A New Class of Skewed Multivariate Distributions

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Abstract

A new two-parameter class of multivariate distributions, which we call "modified multivariate Laplace distribution", has been introduced on the basis of multivariate Laplace distribution. A different from usual parameterisation enables us to have the covariance matrix Σ functionally independent of another parameter $\boldsymbol{\theta}$. In a special case of $\boldsymbol{\theta} = \mathbf{0}$, we have a distribution from elliptical family. We define measures of skewness β_{1p} and kurtosis β_{2p} in terms of the third and fourth moments of the distribution and show that $\beta_{2p} \geq p^2 + \beta_{1p}$ where p is the dimension of the random vector. When $p = 1$, it reduces to the well known inequality in the univariate case. The moments and cumulants up to the fourth order and the skewness and kurtosis measures for the modified multivariate Laplace distribution are derived. Tests and their null and non-null asymptotic distributions are given for testing the mean and skewness parameter $\boldsymbol{\theta}$.

Key words: *characteristic function, kurtosis, matrix derivative, multivariate cumulants, multivariate Laplace distribution, multivariate moments, skewness.*

Proposition 2.4. *Let S be a positive random variable with pdf $f_S(\mathrm{s})$ given in (2.1) which is distributed independently of a random vector $\mathbf{z} \sim N_p(\mathbf{0}, \Sigma - \boldsymbol{\theta}\boldsymbol{\theta}')$. Then*

$$\mathbf{x} = S^{1/2}\mathbf{z} + S\boldsymbol{\theta}$$

is distributed as $ML_p(\boldsymbol{\theta}, \Sigma)$.

PROOF: The statement follows from the representation of the density (2.2).

3. Skewness and kurtosis

To assess the extent of non-normality of this distribution, it would be desirable to evaluate the skewness and kurtosis of this distribution. For this we need to calculate the third and fourth moments or equivalently the third and fourth cumulants. To write these results in a compact form, we need to introduce some notations which are somewhat standard in multivariate analysis literature although variations exist.

A matrix $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n) : m \times n$ with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ is sometimes written as a vector and called vec$\mathbf{A}$, defined by

$$\mathrm{vec}\mathbf{A} = (\mathbf{a}_1', \dots, \mathbf{a}_n')'.$$

Next, the commutation matrix $\mathbf{K}_{p,q} : pq \times pq$ is defined as a partitioned matrix consisting of $p \times q$ blocks $\mathbf{K}_{ij}$ where only the j,i-th element in $\mathbf{K}_{ij}$ equals to one and all the other elements are zeros. An equivalent definition could have been given by the equality

$$\mathbf{K}_{p,q}\mathrm{vec}\mathbf{A} = \mathrm{vec}(\mathbf{A}')$$

for any $\mathbf{A} : p \times q$. To shorten notations the Kroneckerian power $\mathbf{A}^{\otimes k}$ is used to denote k times Kronecker product of matrix $\mathbf{A}$ to itself. For further properties of vec-operator, commutation matrix and related matrix algebra the reader is referred to Schott (1997) or Magnus & Neudecker (1999), for example. Later on much of the presentation will be based on cumulants. The k-th cumulant of a random p-vector $\mathbf{x}$ will be denoted by $\mathbf{c}_k(\mathbf{x})$ where the argument can be skipped in discussions. The cumulants are obtained as the matrix derivatives of the cumulant function

$$\psi_{\mathbf{x}}(\mathbf{t}) = \ln\varphi_{\mathbf{x}}(\mathbf{t}) \tag{3.1}$$

in the following way

$$\mathbf{C}_k = \frac{1}{i^k} \frac{\partial^k \psi_{\mathbf{x}}(\mathbf{t})}{\partial \mathbf{t} \partial \mathbf{t}' \ldots}\bigg|_{\mathbf{t}=0}. \tag{3.2}$$

The relationship between the moments and cumulants is one-to-one and is given in the following proposition due to Kollo (1992).

Proposition 3.1. *Let* $\mathbf{x}$ *be a random p-vector with* $\mathbf{M}_4(\mathbf{x}) < \infty$*. Then*

(i)
$$\mathbf{C}_1(\mathbf{x}) = E(\mathbf{x});$$

(ii)
$$\mathbf{C}_2(\mathbf{x}) = \overline{\mathbf{M}}_2(\mathbf{x}) = D(\mathbf{x}); \tag{3.3}$$

(iii)
$$\mathbf{C}_3(\mathbf{x}) = \overline{\mathbf{M}}_3(\mathbf{x}); \tag{3.4}$$

(iv)
$$\mathbf{C}_4(\mathbf{x}) = \overline{\mathbf{M}}_4(\mathbf{x}) - \overline{\mathbf{M}}_4(N); \tag{3.5}$$

where $\overline{\mathbf{M}}_4(N)$ *is the fourth central moment of a multivariate normal distribution with the same covariance matrix as* $\mathbf{x}$*,*

$$\overline{\mathbf{M}}_4(N) = (\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(D(\mathbf{x}) \otimes D(\mathbf{x})) - vecD(\mathbf{x})vec'D(\mathbf{x}) \tag{3.6}$$

and

$$\begin{aligned}\overline{\mathbf{M}}_3(\mathbf{x}) &= \mathbf{M}_3(\mathbf{x}) - \mathbf{M}_2(\mathbf{x}) \otimes E(\mathbf{x}) - E(\mathbf{x}) \otimes \mathbf{M}_2(\mathbf{x}) \\ &\quad - vec\mathbf{M}_2(\mathbf{x})E(\mathbf{x})' + 2E(\mathbf{x})E(\mathbf{x})' \otimes E(\mathbf{x}).\end{aligned} \tag{3.7}$$

The skewness and kurtosis measures of a univariate random variable X may be defined in terms of the third and fourth cumulants $c_3(Y)$ and $c_4(Y)$ or the moments $m_3(Y)$ and $m_4(Y)$ of a standardized random variable $Y = \frac{X-E(X)}{\sqrt{DX}}$ which has mean 0 and variance 1. The skewness measure is defined by $\beta_1 = \gamma_1^2 = c_3(Y)^2 = m_3(Y)^2$ and the kurtosis measure is defined by $\beta_2 = m_4(Y) = c_4(Y) + 3$. The corresponding quantities for the multivariate case will be defined as follows.

Definition 3.1. *Let* $\mathbf{x}$ *be a random p-vector with mean vector* $\boldsymbol{\theta}$ *and covariance matrix* Σ *and* $\mathbf{y} = \Sigma^{-1/2}(\mathbf{x} - \boldsymbol{\theta})$ *with mean vector* $\mathbf{0}$*, covariance matrix* $\mathbf{I}_p$ *and with* *the third and fourth moments* $\mathbf{M}_3(\mathbf{y})$, $\mathbf{M}_4(\mathbf{y})$ *and cumulants* $\mathbf{C}_3(\mathbf{y})$, $\mathbf{C}_4(\mathbf{y})$. *Then the skewness measure* $\beta_{1p}(\mathbf{x})$ *and kurtosis characteristic* $\beta_{2p}(\mathbf{x})$ *are defined by*

$$\beta_{1p}(\mathbf{x}) = tr[\mathbf{C}_3{}'(\mathbf{y})\mathbf{C}_3(\mathbf{y})] = tr[\mathbf{M}_3{}'(\mathbf{y})\mathbf{M}_3(\mathbf{y})] \tag{3.8}$$

and

$$\beta_{2p}(\mathbf{x}) = tr[\mathbf{M}_4(\mathbf{y})] = tr[\mathbf{C}_4(\mathbf{y})] + p^2 + 2p. \tag{3.9}$$

The last equality in (3.9) is a direct conclusion from (3.5)–(3.6) and basic properties of the trace function.

Proposition 3.2. *Let* $\mathbf{y}$ *and* $\mathbf{z}$ *be i.i.d. random p-vectors with mean vector* $\mathbf{0}$ *and covariance matrix* $\mathbf{I}_p$. *Then*

$$tr[\mathbf{M}_3'(\mathbf{y})\mathbf{M}_3(\mathbf{y})] = E(\mathbf{y}'\mathbf{z})^3$$

and

$$tr[\mathbf{M}_4(\mathbf{y})] = E(\mathbf{y}'\mathbf{y})^2.$$

PROOF: In the proof we use the fact that

$$\mathrm{tr}(\mathbf{A} \otimes \mathbf{B}) = \mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{B}$$

and for vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{ab}' = \mathbf{a} \otimes \mathbf{b}' = \mathbf{b}' \otimes \mathbf{a}.$$

Thus

$$\begin{aligned}\mathrm{tr}[\mathbf{M}_3'(\mathbf{y})\mathbf{M}_3(\mathbf{y})] &= \mathrm{tr}E[(\mathbf{y}' \otimes \mathbf{y} \otimes \mathbf{y}')(\mathbf{z} \otimes \mathbf{z}' \otimes \mathbf{z})] \\ &= E\mathrm{tr}[(\mathbf{y}'\mathbf{z}) \otimes (\mathbf{yz}') \otimes (\mathbf{y}'\mathbf{z})] = E[(\mathbf{y}'\mathbf{z})^2\mathrm{tr}(\mathbf{yz}')] = E(\mathbf{y}'\mathbf{z})^3.\end{aligned}$$

Similarly

$$\begin{aligned}\mathrm{tr}E(\mathbf{y} \otimes \mathbf{y}' \otimes \mathbf{y} \otimes \mathbf{y}') &= \mathrm{tr}E(\mathbf{yy}' \otimes \mathbf{yy}') \\ &= E(\mathrm{tr}[(\mathbf{yy}'))^2 = E(\mathbf{y}'\mathbf{y})^2.\end{aligned}$$

The above proposition shows that the skewness and kurtosis measures defined above through moments are identical to the ones given by Mardia (1970).

