have observations missing on two components, ,..., and last group will have observations on two components. The number of observations is each group will be denoted by $m_p, m_{p-1}, \cdots, m_2$.

These variables are then transformed to y_{ij} as in the previous section. We can write these transformed observations in the following form.

$$
\begin{array}{c}
\\ m_p \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\
m_{p-1} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\
\vdots \\
m_2 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right.
\end{array}
\left[\begin{array}{ccccc}
y_{11} & \cdots & \cdots & \cdots & y_{1\ p-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
y_{m_p 1} & \cdots & \cdots & \cdots & y_{m_p\ p-1} \\
y_{m_{p+1}\ 1} & \cdots & \cdots & y_{m_{p+1}\ p-2} & * \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
y_{m_p+m_{p+1}\ 1} & \cdots & \cdots & y_{m_p+m_{p+1}\ p-2} & * \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
y_{m^*\ 1} & * & \cdots & * & * \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
y_{m^{**}\ 1} & * & \cdots & * & *
\end{array} \right],
$$

where $m^* = (\sum_{j=3}^{p} m_j) + 1$ and $m^{**} = \sum_{j=2}^{p} m_j$. Then $\overline{y}_{\cdot 1}$ is the mean of $m_2 + \cdots + m_p (\equiv \tilde{n}_2)$ observations, while $\overline{y}_{\cdot p-1}$ is the mean of $m_p (\equiv \tilde{n}_p)$ observations. The test statistic in this case can be given by

$$\frac{\sum_{j=1}^{p-1} \widetilde{n}_{j+1}\overline{y}_{\cdot j}^2/(p-1)}{\widehat{\gamma}^2},$$

where

$$\overline{y}_{\cdot j} = \frac{1}{\widetilde{n}_{j+1}} \sum_{i=1}^{\widetilde{n}_{j+1}} y_{ij},$$

$$f\widehat{\gamma}^2 = \sum_{j=1}^{p-1} \sum_{i=1}^{\widetilde{n}_{j+1}} (y_{ij} - \overline{y}_{\cdot j})^2,$$

$$f = \sum_{j=1}^{p-1} \widetilde{n}_{j+1} - (p-1).$$

Thus, large values of this statistic which has an F distribution with $(p-1)$ and f degrees of freedom lead to the rejection of the hypothesis H. Therefore, we can obtain an exact test for the equality of mean components in the missing data at random.

It may be noted that the above procedures are not unique and there are various ways in which i.i.d. observations, under the hypothesis, can be created.

5. EXAMPLES

In this section, we shall discuss two examples to illustrate the method developed in this paper. In the first example, we shall treat three kinds of data, that is, a complete data set taken from SAS weightlifting data, a monotone type missing data and a general missing data. The last two data sets are modified data set from complete SAS weightlifting data, where observations have been dropped with probability 0.3. A second example considers a real data with missing observations.

5.1. A Complete Data

This repeated measures data is a complete data from the SAS Data Set 3.2(a) named WIGHTS: Strength Data Set (see, Littell, et al.[6], p.557). This data set is strengths of the subjects which were measured every other day for two weeks in a weightlifting program(WI) and is given in Table 1.

Table 1. Complete Data WI

Subject #	p							
1	7	84	85	84	83	83	83	84
2	7	74	75	75	76	75	76	76
3	7	83	84	82	81	83	83	82
4	7	86	87	87	87	87	87	86
5	7	82	83	84	85	84	85	86
6	7	79	80	79	79	80	79	80
7	7	79	79	79	81	81	83	83
8	7	87	89	91	90	91	92	92
9	7	81	81	81	82	82	83	83
10	7	82	82	82	84	86	85	87
11	7	79	79	80	81	81	81	81
12	7	79	80	81	82	83	82	82
13	7	83	84	84	84	84	83	83
14	7	81	81	82	84	83	82	85
15	7	78	78	79	79	78	79	79
16	7	83	82	82	84	84	83	84
17	7	80	79	79	81	80	80	80
18	7	80	82	82	82	81	81	81
19	7	85	86	87	86	86	86	86
20	7	77	78	80	81	82	82	82
21	7	80	81	80	81	81	82	83
Sample Means		81.05	81.67	81.90	82.52	82.62	82.71	83.10

Source: SAS Data Set 3.2(a) WI (see, Littell, et al.[6], p.557)