Next we establish a multivariate generalization of the well-known relation between the univariate skewness and kurtosis measures β_1 and β_2 which was not available in Mardia (1970).

Proposition 3.3. *Let $\beta_{1p}(\mathrm{x})$ and $\beta_{2p}(\mathrm{x})$ be the multivariate skewness and kurtosis measures given by (3.8) and (3.9) respectively. Then*

$$\beta_{2p}(\mathrm{x}) \geq p^2 + \beta_{1p}(\mathrm{x}).$$

PROOF: As in Mardia (1970), the inequality

$$E\left[p^2 + A\sum_{i=1}^{p} Y_i - p\sum_{i=1}^{p} Y_i^2\right]^2 > 0$$

gives

$$\beta_{2p}(\mathrm{x}) \geq p^2 + \frac{A^2}{p},$$

where

$$A = E\left[(\sum_{i=1}^{p} Y_i)(\sum_{i=1}^{p} Y_i^2)\right],$$

and Y_i's are the coordinates of y, given in Definition 3.1. Let us show that $\frac{A^2}{p} = \beta_{1p}(\mathrm{x})$. Denote a p-vector $\mathrm{a} = \frac{1}{\sqrt{p}}(1, \ldots, 1)'$. Then

$$\begin{aligned}\frac{A}{\sqrt{p}} &= E[(\mathrm{a}'\mathrm{y})(\mathrm{y}'\mathrm{y}] = E[\mathrm{tr}(\mathrm{y}\mathrm{a}')\mathrm{tr}(\mathrm{y}\mathrm{y}')] \\ &= \mathrm{tr}E[(\mathrm{y}\mathrm{a}') \otimes (\mathrm{y}\mathrm{y}')] = \mathrm{tr}E[(\mathrm{a}\mathrm{y}') \otimes (\mathrm{y}\mathrm{y}')] = \mathrm{tr}(\mathrm{a} \otimes \mathrm{M}_3'(\mathrm{y})).\end{aligned}$$

Hence

$$\begin{aligned}\frac{A^2}{p} &= \mathrm{tr}[(\mathrm{a} \otimes \mathrm{M}_3'(\mathrm{y})]\mathrm{tr}[(\mathrm{a}' \otimes \mathrm{M}_3(\mathrm{y}))] = \mathrm{tr}[(\mathrm{a}\mathrm{a}') \otimes (\mathrm{M}_3'(\mathrm{y})\mathrm{M}_3(\mathrm{y}))] \\ &= \mathrm{tr}(\mathrm{a}\mathrm{a}')\mathrm{tr}[\mathrm{M}_3'(\mathrm{y})\mathrm{M}_3(\mathrm{y}))] = \beta_{1p}(\mathrm{x}).\end{aligned}$$

3.1. Third and fourth cumulants of modified Laplace distribution

If $\mathrm{x} \sim ML_p(\theta, \Sigma)$, the cumulant function (3.1) is of the form:

$$\psi_{\mathrm{x}}(\mathrm{t}) = -\ln\left(1 - i\mathrm{t}'\theta - \frac{1}{2}(\mathrm{t}'\theta)^2 + \frac{1}{2}\mathrm{t}'\Sigma\theta\right). \tag{3.10}$$

The third and fourth cumulants of $\mathbf{x} \sim ML_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ will be presented in the next proposition.

Proposition 3.4. *Let $\mathbf{x} \sim ML_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, then the third and fourth cumulants are given by the equalities:*

$$\mathbf{c}_3(\mathbf{x}) = \boldsymbol{\Sigma} \otimes \boldsymbol{\theta} + \boldsymbol{\theta} \otimes \boldsymbol{\Sigma} + vec\boldsymbol{\Sigma}\boldsymbol{\theta}' - \boldsymbol{\theta}\boldsymbol{\theta}' \otimes \boldsymbol{\theta}; \tag{3.11}$$

$$\begin{aligned}\mathbf{c}_4(\mathbf{x}) &= \overline{\mathbf{M}}_4(N) - 3\boldsymbol{\theta}\boldsymbol{\theta}' \otimes \boldsymbol{\theta}\boldsymbol{\theta}' \\ &\quad + \boldsymbol{\theta}^{\otimes 2} vec'\boldsymbol{\Sigma} + vec\boldsymbol{\Sigma}\boldsymbol{\theta}'^{\otimes 2} + (\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\boldsymbol{\Sigma} \otimes \boldsymbol{\theta}\boldsymbol{\theta}')(\mathbf{I}_{p^2} + \mathbf{K}_{p,p}).\end{aligned} \tag{3.12}$$

$$\overline{\mathbf{M}}_4(\mathbf{x}) = \mathbf{c}_4(\mathbf{x}) + \overline{\mathbf{M}}_4(N),$$

where $\overline{\mathbf{M}}_4(N)$ is defined in (3.6).

PROOF: . The expressions of $\mathbf{c}_3(\mathbf{x})$ and $\mathbf{c}_4(\mathbf{x})$ are obtained by differentiating the cumulant function (3.10). The derivation is somewhat technical and is presented in Appendix 1. The last equality is a straightforward conclusion from (3.5).

Corollary 3.1. *Let $\mathbf{x} \sim ML_p(\mathbf{0}, \boldsymbol{\Sigma})$, then $\mathbf{x}$ is elliptically distributed. Then all odd moments are zero and the second and fourth moments are given by*

$$\begin{aligned} D(\mathbf{x}) &= \boldsymbol{\Sigma}; \\ \overline{\mathbf{M}}_4(\mathbf{x}) &= 2\overline{\mathbf{M}}_4(N); \\ \mathbf{c}_4(\mathbf{x}) &= \overline{\mathbf{M}}_4(N), \end{aligned}$$

where $\overline{\mathbf{M}}_4(N)$ is given by (3.6).

PROOF: We get the expressions of moments and the fourth cumulant directly from Propositions 3.1 and 3.4.

Proposition 3.5. *Let $\mathbf{x} \sim ML_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, then*

$$\beta_{1p}(\mathbf{x}) = a(a^2 - 6a + 3(p+2)) \tag{3.13}$$

and

$$\beta_{2p}(\mathbf{x}) = 2(p+a)(p+2) - 3a^2, \tag{3.14}$$

where

$$a = \boldsymbol{\theta}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}. \tag{3.15}$$

PROOF: We have to find traces of the two matrices following Definition 3.1. Start with the skewness parameter. Let $\mathbf{y} = \mathbf{\Sigma}^{-1/2}(\mathbf{x} - \boldsymbol{\theta})$ and denote

$$\mathbf{m} = \mathbf{\Sigma}^{-1/2}\boldsymbol{\theta}; \tag{3.16}$$

By (3.8) and (3.11) we have

$$\begin{aligned}
\beta_{1p}(\mathbf{x}) &= \mathrm{tr}\left[(\mathbf{m}\mathrm{vec}'\mathbf{I}_p - \mathbf{m}\mathbf{m}' \otimes \mathbf{m}' + \mathbf{I}_p \otimes \mathbf{m}' + \mathbf{m}' \otimes \mathbf{I}_p)\right.\\
&\quad \left.\times(\mathrm{vec}\mathbf{I}_p\mathbf{m}' - \mathbf{m}\mathbf{m}' \otimes \mathbf{m} + \mathbf{I}_p \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{I}_p)\right]\\
&= \mathrm{tr}(\mathbf{A}'\mathbf{A} + \mathbf{B}'\mathbf{B} + \mathbf{C}'\mathbf{C} + \mathbf{D}'\mathbf{D}\\
&\quad - 2\mathbf{A}'\mathbf{B} + 2\mathbf{A}'\mathbf{C} + 2\mathbf{A}'\mathbf{D} - 2\mathbf{B}'\mathbf{C} - 2\mathbf{B}'\mathbf{D} + 2\mathbf{C}'\mathbf{D}),
\end{aligned}$$

where $\mathbf{A} = \mathrm{vec}\mathbf{I}_p\mathbf{m}',\ \mathbf{B} = \mathbf{m}\mathbf{m}' \otimes \mathbf{m},\ \mathbf{C} = \mathbf{I}_p \otimes \mathbf{m},\ \mathbf{D} = \mathbf{m} \otimes \mathbf{I}_p$. Let us find the traces of the terms:

$$\begin{aligned}
&\mathrm{tr}(\mathbf{A}'\mathbf{A}) = \mathrm{tr}(\mathbf{C}'\mathbf{C}) = \mathrm{tr}(\mathbf{D}'\mathbf{D}) = \mathrm{tr}(\mathbf{I}_p \otimes \mathbf{m}'\mathbf{m}) = ap;\\
&\mathrm{tr}(\mathbf{B}'\mathbf{B}) = (\mathbf{m}'\mathbf{m})^3 = a^3;\\
&\mathrm{tr}(\mathbf{A}'\mathbf{B}) = \mathrm{tr}(\mathbf{B}'\mathbf{C}) = \mathrm{tr}(\mathbf{B}'\mathbf{D}) = \mathrm{tr}(\mathbf{m}\mathbf{m}' \otimes \mathbf{m}\mathbf{m}') = (\mathrm{tr}(\mathbf{m}\mathbf{m}'))^2 = a^2;\\
&\mathrm{tr}(\mathbf{A}'\mathbf{C}) = \mathrm{tr}(\mathbf{A}'\mathbf{D}) = \mathrm{tr}(\mathbf{C}'\mathbf{D}) = \mathrm{tr}(\mathbf{m}\mathbf{m}') = a.
\end{aligned}$$