In the case of the complete data, we can also use the method of this paper. From data of Table 1, we obtain the sample covariance matrix $\boldsymbol{S}$ given by

$$\boldsymbol{S} = \begin{bmatrix} 9.65 & 10.12 & 10.00 & 8.47 & 9.22 & 8.76 & 8.65 \\ 10.12 & 11.23 & 11.17 & 9.03 & 9.92 & 9.55 & 9.23 \\ 10.00 & 11.17 & 12.09 & 10.10 & 10.81 & 10.52 & 10.26 \\ 8.47 & 9.03 & 10.10 & 9.26 & 9.76 & 9.41 & 9.60 \\ 9.22 & 9.92 & 10.81 & 9.76 & 11.15 & 10.54 & 10.64 \\ 8.76 & 9.55 & 10.52 & 9.41 & 10.54 & 10.71 & 10.53 \\ 8.65 & 9.23 & 10.26 & 9.60 & 10.64 & 10.53 & 11.19 \end{bmatrix},$$

which is based on $20(= 21 - 1)$ degrees of freedom.

Making the transformation $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$, we obtain the following Table 2, giving the values of the transformed variables:

$$\boldsymbol{Y}(= \{y_{ij}\}) = \begin{bmatrix} \boldsymbol{y}_1' \\ \vdots \\ \boldsymbol{y}_n' \end{bmatrix} = \begin{bmatrix} 0.707 & -0.408 & -1.155 & -0.894 & -0.730 & 0.309 \\ 0.707 & 0.408 & 1.155 & 0.000 & 0.913 & 0.772 \\ 0.707 & -1.225 & -1.732 & 0.447 & 0.365 & -0.617 \\ 0.707 & 0.408 & 0.289 & 0.224 & 0.183 & -0.772 \\ 0.707 & 1.225 & 1.732 & 0.447 & 1.278 & 2.006 \\ 0.707 & -0.408 & -0.289 & 0.671 & -0.365 & 0.617 \\ 0.000 & 0.000 & 1.732 & 1.342 & 2.921 & 2.469 \\ 1.414 & 2.449 & 0.866 & 1.565 & 2.191 & 1.852 \\ 0.000 & 0.000 & 0.866 & 0.671 & 1.461 & 1.234 \\ 0.000 & 0.000 & 1.732 & 3.130 & 1.643 & 3.240 \\ 0.000 & 0.816 & 1.443 & 1.118 & 0.913 & 0.772 \\ 0.707 & 1.225 & 1.732 & 2.236 & 0.913 & 0.772 \\ 0.707 & 0.408 & 0.289 & 0.224 & -0.730 & -0.617 \\ 0.000 & 0.816 & 2.309 & 0.894 & -0.183 & 2.623 \\ 0.000 & 0.816 & 0.577 & -0.447 & 0.548 & 0.463 \\ -0.707 & -0.408 & 1.443 & 1.118 & 0.000 & 0.926 \\ -0.707 & -0.408 & 1.443 & 0.224 & 0.183 & 0.154 \\ 1.414 & 0.816 & 0.577 & -0.447 & -0.365 & -0.309 \\ 0.707 & 1.225 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.707 & 2.041 & 2.309 & 2.683 & 2.191 & 1.852 \\ 0.707 & -0.408 & 0.577 & 0.447 & 1.278 & 2.006 \end{bmatrix},$$

and $f = 120$, $\widehat{\gamma}^2 = 0.932$, and

$$\frac{n \sum_{j=1}^{p-1} \overline{y}_{\cdot j}^2 / (p-1)}{\widehat{\gamma}^2} = 11.423.$$

This gives the value of the statistic $11.423 > F_{6,120,0.05} = 2.175$. Thus, the hypothesis $H : \mu_1 = \cdots = \mu_7$ is rejected.