After adding the terms together we have

$$\begin{aligned}
\beta_{1p}(\mathbf{x}) &= 3pa - 6a^2 + a^3 + 6a\\
&= a^3 - 6a^2 + 6a + 3pa.
\end{aligned}$$

The expression for the kurtosis measure $\beta_{2p}(\mathbf{x})$ is obtained in the same way. By (3.9) and (3.12)

$$\begin{aligned}
\beta_{2p}(\mathbf{x}) &= \mathrm{tr}\{2[(\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{I}_p \otimes \mathbf{I}_p) + \mathrm{vec}\mathbf{I}_p\mathrm{vec}'\mathbf{I}_p] - 3\mathbf{m}\mathbf{m}' \otimes \mathbf{m}\mathbf{m}'\\
&\quad + (\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{I}_p \otimes \mathbf{m}\mathbf{m}')(\mathbf{I}_{p^2} + \mathbf{K}_{p,p}) + \mathbf{m}^{\otimes 2}\mathrm{vec}'\mathbf{I}_p + \mathrm{vec}\mathbf{I}_p\mathbf{m}'^{\otimes 2}\}\\
&= 2\mathrm{tr}(\mathbf{A}_1 + \mathbf{A}_2 - 3\mathbf{A}_3 + \mathbf{A}_4 + 2\mathbf{A}_5),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{A}_1 &= (\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{I}_p \otimes \mathbf{I}_p);\\
\mathbf{A}_2 &= \mathrm{vec}\mathbf{I}_p\mathrm{vec}'\mathbf{I}_p;\\
\mathbf{A}_3 &= \mathbf{m}\mathbf{m}' \otimes \mathbf{m}\mathbf{m}';\\
\mathbf{A}_4 &= (\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{I}_p \otimes \mathbf{m}\mathbf{m}')(\mathbf{I}_{p^2} + \mathbf{K}_{p,p});\\
\mathbf{A}_5 &= \mathbf{m}^{\otimes 2}\mathrm{vec}'\mathbf{I}_p.
\end{aligned}$$

Let us find traces of the terms $\mathbf{A}_i$. The first term equals

$$\mathrm{tr}\mathbf{A}_1 = \mathrm{tr}(\mathbf{I}_p \otimes \mathbf{I}_p) + \mathrm{tr}[\mathbf{K}_{p,p}(\mathbf{I}_p \otimes \mathbf{I}_p)] = (\mathrm{tr}(\mathbf{I}_p))^2 + \mathrm{tr}(\mathbf{I}_p) = p^2 + p.$$

The second term is

$$\mathrm{tr}\mathbf{A}_2 = \mathrm{tr}(\mathrm{vec}\mathbf{I}_p \mathrm{vec}'\mathbf{I}_p) = \mathrm{vec}'(\mathbf{I}_p)\mathrm{vec}(\mathbf{I}_p) = p.$$

The third term equals

$$\mathrm{tr}\mathbf{A}_3 = \mathrm{tr}(\mathbf{mm}' \otimes \mathbf{mm}') = (\mathrm{tr}(\mathbf{mm}'))^2 = a^2.$$

The fourth term is the most complicated:

$$\begin{aligned}
\mathbf{A}_4 &= \mathrm{tr}[(\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{I}_p \otimes \mathbf{mm}')(\mathbf{I}_{p^2} + \mathbf{K}_{p,p})] \\
&= 2\mathrm{tr}(\mathbf{I}_m \otimes \mathbf{mm}') + 2\mathrm{tr}[\mathbf{K}_{p,p})(\mathbf{I}_p \otimes \mathbf{mm}')] = 2\mathrm{tr}(\mathbf{I}_p)\mathrm{tr}(\mathbf{mm}') + 2\mathrm{tr}(\mathbf{I}_p\mathbf{mm}') \\
&= 2ap + 2a.
\end{aligned}$$

The fifth term is

$$\mathbf{A}_5 = \mathrm{tr}(\mathbf{m}^{\otimes 2}\mathrm{vec}'\mathbf{I}_p) = \mathrm{vec}'(\mathbf{I}_p)\mathbf{m}^{\otimes 2} = a.$$

After adding the terms we have

$$\beta_{2p}(\mathbf{x}) = 2(p^2 + p + p) - 3a^2 + 2a + 2a(p+1) = 2(p+a)(p+2) - 3a^2.$$

Corollary 3.2. *Let* $\mathbf{x} \sim ML_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, *then*

$$\mathrm{tr}(\mathbf{C}_4(\mathbf{x})) = (p + 2a)(p+2) - 3a^2.$$

Remark 1. Since $a < p$, it follows that $\beta_{2p}(\mathbf{x})$ is always positive for the modified multivariate Laplace distribution.

4. Estimation of parameters

In this section, we first display in subsection 4.1 some simulated data for $p = 1$ and $p = 2$. In subsection 4.2, we use the method of moments to estimate the mean, the covariance matrix and the skewness and kurtosis measures.

4.1. Examples of simulated data

We shall first display (Figure 1a) generated data from the above model using Proposition 2.3 for $p = 1$. We choose $\theta = 1$ and $\sigma^2 = 4$. The skewness and kurtosis measures are $\beta_{11} = 1.8594$ and $\beta_{21} = 7.3125$ respectively. A second display in Figure 1b is for $\theta = \frac{1}{2}$ and $\sigma^2 = 4$ with the skewness and kurtosis measures $\beta_{11} = 0.5393$ and $\beta_{21} = 6.3633$ respectively. In all cases considered in this subsection the number of replications is 10000.

Figure 1a. Density estimate of $ML_1(1, 4)$

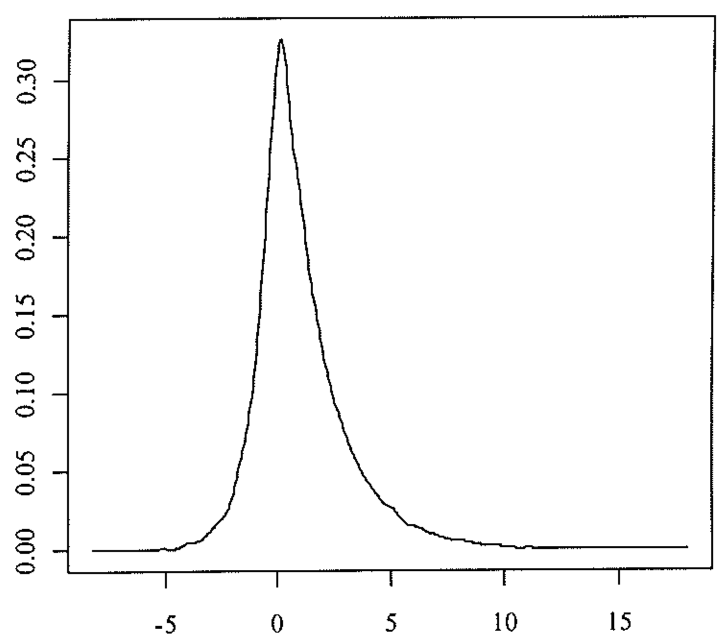
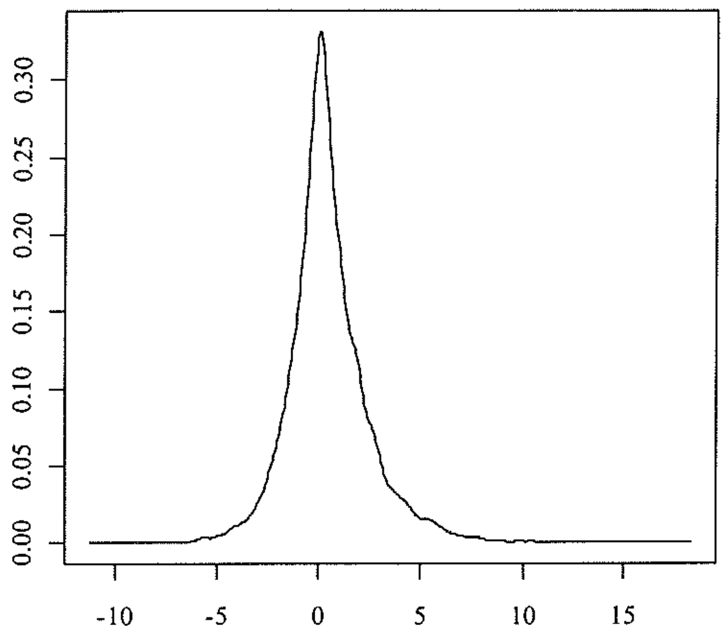


Figure 1b. Density estimate of $ML_1(0.5, 4)$



Next we consider $p = 2$. Here we consider the following two cases:

$$\text{(i)} \qquad \theta_1 = (1,1)' \qquad \text{and} \qquad \Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix};$$

$$\text{(ii)} \qquad \theta_2 = (\frac{1}{2}, \frac{1}{2})' \qquad \text{and} \qquad \Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}.$$

The skewness and kurtosis measures are in the case (i) equal to $\beta_{12} = 5.0845$ and $\beta_{22} = 19.5918$ respectively. In the case (ii) the corresponding characteristics are $\beta_{12} = 1.5948$ and $\beta_{22} = 17.0816$. The empirical density functions of the simulated data are displayed via contour lines in Figures 2a and 2b.