In order to find out what caused the rejection or otherwise, we may wish to examine simultaneous confidence intervals for contrasts $\boldsymbol{a}'\boldsymbol{\mu}$, such that $\boldsymbol{a}'\mathbf{1} = 0$, where $\boldsymbol{a} \neq \mathbf{0}$ and $\mathbf{1}$ is a p-vector of ones. In the complete sample case we can obtain Tukey's, Scheffé's and Bonferroni's confidence intervals. We have chosen eleven contrasts of the type $\mu_i - \mu_j$. Thus $\ell = 11$, $p = 7$, $(\sum |a_i|)^2 = 4$ and $\sum a_i^2 = 2$. Thus, from Bhargava and Srivastava [3]. we find that

$$\frac{c_1^2}{4c(p-1)} \cdot \frac{[\sum_{j=1}^{p} |a_j|]^2}{\sum_{j=1}^{p} a_j^2} = 0.689 < 1,$$

where $c = q_{p,f,\alpha} = q_{7,120,0.05} = 4.241$ is the upper α percentiles of Studentized range on p and f d.f. and $c = F_{p-1,f,\alpha} = F_{6,120,0.05} = 2.175$. Thus, Tukey's simultaneous confidence intervals will be shorter than Scheffé's. Also, since $t^2_{120,0.05/22} = 8.364 < 6 \times 2.175$, we find that Bonferroni will be shorter than Scheffé but larger than Tukey. Clearly Tukey's type is a winner. However, unfortunately, we do not have Tukey's type confidence intervals available for the incomplete samples. Since most readers, we felt are familiar with Scheffé's type of confidence intervals, we chose to present Scheffé's type of confidence intervals even though it gives somewhat wider confidence intervals than Bonferroni's(and Tukey's in complete samples) in the examples we discuss in this paper.

To find Scheffé's confidence intervals for $\boldsymbol{a}'\boldsymbol{\mu}$, we need to find $\boldsymbol{b}$ such that $\boldsymbol{a}' = \boldsymbol{b}'\boldsymbol{C}$ and $\boldsymbol{C}\mathbf{1} = \mathbf{0}$. As an example, suppose we wish to find a 95% confidence intervals for $\mu_1 - \mu_3$. By choosing $b_1 = -\sqrt{2}/2$, $b_2 = -\sqrt{6}/2$, $b_3 = \cdots = b_6 = 0$, we find that $\boldsymbol{a} = (1, 0, -1, 0, 0, 0, 0)'$. Table 2 gives some Scheffé's type simultaneous confidence intervals. From the results in Table 2, it can be seen that the sources of significant differences among μ_i's are the differences between μ_1 and μ_4, μ_1 and μ_6, and the difference between μ_2 and μ_7.

Table 2. Simultaneous Confidence Intervals for the Complete Data WI

$a'\mu$	Confidence Intervals at Level 0.05				
$\mu_1 - \mu_2$	−0.61905	±	1.076	(−1.695,	0.457)
$\mu_1 - \mu_4$	−1.47619	±	1.076	(−2.552,	−0.400)
$\mu_1 - \mu_6$	−1.66667	±	1.076	(−2.743,	−0.590)
$\mu_2 - \mu_3$	−0.23810	±	1.076	(−1.314,	0.838)
$\mu_2 - \mu_5$	−0.95238	±	1.076	(−2.029,	0.124)
$\mu_2 - \mu_7$	−1.42857	±	1.076	(−2.505,	−0.352)
$\mu_3 - \mu_4$	−0.61905	±	1.076	(−1.695,	0.457)
$\mu_3 - \mu_6$	−0.80952	±	1.076	(−1.886,	0.267)
$\mu_4 - \mu_5$	−0.09524	±	1.076	(−1.171,	0.981)
$\mu_4 - \mu_7$	−0.57143	±	1.076	(−1.648,	0.505)
$\mu_5 - \mu_6$	−0.09524	±	1.076	(−1.171,	0.981)

5.2. A Monotone Type Data

To obtain missing observations, the weightlifting data set is modified by randomly deleting strength measurements each with probability 0.3 except the measurements of first and second days. Missing observations are designated by '*'. The modified data set is given in Table 5 which is used as general case data. Here, in order to obtain a monotone-type missing data set, the data set of Table 5 is modified by deleting less than or equal to the first missing observation for each subject, which is given in Table 3. For example, we have modified that $(x_{i1}, x_{i2}, x_{i3}, *, x_{i5}, *, x_{i6})$ changes to $(x_{i1}, x_{i2}, x_{i3}, *, *, *, *)$. Table 3 gives a orderly monotone type data set obtained from non monotone type data of Table 5.

Table 3. Monotone-type Missing Data Based on the Data WI

Subject #	p							
5	7	82	83	84	85	84	85	86
6	7	79	80	79	79	80	79	80
7	7	79	79	79	81	81	83	83
12	7	79	80	81	82	83	82	82
14	7	81	81	82	84	83	82	85
15	7	78	78	79	79	78	79	79
18	7	80	82	82	82	81	81	81
19	7	85	86	87	86	86	86	86
2	6	74	75	75	76	75	76	*
3	6	83	84	82	81	83	83	*
10	6	82	82	82	84	86	85	*
16	5	83	82	82	84	84	*	*
1	4	84	85	84	83	*	*	*
11	4	79	79	80	81	*	*	*
4	3	86	87	87	*	*	*	*
8	3	87	89	91	*	*	*	*
21	3	80	81	80	*	*	*	*
9	2	81	81	*	*	*	*	*
13	2	83	84	*	*	*	*	*
17	2	80	79	*	*	*	*	*
20	2	77	78	*	*	*	*	*
Sample Means		81.05	81.67	82.12	81.93	82.00	81.91	82.75

From this data of Table 3, we obtain

$$
\{y_{ij}\} = \begin{bmatrix}
0.707 & 1.225 & 1.732 & 0.447 & 1.278 & 2.006 \\
0.707 & -0.408 & -0.289 & 0.671 & -0.365 & 0.617 \\
0.000 & 0.000 & 1.732 & 1.342 & 2.921 & 2.469 \\
0.707 & 1.225 & 1.732 & 2.236 & 0.913 & 0.772 \\
0.000 & 0.816 & 2.309 & 0.894 & -0.183 & 2.623 \\
0.000 & 0.816 & 0.577 & -0.447 & 0.548 & 0.463 \\
1.414 & 0.816 & 0.577 & -0.447 & -0.365 & -0.309 \\
0.707 & 1.225 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.707 & 0.408 & 1.155 & 0.000 & 0.913 & * \\
0.707 & -1.225 & -1.732 & 0.447 & 0.365 & * \\
0.000 & 0.000 & 1.732 & 3.130 & 1.643 & * \\
-0.707 & -0.408 & 1.443 & 1.118 & * & * \\
0.707 & -0.408 & -1.155 & * & * & * \\
0.000 & 0.816 & 1.443 & * & * & * \\
0.707 & 0.408 & * & * & * & * \\
1.414 & 2.449 & * & * & * & * \\
0.707 & -0.408 & * & * & * & * \\
0.000 & * & * & * & * & * \\
0.707 & * & * & * & * & * \\
-0.707 & * & * & * & * & * \\
0.707 & * & * & * & * & *
\end{bmatrix},
$$

$f = 77$, $\hat{\gamma}^2 = 0.899$, and the value of the test statistic $7.094 > F_{6,77,0.05} = 2.219$. Thus, the hypothesis H is rejected and the Scheffé's type simultaneous confidence intervals for contrasts $\boldsymbol{a}'\boldsymbol{\mu}$ for all non-null vector $\boldsymbol{a}$ such that $\boldsymbol{a}'\mathbf{1} = 0$ can be obtained. Table 4 gives the some cases of the simultaneous confidence intervals. It can be seen from Table 4 the sources of significant differences among μ_i's are the differences between μ_1 and μ_4, μ_1 and μ_6, and the difference between μ_2 and μ_7. These confidence intervals are comparable with the complete data case.

Table 4. Simultaneous Confidence Intervals for Monotone-type Missing Data

$\boldsymbol{a}'\boldsymbol{\mu}$	Confidence Intervals at Level 0.05				
$\mu_1 - \mu_2$	−0.61905	±	1.068	(−1.687,	0.4449)
$\mu_1 - \mu_4$	−1.41457	±	1.242	(−2.657,	−0.172)
$\mu_1 - \mu_6$	−1.65677	±	1.353	(−3.009,	−0.304)
$\mu_2 - \mu_3$	−0.21989	±	1.158	(−1.378,	0.938)
$\mu_2 - \mu_5$	−0.97409	±	1.312	(−2.286,	0.338)
$\mu_2 - \mu_7$	−1.56803	±	1.519	(−3.087,	−0.049)
$\mu_3 - \mu_4$	−0.57563	±	1.269	(−1.844,	0.693)
$\mu_3 - \mu_6$	−0.81784	±	1.377	(−2.195,	0.559)
$\mu_4 - \mu_5$	−0.17857	±	1.374	(−1.553,	1.196)
$\mu_4 - \mu_7$	−0.77251	±	1.573	(−2.345,	0.800)
$\mu_5 - \mu_6$	−0.06364	±	1.451	(−1.514,	1.387)

5.3. A General Case: Missing Data at Random

As described in subsection 5.2, the data of Table 5 is the general case of missing data.