Figure 2a. Contour lines of the density estimate $ML_2(\boldsymbol{\theta}_1, \boldsymbol{\Sigma})$

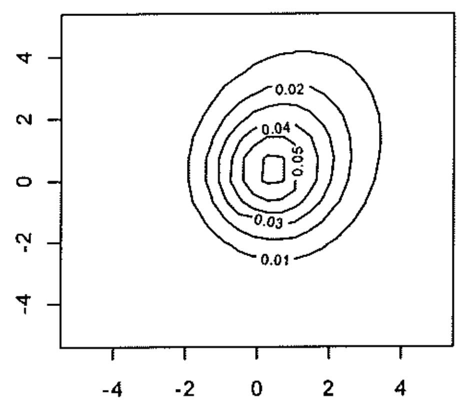
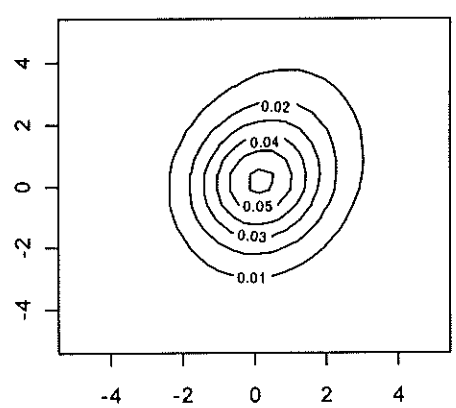


Figure 2b. Contour lines of the density estimate $ML_2(\boldsymbol{\theta}_2, \boldsymbol{\Sigma})$



4.2. Parameter estimation

We shall use the method of moments to estimate the parameters. We suppose that

$\mathbf{x}_1, \ldots, \mathbf{x}_n$ are i.i.d. as $ML_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$. We estimate the mean vector $\boldsymbol{\theta}$ and covariance matrix $\boldsymbol{\Sigma}$ by

$$\overline{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i$$

and

$$\mathbf{S} = \frac{1}{n-1} \sum_{i=1}^{n} (\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})'$$

respectively. To estimate the skewness nad kurtosis measures, we define the random vectors

$$\mathbf{y}_i = \mathbf{S}^{-1/2}(\mathbf{x}_i - \overline{\mathbf{x}}), \qquad i = 1, \ldots, n,$$

where $\mathbf{S}^{-1/2}$ is any square root of $\mathbf{S}^{-1}$ such that $\mathbf{S}^{-1} = \mathbf{S}^{-1/2}(\mathbf{S}^{-1/2})'$. Let

$$\widehat{\mathbf{M}}_3 = \frac{1}{n} \sum_{i=1}^{n} (\mathbf{y}_i \otimes \mathbf{y}_i' \otimes \mathbf{y}_i) \tag{4.1}$$

and

$$\widehat{\mathbf{M}}_4 = \frac{n+1}{n-1} \frac{1}{n} \sum_{i=1}^{n} (\mathbf{y}_i \otimes \mathbf{y}_i' \otimes \mathbf{y}_i \otimes \mathbf{y}_i'). \tag{4.2}$$

The scalar multiplying factor in (4.2) is chosen such that $\widehat{\mathbf{M}}_4$ is an unbiased estimator of $\mathbf{M}_4$ when the observations are independent and normally distributed, see Mardia (1970). The skewness measure β_{1p} and the kurtosis measure β_{2p} are estimated by

$$\hat{\beta}_{1p} = \operatorname{tr}[\widehat{\mathbf{M}}_3' \widehat{\mathbf{M}}_3] \tag{4.3}$$

and

$$\hat{\beta}_{2p} = \operatorname{tr}[\widehat{\mathbf{M}}_4]. \tag{4.4}$$

When $\mathbf{x} \sim ML_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, the skewness measure β_{1p} and the kurtosis measure β_{2p} are expressed through the quantity $a = \boldsymbol{\theta}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\theta}$. An estimate of a can be given by

$$\hat{a} = \overline{\mathbf{x}}' \mathbf{S}^{-1} \overline{\mathbf{x}}. \tag{4.5}$$

Thus, when $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are i.i.d. as $ML_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, the skewness and kurtosis measures can also be estimated by

$$\tilde{\beta}_{1p} = \hat{a}(\hat{a}^2 - 6\hat{a} + 3(p+2)) \tag{4.6}$$

and

$$\tilde{\beta}_{2p} = 2(p+\hat{a})(p+2) - 3\hat{a}^2 \tag{4.7}$$

respectively. There is, however, a shortcoming in using these estimates as they are not location invariant while (4.3) and (4.4) are.

5. Tests of hypothesis

In this section we shall consider the problem of testing the hypothesis that $\boldsymbol{\theta} = \mathbf{0}$ when $\mathbf{x} \sim ML_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, that is $\mathbf{x}$ has elliptical distribution, $\mathbf{x} \sim ML_p(\mathbf{0}, \boldsymbol{\Sigma})$. In the elliptical model, tests of hypothesis such as $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$ and $\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_p$ have been considered by Purkayastha & Srivastava (1995) and Nagao & Srivastava (1992). However, when $\boldsymbol{\theta} \neq \mathbf{0}$, tests have not yet been developed. We shall consider the problem of testing the hypothesis that the skewness β_{1p} for the modified Laplace distribution is zero. From Proposition 3.5 it follows that the skewness for this distribution is zero if and only if $a = \boldsymbol{\theta}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta} = 0$. Thus, we can either use a test statistic based on the skewness statistic (4.6) or a test statistic $\hat{a}$ given in (4.5). The former has the advantage that this statistic is location invariant while the statistic (4.5) is not. However, a power comparison for $p = 1$ by simulation showed that the test statistic (4.6) has a very low power compared to the test based on $\hat{a}$. We shall therefore only consider the test based on $\hat{a}$ in the next subsection.

5.1. Distributions for testing about the parameter $\boldsymbol{\theta}$

Since we estimate $\boldsymbol{\theta}$ by $\overline{\mathbf{x}}$ and $\boldsymbol{\Sigma}$ by $\mathbf{S}$, a reasonable test for the hypothesis

$$H : \boldsymbol{\theta} = \mathbf{0} \qquad \text{against} \qquad A : \boldsymbol{\theta} \neq \mathbf{0}$$

can be based on the test statistic

$$T^2 = n\overline{\mathbf{x}}'\mathbf{S}^{-1}\overline{\mathbf{x}}.$$

Unfortunately the statistic has considerable bias for both normal and non-normal populations (see Vilismäe, Kollo (1988), Kollo (1990), for example) which has to be taken into acoount in the simulation. Under the hypothesis that $\mathbf{x} \sim ML_p(\mathbf{0}, \boldsymbol{\Sigma})$, the asymptotic expansion of its cumulative distribution function is given by Iwashita

(1997):

$$P(T^2 \leq x) = P(\chi_p^2 \leq x) + \frac{p(p+1)}{2n}\left\{P(\chi_{p+2}^2 \leq x) - P(\chi_p^2 \leq x)\right\} + o(n^{-1}),$$

where χ_f^2 denotes the chi-square random variable with f degrees of freedom. The non-null distribution we can characterize by the first order approximations through the asymptotic non-central chi-square distribution or the asymptotic normal distribution. From law of large numbers and central limit theorem it follows that for the alternative close to the null hypothesis ($\boldsymbol{\theta} = \mathbf{0}$ versus $\boldsymbol{\theta}/\sqrt{n}$),

$$T^2 \xrightarrow{\mathcal{D}} \chi_p^2(a),$$

where $\chi_p^2(a)$ denotes the noncentral chi-square distribution with p degrees of freedom and non-centrality parameter a. In practical applications it is more convenient to use asymptotic normal distribution which is given in the next statement.

Proposition 5.1. *Let* $\mathbf{x} \sim ML_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$*. Then the following convergence holds, if* $n \to \infty$ *and* $\boldsymbol{\theta} \neq \mathbf{0}$*:*

$$\sqrt{n}(\overline{\mathbf{x}}'\mathbf{S}^{-1}\overline{\mathbf{x}} - \boldsymbol{\theta}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}) \xrightarrow{\mathcal{D}} N(0, \omega_{\boldsymbol{\theta}}^2), \tag{5.1}$$

where

$$\omega_{\boldsymbol{\theta}}^2 = a(4 - 6a + 10a^2 - 3a^3) \tag{5.2}$$

and a *is given by (3.15).*

PROOF: The statement follows as a special case from the proof of Proposition 5.2, given in Appendix 2, if $\boldsymbol{\theta}_0 = \mathbf{0}$. The convergence (5.1) gives us general picture about the power behaviour of the test, the same time it is known from simulation experiments that the convergence (5.1) can be rather slow for small values of $\boldsymbol{\theta}$.

Consider now the hypothesis

$$H : \boldsymbol{\theta} = \boldsymbol{\theta}_0 \qquad \text{against} \qquad A : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0. \tag{5.3}$$

It means that we are going to test about the distribution with certain skewness and kurtosis structure. Consider the statistic

$${T_{\boldsymbol{\theta}_0}}^2 = n(\overline{\mathbf{x}} - \boldsymbol{\theta}_0)'\mathbf{S}^{-1}(\overline{\mathbf{x}} - \boldsymbol{\theta}_0).$$

Under the null-hypothesis H in (5.3) the statistic ${T_{\theta_0}}^2$ has an asymptotic chi-square distribution, if $n \to \infty$:

$$n(\overline{\mathrm{x}} - \theta_0)'\mathrm{S}^{-1}(\overline{\mathrm{x}} - \theta_0) \xrightarrow{\mathcal{D}} \chi^2_p.$$

For the non-null case we can establish again an asymptotic normal distribution.