Table 5. Missing Data at Random Based on the Data WI

Subject #	p							
1	6	84	85	84	83	*	83	84
2	6	74	75	75	76	75	76	*
3	6	83	84	82	81	83	83	*
4	5	86	87	87	*	87	87	*
5	7	82	83	84	85	84	85	86
6	7	79	80	79	79	80	79	80
7	7	79	79	79	81	81	83	83
8	6	87	89	91	*	91	92	92
9	4	81	81	*	*	82	*	83
10	6	82	82	82	84	86	85	*
11	6	79	79	80	81	*	81	81
12	7	79	80	81	82	83	82	82
13	6	83	84	*	84	84	83	83
14	7	81	81	82	84	83	82	85
15	7	78	78	79	79	78	79	79
16	6	83	82	82	84	84	*	84
17	3	80	79	*	*	80	*	*
18	7	80	82	82	82	81	81	81
19	7	85	86	87	86	86	86	86
20	5	77	78	*	81	82	*	82
21	5	80	81	80	*	*	82	83
Sample Means		81.05	81.67	82.12	82.00	82.78	82.88	83.38

From the data of Table 5, we obtain

$$\{y_{ij}\} = \begin{bmatrix} 0.707 & -0.408 & * & * & -0.894 & 0.183 \\ 0.707 & 0.408 & 1.155 & 0.000 & 0.913 & * \\ 0.707 & * & * & 0.447 & 0.365 & * \\ 0.707 & 0.408 & * & 0.289 & 0.224 & * \\ 0.707 & 1.225 & 1.732 & 0.447 & 1.278 & 2.006 \\ 0.707 & -0.408 & -0.289 & 0.671 & -0.365 & 0.617 \\ 0.000 & 0.000 & 1.732 & 1.342 & 2.921 & 2.469 \\ 1.414 & 2.449 & * & 1.732 & 2.236 & 1.826 \\ 0.000 & * & * & 0.816 & * & 1.443 \\ 0.000 & 0.000 & 1.732 & 3.130 & 1.643 & * \\ 0.000 & 0.816 & 1.443 & * & 1.118 & 0.913 \\ 0.707 & 1.225 & 1.732 & 2.236 & 0.913 & 0.772 \\ 0.707 & * & 0.408 & 0.289 & -0.671 & -0.548 \\ 0.000 & 0.816 & 2.309 & 0.894 & -0.183 & 2.623 \\ 0.000 & 0.816 & 0.577 & -0.447 & 0.548 & 0.463 \\ -0.707 & -0.408 & 1.443 & 1.118 & * & 0.913 \\ -0.707 & * & * & 0.408 & * & * \\ 1.414 & 0.816 & 0.577 & -0.447 & -0.365 & -0.309 \\ 0.707 & 1.225 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.707 & * & 2.858 & 2.887 & * & 2.236 \\ 0.707 & -0.408 & * & * & 1.443 & 2.012 \end{bmatrix}.$$

As for the calculation of $\{y_{ij}\}$, we can obtain it by the transformation of the observations except for the missing observations described as in Section 4. Hence, we have $f = 99$, $\hat{\gamma}^2 = 0.943$ and the value of the test statistic $10.775 > F_{6,99,0.05} = 2.192$. Hence the hypothesis H is also rejected in this case.

5.4. A Real Missing Data

In this example, a real data of the cholesterol levels for a treatment group studied at times $0, 6, 12, 20$ and 24 months is taken from Wei and Lachin(1984)(see, Srivastava and Carter(1986)). Some observations are incomplete, the monotone-type data can be obtained by ignoring non-monotone missing observations type, that is, we delete the observations type such as $(x_{i1}, x_{i2}, *, *, x_{i5})$. A complete part and incomplete parts in this data are shown in Table 6 and Table 7, respectively.