Proposition 5.2. *Let* $\mathrm{x} \sim ML_p(\theta, \Sigma)$. *Then the following convergence holds, if* $n \to \infty$ *and* $\theta \neq \theta_0$*:*

$$\sqrt{n}\left[(\overline{\mathrm{x}} - \theta_0)'\mathrm{S}^{-1}(\overline{\mathrm{x}} - \theta) - (\theta - \theta_0)'\Sigma^{-1}(\theta - \theta_0)\right] \xrightarrow{\mathcal{D}} N(0, \omega^2_{\theta_0}), \tag{5.4}$$

where

$$\omega^2_{\theta_0} = 4a_2 + 6a_2^2 + 6a_1^2 a_2 - 12a_1 a_2 + 4a_1^3 - 3a_1^4, \tag{5.5}$$

a_1 *and* a_2 *are given by the following equalities:*

$$a_1 = \theta'\Sigma^{-1}(\theta - \theta_0), \tag{5.6}$$

$$a_2 = (\theta - \theta_0)'\Sigma^{-1}(\theta - \theta_0). \tag{5.7}$$

The proof of Proposition 5.2 is presented in Appendix 2.

5.2. Tests for parameter θ

The power of the test to decide about the hypothesis

$$H : \theta = 0 \qquad \text{against} \qquad A : \theta > 0$$

was examined in a simulation experiment. We consider the modified Laplace distributions $ML_2(\theta, \Sigma)$ with the parameters:

$$\theta = c(1,1)' \qquad \text{and} \qquad \Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

where the real number c is varied in the experiment and generate samples of size $n = 25,\ 50,\ 100,\ 200$. The number of replications $k = 600$ in all experiments. In the following tables (Table 5.1 – Table 5.4) we shall present the estimated values of the power of the test from simulation and its values obtained via the asymptotic normal non-null distribution on the confidence level $\alpha = 0.05$. The case $\alpha = 0.02$ is added in Table 5.3. In the tables $p(\alpha)$ denotes the value of power from simulation on the confidence level α and $p(as., \alpha)$ stands for the estimate obtained from (5.1).

Table 5.1 Comparison of powers of the test ($c = 0.1$, $a = 0.0057$, $\beta_{12} = 0.0682$)

n	25	50	100	200
p(0.05)	0.018	0.050	0.038	0.060
p(as., 0.05)	0.000	0.000	0.000	0.001

Table 5.2 Comparison of powers of the test ($c = 0.25$, $a = 0.0357$, $\beta_{12} = 0.4208$)

n	25	50	100	200
p(0.05)	0.055	0.100	0.230	0.532
p(as., 0.05)	0.000	0.042	0.249	0.571

Table 5.3. Comparison of powers of the test ($c = 0.5$, $a = 0.1429$, $\beta_{12} = 1.5947$)

n	25	50	100	200
p(0.05)	0.215	0.488	0.873	1.000
p(as., 0.05)	0.186	0.568	0.880	0.989
p(0.02)	0.113	0.377	0.777	0.995
p(as., 0.02)	0.073	0.415	0.819	0.983

Table 5.4. Comparison of powers of the test ($c = 1.0$, $a = 0.5714$, $\beta_{12} = 5.0846$)

n	25	50
p(0.05)	0.857	1.000
p(as., 0.05)	0.877	0.989

As it is expected, the power of the test is growing with the sample size in all considered cases. One can conclude from the tables that the test based on asymptotic normality is conservative for small values of power. For bigger values of power estimates it starts to be too optimistic. This behaviour can be explained by the fact that the modified Laplace distribution has heavier tails than a normal distribution. For the considered sample sizes our test is not able to discover the difference 0.1 from the theoretical expectation (Table 5.1). In the case $c = 1.0$ the difference of the mean vector from zero is so big that this is discovered for all considered sample sizes. How the power depends on the confidence level is demonstrated in Table 5.3.

Appendix 1. Proof of Proposition 3.4

To get an expression for $\mathbf{C}_3(\mathbf{x})$ we have to differentiate the cumulant function (3.9) three times. From the proof of Proposition 2.1 we get the second derivative if we take into account the relation between $\varphi_{\mathbf{x}}(\mathbf{t})$ and $\psi_{\mathbf{x}}(\mathbf{t})$:

$$\frac{\partial^2\psi_{\mathbf{x}}(\mathbf{t})}{\partial\mathbf{t}\partial\mathbf{t}'} = i^2[\boldsymbol{\Sigma}_1\varphi_{\mathbf{x}}(\mathbf{t}) + (\boldsymbol{\theta}\boldsymbol{\theta}' - \boldsymbol{\Sigma}_1\mathbf{t}\mathbf{t}'\boldsymbol{\Sigma}_1 + i(\boldsymbol{\Sigma}_1\mathbf{t}\boldsymbol{\theta}' + \boldsymbol{\theta}\mathbf{t}'\boldsymbol{\Sigma}_1))\varphi_{\mathbf{x}}(\mathbf{t})^2],$$

where $\boldsymbol{\Sigma}_1$ is given by (2.7). The third derivative we get from the following chain of equalities:

$$\begin{aligned}
\frac{\partial^3\psi_{\mathbf{x}}(\mathbf{t})}{\partial\mathbf{t}\partial\mathbf{t}'\partial\mathbf{t}} &= i^2\frac{\partial}{\partial\mathbf{t}}[\boldsymbol{\Sigma}_1\varphi_{\mathbf{x}}(\mathbf{t}) + (\boldsymbol{\theta}\boldsymbol{\theta}' - \boldsymbol{\Sigma}_1\mathbf{t}\mathbf{t}'\boldsymbol{\Sigma}_1 + i(\boldsymbol{\Sigma}_1\mathbf{t}\boldsymbol{\theta}' + \boldsymbol{\theta}\mathbf{t}'\boldsymbol{\Sigma}_1))\varphi_{\mathbf{x}}(\mathbf{t})^2]\\
&= i^2\left\{\frac{\partial[\varphi_{\mathbf{x}}(\mathbf{t})\otimes(\boldsymbol{\Sigma}_1 + \boldsymbol{\theta}\varphi_{\mathbf{x}}(\mathbf{t})\boldsymbol{\theta}')]}{\partial\mathbf{t}} - \frac{\partial(\boldsymbol{\Sigma}_1\mathbf{t}{\varphi_{\mathbf{x}}}^2(\mathbf{t}))}{\partial\mathbf{t}}\mathbf{t}'\boldsymbol{\Sigma}_1 - (\mathbf{I}_p\otimes\boldsymbol{\Sigma}_1\mathbf{t}{\varphi_{\mathbf{x}}}^2(\mathbf{t}))\frac{\partial(\mathbf{t}'\boldsymbol{\Sigma}_1)}{\partial\mathbf{t}}\right.\\
&+ i\left[\frac{\partial(\boldsymbol{\Sigma}_1\mathbf{t})}{\partial\mathbf{t}}{\varphi_{\mathbf{x}}}^2(\mathbf{t})\boldsymbol{\theta}' + (\mathbf{I}_p\otimes\boldsymbol{\Sigma}_1\mathbf{t})\frac{\partial({\varphi_{\mathbf{x}}}^2(\mathbf{t})\boldsymbol{\theta}')}{\partial\mathbf{t}}\right.\\
&+ \left.\left.\frac{\partial(\boldsymbol{\theta}{\varphi_{\mathbf{x}}}^2(\mathbf{t}))}{\partial\mathbf{t}}\mathbf{t}'\boldsymbol{\Sigma}_1 + (\mathbf{I}_p\otimes\boldsymbol{\theta}{\varphi_{\mathbf{x}}}^2(\mathbf{t}))\frac{\partial(\mathbf{t}'\boldsymbol{\Sigma}_1)}{\partial\mathbf{t}}\right]\right\}\\
&= i^2\left\{\frac{\partial\varphi_{\mathbf{x}}(\mathbf{t})}{\partial\mathbf{t}}\otimes(\boldsymbol{\Sigma}_1 + \boldsymbol{\theta}2\varphi_{\mathbf{x}}(\mathbf{t})\boldsymbol{\theta}') - \frac{\partial(\boldsymbol{\Sigma}_1\mathbf{t})}{\partial\mathbf{t}}{\varphi_{\mathbf{x}}}^2(\mathbf{t})\mathbf{t}'\boldsymbol{\Sigma}_1 - (\mathbf{I}_p\otimes\boldsymbol{\Sigma}_1\mathbf{t})\frac{\partial({\varphi_{\mathbf{x}}}^2(\mathbf{t}))}{\partial\mathbf{t}}\mathbf{t}'\boldsymbol{\Sigma}_1\right.\\
&- (\mathbf{I}_p\otimes\boldsymbol{\Sigma}_1\mathbf{t}){\varphi_{\mathbf{x}}}^2(\mathbf{t})\boldsymbol{\Sigma}_1 + i\left[\mathrm{vec}\boldsymbol{\Sigma}_1\boldsymbol{\theta}'{\varphi_{\mathbf{x}}}^2(\mathbf{t}) + (\mathbf{I}_p\otimes\boldsymbol{\Sigma}_1\mathbf{t})2{\varphi_{\mathbf{x}}}^3(\mathbf{t})(i\boldsymbol{\theta} - \boldsymbol{\Sigma}_1\mathbf{t})\boldsymbol{\theta}'\right.\\
&+ \left.\left.(\mathbf{I}_p\otimes\boldsymbol{\theta})2{\varphi_{\mathbf{x}}}^3(\mathbf{t})(i\boldsymbol{\theta} - \boldsymbol{\Sigma}_1\mathbf{t})\boldsymbol{\theta}'\boldsymbol{\Sigma}_1 + (\mathbf{I}_p\otimes\boldsymbol{\theta}){\varphi_{\mathbf{x}}}^2(\mathbf{t})\boldsymbol{\Sigma}_1\right]\right\}\\
&= i^2\left\{{\varphi_{\mathbf{x}}}^2(\mathbf{t})(i\boldsymbol{\theta} - \boldsymbol{\Sigma}_1\mathbf{t})\otimes(\boldsymbol{\Sigma}_1 + 2\boldsymbol{\theta}\varphi_{\mathbf{x}}(\mathbf{t})\boldsymbol{\theta}') - \mathrm{vec}\boldsymbol{\Sigma}_1{\varphi_{\mathbf{x}}}^2(\mathbf{t})\mathbf{t}'\boldsymbol{\Sigma}_1\right.\\
&-2{\varphi_{\mathbf{x}}}^3(\mathbf{t})(\mathbf{I}_p\otimes\boldsymbol{\Sigma}_1\mathbf{t})(i\boldsymbol{\theta} - \boldsymbol{\Sigma}_1\mathbf{t})\boldsymbol{\theta}'\boldsymbol{\Sigma}_1 - {\varphi_{\mathbf{x}}}^2(\mathbf{t})(\boldsymbol{\Sigma}_1\otimes\boldsymbol{\Sigma}_1\mathbf{t})\\
&+ i\left[{\varphi_{\mathbf{x}}}^2(\mathbf{t})\mathrm{vec}\boldsymbol{\Sigma}_1\boldsymbol{\theta}' + 2{\varphi_{\mathbf{x}}}^3(\mathbf{t})((i\boldsymbol{\theta} - \boldsymbol{\Sigma}_1\mathbf{t})\otimes\boldsymbol{\Sigma}_1\mathbf{t})\boldsymbol{\theta}'\right.\\
&+ \left.\left.2{\varphi_{\mathbf{x}}}^3(\mathbf{t})((i\boldsymbol{\theta} - \boldsymbol{\Sigma}_1\mathbf{t})\otimes\boldsymbol{\theta})\mathbf{t}'\boldsymbol{\Sigma}_1 + {\varphi_{\mathbf{x}}}^2(\mathbf{t})(\boldsymbol{\Sigma}_1\otimes\boldsymbol{\theta})\right]\right\}\\
&= i^2\left\{{\varphi_{\mathbf{x}}}^2(\mathbf{t})\left[i\boldsymbol{\theta}\otimes\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_1\mathbf{t}\otimes\boldsymbol{\Sigma}_1 + 2i\varphi_{\mathbf{x}}(\mathbf{t})\boldsymbol{\theta}\otimes\boldsymbol{\theta}\boldsymbol{\theta}' - 2\varphi_{\mathbf{x}}(\mathbf{t})\boldsymbol{\Sigma}_1\mathbf{t}\otimes\boldsymbol{\theta}\boldsymbol{\theta}' - \mathrm{vec}\boldsymbol{\Sigma}_1\mathbf{t}'\boldsymbol{\Sigma}_1\right]\right.\\
&-2{\varphi_{\mathbf{x}}}^3(\mathbf{t})\left[(i\boldsymbol{\theta}\otimes\boldsymbol{\Sigma}_1\mathbf{t})\mathbf{t}'\boldsymbol{\Sigma}_1 - (\boldsymbol{\Sigma}_1\mathbf{t}\otimes\boldsymbol{\Sigma}_1\mathbf{t})\mathbf{t}'\boldsymbol{\Sigma}_1\right] - {\varphi_{\mathbf{x}}}^2(\mathbf{t})(\boldsymbol{\Sigma}_1\otimes\boldsymbol{\Sigma}_1\mathbf{t})\\
&+ i{\varphi_{\mathbf{x}}}^2(\mathbf{t})\mathrm{vec}\boldsymbol{\Sigma}_1\boldsymbol{\theta}' + i^2 2{\varphi_{\mathbf{x}}}^3(\mathbf{t})(\boldsymbol{\theta}\otimes\boldsymbol{\Sigma}_1\mathbf{t})\boldsymbol{\theta}' - 2i{\varphi_{\mathbf{x}}}^3(\mathbf{t})(\boldsymbol{\Sigma}_1\mathbf{t}\otimes\boldsymbol{\Sigma}_1\mathbf{t})\boldsymbol{\theta}'\\
&\left.+2i^2{\varphi_{\mathbf{x}}}^3(\mathbf{t})(\boldsymbol{\theta}\otimes\boldsymbol{\theta})\mathbf{t}'\boldsymbol{\Sigma}_1 - 2i{\varphi_{\mathbf{x}}}^3(\mathbf{t})(\boldsymbol{\Sigma}_1\mathbf{t}\otimes\boldsymbol{\theta})\mathbf{t}'\boldsymbol{\Sigma}_1 + i{\varphi_{\mathbf{x}}}^2(\mathbf{t})(\boldsymbol{\Sigma}_1\otimes\boldsymbol{\theta})\right\}.
\end{aligned}$$

The expression of $\mathbf{C}_3(\mathbf{x})$ we get applying (3.2) to the obtained expression:

$$\begin{aligned}
\mathbf{C}_3(\mathbf{x}) &= \frac{1}{i^3}\frac{\partial^3\psi_{\mathbf{x}}(\mathbf{t})}{\partial\mathbf{t}\partial\mathbf{t}'\partial\mathbf{t}}\bigg|_{\mathbf{t}=0}\\
&= \boldsymbol{\theta}\otimes\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_1\otimes\boldsymbol{\theta} + \mathrm{vec}\boldsymbol{\Sigma}_1\boldsymbol{\theta}' + 2\boldsymbol{\theta}\boldsymbol{\theta}'\otimes\boldsymbol{\theta}. \qquad \text{(A.1)}
\end{aligned}$$

It remains to replace $\mathbf{\Sigma}_1$ by $\mathbf{\Sigma}$ from (2.7):

$$\mathbf{C}_3(\mathbf{x}) = \boldsymbol{\theta} \otimes \mathbf{\Sigma} + \mathbf{\Sigma} \otimes \boldsymbol{\theta} + \mathrm{vec}\mathbf{\Sigma}\boldsymbol{\theta}' - \boldsymbol{\theta}\boldsymbol{\theta}' \otimes \boldsymbol{\theta}.$$

We shall get the fourth cumulant as the result of next differentiation. We shall exclude those terms from our consideration during derivation which will turn into zero at the point $\mathbf{t} = \mathbf{0}$.