Table 6. A Complete Part in Data of the Cholesterol Levels

Subject #	p					
1	5	178	246	295	228	274
2	5	254	260	278	245	340
3	5	185	232	215	220	292
4	5	219	268	241	260	320
5	5	205	232	265	242	230
6	5	182	213	173	200	193
7	5	310	334	290	286	248
8	5	191	204	227	228	196
9	5	245	270	209	255	213
10	5	229	200	238	259	221
11	5	245	293	261	297	231
12	5	240	313	251	307	291
13	5	234	281	277	235	210
14	5	210	252	275	235	237
15	5	275	231	285	238	251
16	5	269	332	300	320	335
17	5	148	180	184	231	184
18	5	181	194	212	217	205
19	5	165	241	250	249	312
20	5	293	276	276	278	306
21	5	195	190	205	217	238
22	5	210	230	249	240	194
23	5	212	224	246	271	256
24	5	243	271	304	273	318
25	5	259	279	296	262	283
26	5	202	214	192	239	172
27	5	184	192	205	253	217
28	5	238	272	297	282	251
29	5	263	283	248	334	271
30	5	144	226	261	227	283
31	5	220	272	222	246	253
32	5	225	260	253	202	265
33	5	307	252	316	258	283
34	5	313	300	313	317	397
35	5	206	177	194	194	212
36	5	285	291	291	268	260

Table 7. Incomplete Parts in Data of the Cholesterol Levels

Subject #	p					
37	4	224	273	242	274	*
38	4	231	252	267	299	*
39	4	268	296	314	330	*
40	4	284	288	268	261	*
41	4	217	231	276	257	*
42	4	209	200	269	233	*
43	4	200	261	264	300	*
44	3	201	219	220	*	*
45	3	202	186	253	*	*
46	3	209	207	167	*	*
47	3	212	253	225	*	*
48	3	276	326	304	*	*
49	3	163	179	199	*	*
50	3	239	243	265	*	*
51	3	204	203	198	*	*
52	3	247	211	225	*	*
53	3	195	250	272	*	*
54	3	228	228	279	*	*
55	3	290	264	260	*	*
56	2	250	269	*	*	*
57	2	175	214	*	*	*
58	2	260	268	*	*	*
59	2	197	218	*	*	*
60	2	248	262	*	*	*
Sample Means*		226.55	246.43	252.02	257.37	256.72

*Sample Means: A average for a complete and incomplete data of Table 6 and 7

In this example, both for testing equality of means and constructing simultaneous confidence intervals for the contrasts in the means are done from the data set which has deleted observations from 5 subjects with non-monotone data, Thus, we are considering the data of 60 subjects instead of 65 subjects, although the testing method of this paper can be used for the general non-monotone type data. From the data, we obtain $f = 190$, $\hat{\gamma}^2 = 655.475$. Here, the value of the statistic $11.205 > F_{4,190,0.05} = 2.419$. Thus, the result shows the hypothesis $H : \mu_1 = \cdots = \mu_5$ is rejected at level 0.05. Also, Table 8 gives the simultaneous confidence intervals for $\mu_i - \mu_j$, $1 \le i < j \le 5$. It can be seen from Table 8 that the main source of significant differences among μ_i's is large differences between μ_1 and the others. This implies that time has no effect. Without studying confidence intervals, this information is lost.

Table 8. Simultaneous Confidence Intervals for the Cholesterol Levels Data

$\boldsymbol{a}'\boldsymbol{\mu}$	Confidence Intervals at Level 0.05				
$\mu_1 - \mu_2$	−19.88333	±	14.541	(−34.424,	−5.343)
$\mu_1 - \mu_3$	−15.10530	±	8.490	(−23.595,	−6.615)
$\mu_1 - \mu_4$	−27.69445	±	16.394	(−44.088,	−11.301)
$\mu_1 - \mu_5$	−29.44703	±	17.453	(−46.900,	−11.994)
$\mu_2 - \mu_3$	4.77803	±	8.490	(−3.712,	13.268)
$\mu_2 - \mu_4$	−7.81112	±	16.394	(−24.205,	8.583)
$\mu_2 - \mu_5$	−9.56370	±	17.453	(−27.017,	7.889)
$\mu_3 - \mu_4$	−2.26187	±	16.540	(−18.802,	14.278)
$\mu_3 - \mu_5$	−4.01446	±	17.590	(−21.605,	13.576)
$\mu_4 - \mu_5$	−1.75258	±	18.190	(−19.942,	16.437)

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