$$\begin{aligned}
\mathbf{C}_4(\mathbf{x}) &= \frac{1}{i^4}\frac{\partial^4 \psi_{\mathbf{x}}(\mathbf{t})}{\partial \mathbf{t}\partial \mathbf{t}'\partial \mathbf{t}\partial \mathbf{t}'}\bigg|_{\mathbf{t}=\mathbf{0}} \\
&= \frac{1}{i}\frac{\partial}{\partial \mathbf{t}'}\left\{\varphi_{\mathbf{x}}{}^2(\mathbf{t})\left[\boldsymbol{\theta} \otimes \mathbf{\Sigma}_1 + i\mathbf{\Sigma}_1\mathbf{t} \otimes \mathbf{\Sigma}_1 + 2\varphi_{\mathbf{x}}(\mathbf{t})\boldsymbol{\theta} \otimes \boldsymbol{\theta}\boldsymbol{\theta}' + 2i\varphi_{\mathbf{x}}(\mathbf{t})\mathbf{\Sigma}_1\mathbf{t} \otimes \boldsymbol{\theta}\boldsymbol{\theta}'\right.\right. \\
&+ \left. i\mathrm{vec}\mathbf{\Sigma}_1\mathbf{t}'\mathbf{\Sigma}_1\right] + i\varphi_{\mathbf{x}}{}^2(\mathbf{t})(\mathbf{\Sigma}_1 \otimes \mathbf{\Sigma}_1\mathbf{t}) + \varphi_{\mathbf{x}}{}^2(\mathbf{t})\mathrm{vec}\mathbf{\Sigma}_1\boldsymbol{\theta}' + 2i\varphi_{\mathbf{x}}{}^3(\mathbf{t})(\boldsymbol{\theta} \otimes \mathbf{\Sigma}_1\mathbf{t})\boldsymbol{\theta}' \\
&+ \left. 2i\varphi_{\mathbf{x}}{}^3(\mathbf{t})(\boldsymbol{\theta} \otimes \boldsymbol{\theta})\mathbf{t}'\mathbf{\Sigma}_1 + \varphi_{\mathbf{x}}{}^2(\mathbf{t})(\mathbf{\Sigma}_1 \otimes \boldsymbol{\theta})\right\}_{\mathbf{t}=\mathbf{0}} \\
&= \frac{1}{i}\frac{\partial}{\partial \mathbf{t}'}\left\{\varphi_{\mathbf{x}}{}^2(\mathbf{t})\left[\boldsymbol{\theta} \otimes \mathbf{\Sigma}_1 + \mathbf{\Sigma}_1 \otimes \boldsymbol{\theta} + \mathrm{vec}\mathbf{\Sigma}_1\boldsymbol{\theta}'\right]\right. \\
&+ i\varphi_{\mathbf{x}}{}^2(\mathbf{t})\left[\mathbf{\Sigma}_1\mathbf{t} \otimes \mathbf{\Sigma}_1 + \mathrm{vec}\mathbf{\Sigma}_1\mathbf{t}'\mathbf{\Sigma}_1 + \mathbf{\Sigma}_1 \otimes \mathbf{\Sigma}_1\mathbf{t}\right] \\
&+ \left. 2\varphi_{\mathbf{x}}{}^3(\mathbf{t})\left[\boldsymbol{\theta} \otimes \boldsymbol{\theta}\boldsymbol{\theta}' + i\mathbf{\Sigma}_1\mathbf{t} \otimes \boldsymbol{\theta}\boldsymbol{\theta}' + i(\boldsymbol{\theta} \otimes \boldsymbol{\theta})\mathbf{t}'\mathbf{\Sigma}_1 + i(\boldsymbol{\theta}\boldsymbol{\theta}' \otimes \mathbf{\Sigma}_1\mathbf{t})\right]\right\}_{\mathbf{t}=\mathbf{0}} \\
&= \frac{1}{i}\left\{2\varphi_{\mathbf{x}}{}^3(\mathbf{t})(i\boldsymbol{\theta}' - \mathbf{t}\mathbf{\Sigma}_1) \otimes \left[\boldsymbol{\theta} \otimes \mathbf{\Sigma}_1 + \mathbf{\Sigma}_1 \otimes \boldsymbol{\theta} + \mathrm{vec}\mathbf{\Sigma}_1\boldsymbol{\theta}'\right]\right. \\
&+ i\varphi_{\mathbf{x}}{}^2(\mathbf{t})\left[\mathrm{vec}\mathbf{\Sigma}_1\mathrm{vec}'\mathbf{\Sigma}_1 + (\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{\Sigma}_1 \otimes \mathbf{\Sigma}_1)\right] \\
&+ \left. 6\varphi_{\mathbf{x}}{}^4(\mathbf{t})(i\boldsymbol{\theta}' - \mathbf{t}'\mathbf{\Sigma}_1) \otimes \boldsymbol{\theta} \otimes \boldsymbol{\theta}\boldsymbol{\theta}' + 2i\varphi_{\mathbf{x}}{}^3(\mathbf{t})\left[(\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{\Sigma}_1 \otimes \boldsymbol{\theta}\boldsymbol{\theta}') + \boldsymbol{\theta}^{\otimes 2}\mathrm{vec}'\mathbf{\Sigma}_1\right]\right\}_{\mathbf{t}=\mathbf{0}} \\
&= 6\boldsymbol{\theta}\boldsymbol{\theta}' \otimes \boldsymbol{\theta}\boldsymbol{\theta}' + (\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{\Sigma}_1 \otimes \mathbf{\Sigma}_1) + \mathrm{vec}\mathbf{\Sigma}_1\mathrm{vec}'\mathbf{\Sigma}_1 \\
&+ 2\boldsymbol{\theta}\boldsymbol{\theta}' \otimes \mathbf{\Sigma}_1 + 2\boldsymbol{\theta}' \otimes \mathbf{\Sigma}_1 \otimes \boldsymbol{\theta} + 2\boldsymbol{\theta}' \otimes (\mathrm{vec}\mathbf{\Sigma}_1\boldsymbol{\theta}') \\
&+ 2(\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{\Sigma}_1 \otimes \boldsymbol{\theta}\boldsymbol{\theta}') + 2\boldsymbol{\theta}^{\otimes 2}\mathrm{vec}'\mathbf{\Sigma}_1 \\
&= (\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{\Sigma}_1 \otimes \mathbf{\Sigma}_1) + \mathrm{vec}\mathbf{\Sigma}_1\mathrm{vec}'\mathbf{\Sigma}_1 + 6\boldsymbol{\theta}\boldsymbol{\theta}' \otimes \boldsymbol{\theta}\boldsymbol{\theta}' \\
&+ 2(\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{\Sigma}_1 \otimes \boldsymbol{\theta}\boldsymbol{\theta}')(\mathbf{I}_{p^2} + \mathbf{K}_{p,p}) + 2\boldsymbol{\theta}^{\otimes 2}\mathrm{vec}'\mathbf{\Sigma}_1 + 2\mathrm{vec}\mathbf{\Sigma}_1\boldsymbol{\theta}'^{\otimes 2}.
\end{aligned}$$

Let us present the obtained expression

$$\begin{aligned}
\mathbf{C}_4(\mathbf{x}) &= (\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{\Sigma}_1 \otimes \mathbf{\Sigma}_1) + \mathrm{vec}\mathbf{\Sigma}_1\mathrm{vec}'\mathbf{\Sigma}_1 + 6\boldsymbol{\theta}\boldsymbol{\theta}' \otimes \boldsymbol{\theta}\boldsymbol{\theta}' \\
&+ 2(\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\mathbf{\Sigma}_1 \otimes \boldsymbol{\theta}\boldsymbol{\theta}')(\mathbf{I}_{p^2} + \mathbf{K}_{p,p}) + 2\boldsymbol{\theta}^{\otimes 2}\mathrm{vec}'\mathbf{\Sigma}_1 + 2\mathrm{vec}\mathbf{\Sigma}_1\boldsymbol{\theta}'^{\otimes 2}
\end{aligned} \tag{A.2}$$

through the initial parameters $\boldsymbol{\theta}$ and $\boldsymbol{\Sigma}$:

$$\begin{aligned}
\mathbf{C}_4(\mathbf{x}) &= (\mathbf{I}_{p^2} + \mathbf{K}_{p,p})((\boldsymbol{\Sigma} - \boldsymbol{\theta}\boldsymbol{\theta}') \otimes (\boldsymbol{\Sigma} - \boldsymbol{\theta}\boldsymbol{\theta}')) + \mathrm{vec}(\boldsymbol{\Sigma} - \boldsymbol{\theta}\boldsymbol{\theta}')\mathrm{vec}'(\boldsymbol{\Sigma} - \boldsymbol{\theta}\boldsymbol{\theta}') \\
&+ 6\boldsymbol{\theta}\boldsymbol{\theta}' \otimes \boldsymbol{\theta}\boldsymbol{\theta}' + 2(\mathbf{I}_{p^2} + \mathbf{K}_{p,p})((\boldsymbol{\Sigma} - \boldsymbol{\theta}\boldsymbol{\theta}') \otimes \boldsymbol{\theta}\boldsymbol{\theta}')(\mathbf{I}_{p^2} + \mathbf{K}_{p,p}) \\
&+ 2\boldsymbol{\theta}^{\otimes 2}\mathrm{vec}'(\boldsymbol{\Sigma} - \boldsymbol{\theta}\boldsymbol{\theta}') + 2\mathrm{vec}(\boldsymbol{\Sigma} - \boldsymbol{\theta}\boldsymbol{\theta}')\boldsymbol{\theta}'^{\otimes 2} \\
&= (\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}) + \mathrm{vec}\boldsymbol{\Sigma}\mathrm{vec}'\boldsymbol{\Sigma} - 3\boldsymbol{\theta}\boldsymbol{\theta}' \otimes \boldsymbol{\theta}\boldsymbol{\theta}' \\
&+ (\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\boldsymbol{\Sigma} \otimes \boldsymbol{\theta}\boldsymbol{\theta}')(\mathbf{I}_{p^2} + \mathbf{K}_{p,p}) + \boldsymbol{\theta}^{\otimes 2}\mathrm{vec}'\boldsymbol{\Sigma} + \mathrm{vec}\boldsymbol{\Sigma}\boldsymbol{\theta}'^{\otimes 2}.
\end{aligned}$$

The statements of Proposition 3.4 are proved.

Appendix 2. Proof of Proposition 5.2

Let us denote

$$\mathbf{z} = \sqrt{n}\left[(\overline{\mathbf{x}} - \boldsymbol{\theta}_0)'\mathbf{S}^{-1}(\overline{\mathbf{x}} - \boldsymbol{\theta}) - (\boldsymbol{\theta} - \boldsymbol{\theta}_0)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)\right].$$

Then the statement of Proposition 5.2 is of the form

$$\mathbf{z} \xrightarrow{\mathcal{D}} N(0, \omega^2_{\boldsymbol{\theta}_0}),$$

where $\omega^2_{\boldsymbol{\theta}_0}$, a_1 and a_2 are given by equalities (5.5)–(5.7). The proof is based on a classical result about asymptotic normality (see Srivastava & Khatri (1979), p. 59, for example). If for an p-vector $\mathbf{x}_n$

$$\sqrt{n}(\mathbf{x}_n - \mathbf{b}) \xrightarrow{\mathcal{D}} N(\mathbf{0}, \boldsymbol{\Sigma}),$$

then for any smooth function $\mathbf{g}(\cdot) : \mathbb{R}^p \longrightarrow \mathbb{R}^q$

$$\sqrt{n}[\mathbf{g}(\mathbf{x}_n) - \mathbf{g}(\mathbf{b})] \xrightarrow{\mathcal{D}} N_q(\mathbf{0}, \boldsymbol{\xi}'\boldsymbol{\Sigma}\boldsymbol{\xi}),$$

where

$$\boldsymbol{\xi} = \left.\frac{\partial(\mathbf{g}(\mathbf{x}))'}{\partial\mathbf{x}}\right|_{\mathbf{x}=\mathbf{b}} \neq \mathbf{0}.$$

Take as vectors $\mathbf{x}_n$ the following:

$$\mathbf{v} = \mathbf{v}(n) = \begin{pmatrix} \overline{\mathbf{x}} - \boldsymbol{\theta}_0 \\ \mathrm{vec}\mathbf{S} \end{pmatrix}$$

and consider the point

$$\mathbf{v}_0 = \begin{pmatrix} \boldsymbol{\theta} - \boldsymbol{\theta}_0 \\ \mathrm{vec}\boldsymbol{\Sigma} \end{pmatrix}.$$

We have to find the derivative

$$
\begin{aligned}
&\left.\frac{\partial[(\overline{\mathbf{x}}-\theta_0)'\mathbf{S}^{-1}(\overline{\mathbf{x}}-\theta_0)]}{\partial \mathbf{v}}\right|_{\mathbf{v}=\mathbf{v}_0}\\
&= \left.\left[(\overline{\mathbf{x}}-\theta_0)'\mathbf{S}^{-1}\frac{\partial(\overline{\mathbf{x}}-\theta_0)'}{\partial \mathbf{v}} + (\overline{\mathbf{x}}-\theta_0)'\frac{\partial(\mathbf{S}^{-1}(\overline{\mathbf{x}}-\theta_0))}{\partial \mathbf{v}}\right]\right|_{\mathbf{v}=\mathbf{v}_0}\\
&= \left.\left\{\left((\overline{\mathbf{x}}-\theta_0)'\mathbf{S}^{-1}|\mathbf{0}\right) + (\overline{\mathbf{x}}-\theta_0)'\left[(\overline{\mathbf{x}}-\theta_0)'\otimes \mathbf{I}_p\frac{\partial \mathbf{S}^{-1}}{\partial \mathbf{v}} + \mathbf{S}^{-1}\frac{\partial(\overline{\mathbf{x}}-\theta_0)}{\partial \mathbf{v}}\right]\right\}\right|_{\mathbf{v}=\mathbf{v}_0}\\
&= \left[\left(2(\overline{\mathbf{x}}-\theta_0)'\mathbf{S}^{-1}|\mathbf{0}\right) - \left(\mathbf{0}|(\overline{\mathbf{x}}-\theta_0)'\mathbf{S}^{-1}\otimes(\overline{\mathbf{x}}-\theta_0)'\mathbf{S}^{-1}\right)\right]\Big|_{\mathbf{v}=\mathbf{v}_0}\\
&= \left(2(\theta-\theta_0)'\Sigma^{-1}| - (\theta-\theta_0)'\Sigma^{-1}\otimes(\theta-\theta_0)'\Sigma^{-1}\right).
\end{aligned}
$$

Denote

$$\xi_{11} = 2(\theta-\theta_0)'\Sigma^{-1}, \qquad \xi_{12} = -(\theta-\theta_0)'\Sigma^{-1}\otimes(\theta-\theta_0)'\Sigma^{-1}.$$

From Parring (1979)

$$\sqrt{n}(\mathbf{v}-\mathbf{v}_0) \xrightarrow{\mathcal{D}} N(\mathbf{0},\Pi),$$

where

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix}$$

with

$$\Pi_{11} = \Sigma; \qquad \Pi_{21} = (\Pi_{12})' = \overline{\mathbf{M}}_3(\mathbf{x})$$

and

$$\Pi_{22} = \overline{\mathbf{M}}_4(\mathbf{x}),$$

where $\overline{\mathbf{M}}_4(\mathbf{x})$ is given by (3.12). The product

$$\xi'\Pi\xi = \xi_{11}'\Pi_{11}\xi_{11} + \xi_{12}'\Pi_{21}\xi_{11} + \xi_{11}'\Pi_{12}\xi_{12} + \xi_{12}'\Pi_{22}\xi_{12}. \tag{5.8}$$

Let us find the terms in (5.8) one by one.

$$\xi_{11}'\Pi_{11}\xi_{11} = 4(\theta-\theta_0)'\Sigma^{-1}(\theta-\theta_0) = 4a_2.$$

The second term equals

$$
\begin{aligned}
\xi_{12}'\Pi_{21}\xi_{11} &= \xi_{11}'\Pi_{12}\xi_{12}\\
&= -(\theta-\theta_0)'\Sigma^{-1}\otimes(\theta-\theta_0)'\Sigma^{-1}\\
&\times[\mathrm{vec}\Sigma\theta' + \theta\otimes\Sigma + \Sigma\otimes\theta - \theta\theta'\otimes\theta]2\Sigma^{-1}(\theta-\theta_0)\\
&= \left[-\mathrm{vec}((\theta-\theta_0)'\Sigma^{-1}(\theta-\theta_0))\theta' - (\theta-\theta_0)'\Sigma^{-1}\theta(\theta-\theta_0)'\right.\\
&\left. -(\theta-\theta_0)'(\theta-\theta_0)'\Sigma^{-1}\theta + (\theta-\theta_0)'\Sigma^{-1}\theta\theta'(\theta-\theta_0)'\Sigma^{-1}\theta\right]2\Sigma^{-1}(\theta-\theta_0).
\end{aligned}
$$

In notations (5.6) and (5.7) we have

$$\xi_{12}'\Pi_{21}\xi_{11} = -6a_1a_2 + 2a_1^3.$$

The last term splits into five terms

$$\xi_{12}'\Pi_{22}\xi_{12} = \xi_{12}'2\overline{\mathbf{M}}_4(N)\xi_{12} - 3\xi_{12}'(\theta\theta' \otimes \theta\theta')\xi_{12} + \xi_{12}'\theta^{\otimes 2}\mathrm{vec}'\boldsymbol{\Sigma}\xi_{12} + \xi_{12}'\mathrm{vec}\boldsymbol{\Sigma}\theta'^{\otimes 2}\xi_{12}$$
$$+ \xi_{12}'(\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\boldsymbol{\Sigma} + \theta\theta')(\mathbf{I}_{p^2} + \mathbf{K}_{p,p})\xi_{12} = 2A_1 - 3A_2 + A_3 + A_3' + A_4.$$

Let us find these one by one.

$$\begin{aligned}
2A_1 &= 2((\theta-\theta_0)'\boldsymbol{\Sigma}^{-1} \otimes (\theta-\theta_0)'\boldsymbol{\Sigma}^{-1})[(\mathbf{I}_{p^2} + K_{p,p})(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}) + \mathrm{vec}\boldsymbol{\Sigma}\mathrm{vec}'\boldsymbol{\Sigma}] \\
&\quad \times \left(\boldsymbol{\Sigma}^{-1}(\theta-\theta_0) \otimes \boldsymbol{\Sigma}^{-1}(\theta-\theta_0)\right) \\
&= 2((\theta-\theta_0)'\boldsymbol{\Sigma}^{-1} \otimes (\theta-\theta_0)'\boldsymbol{\Sigma}^{-1}) \\
&\quad \times [(\mathbf{I}_{p^2} + K_{p,p})((\theta-\theta_0)\otimes(\theta-\theta_0)) + \mathrm{vec}\boldsymbol{\Sigma}(\theta-\theta_0)'\boldsymbol{\Sigma}^{-1}(\theta-\theta_0)] \\
&= 2(2a_2^2 + a_2^2) = 6a_2^2.
\end{aligned}$$

The second term equals

$$\begin{aligned}
3A_2 &= 3((\theta-\theta_0)'\boldsymbol{\Sigma}^{-1} \otimes (\theta-\theta_0)'\boldsymbol{\Sigma}^{-1})(\theta\theta' \otimes \theta\theta')\left(\boldsymbol{\Sigma}^{-1}(\theta-\theta_0) \otimes \boldsymbol{\Sigma}^{-1}(\theta-\theta_0)\right) \\
&= 3((\theta-\theta_0)'\boldsymbol{\Sigma}^{-1}\theta)((\theta-\theta_0)'\boldsymbol{\Sigma}^{-1}\theta)(\theta'\boldsymbol{\Sigma}^{-1}(\theta-\theta_0))(\theta'\boldsymbol{\Sigma}^{-1}(\theta-\theta_0)) = 3a_1^4.
\end{aligned}$$

The third term gives us

$$\begin{aligned}
A_3 &= ((\theta-\theta_0)'\boldsymbol{\Sigma}^{-1} \otimes (\theta-\theta_0)'\boldsymbol{\Sigma}^{-1})(\theta\otimes\theta)\mathrm{vec}'\boldsymbol{\Sigma}(\ \boldsymbol{\Sigma}^{-1}(\theta-\theta_0) \otimes \boldsymbol{\Sigma}^{-1}(\theta-\theta_0)) \\
&= a_1^2\mathrm{vec}'((\theta-\theta)'\boldsymbol{\Sigma}^{-1}(\theta-\theta)) = a_1^2a_2.
\end{aligned}$$

The last term equals

$$\begin{aligned}
A_4 &= ((\theta-\theta_0)'\boldsymbol{\Sigma}^{-1} \otimes (\theta-\theta_0)'\boldsymbol{\Sigma}^{-1})(\mathbf{I}_{p^2} + \mathbf{K}_{p,p})(\boldsymbol{\Sigma} + \theta\theta')(\mathbf{I}_{p^2} + \mathbf{K}_{p,p}) \\
&\quad \times \left(\boldsymbol{\Sigma}^{-1}(\theta-\theta_0) \otimes \boldsymbol{\Sigma}^{-1}(\theta-\theta_0)\right) \\
&= 4((\theta-\theta_0)'\boldsymbol{\Sigma}^{-1} \otimes (\theta-\theta_0)'\boldsymbol{\Sigma}^{-1})(\theta-\theta_0) \otimes (\theta\theta'\boldsymbol{\Sigma}^{-1}(\theta-\theta_0)) \\
&= 4(\theta-\theta_0)'\boldsymbol{\Sigma}^{-1}(\theta-\theta_0) \otimes (\theta-\theta_0)'\boldsymbol{\Sigma}^{-1}\theta\theta'\boldsymbol{\Sigma}^{-1}(\theta-\theta_0) = 4a_2a_1^2.
\end{aligned}$$

From here

$$\xi_{12}'\Pi_{22}\xi_{12} = 6a_2^2 - 3a_1^4 + 6a_1^2a_2$$

and the final expression for the asymptotic variance is

$$\omega_{\theta_0}^2 = 4a_2 + 6a_2^2 + 6a_1^2 a_2 + 4a_1^3 - 12a_1 a_2 - 3a_1^4.$$

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