Computational procedure for the steady-state analysis of a finite-capacity-bulk-service double-ended queueing system

Sudha Jain
Department of Statistics
University of Toronto

and

T. S. S. Srinivasa Rao
Institute for Systems Studies and Analysis
Defence R & D Organization
Metcalfe House

Computational procedure for the steady-state analysis of a finite-capacity bulk-service double-ended queueing system

Sudha Jain\textsuperscript{a,*} \textsuperscript{,} T. S. S. Srinivasa Rao\textsuperscript{b,2}

\textsuperscript{a}Department of Statistics, University of Toronto, Toronto, Ontario, M5S 3G3, Canada

\textsuperscript{b}Institute for Systems Studies and Analyses, Defence R & D Organisation, Metcalfe House, Delhi-110 054, India.

Abstract

This paper studies the steady-state analysis of a finite-capacity bulk-service double-ended queueing system in which either customers wait in queue for service or idle servers queue up for customers. The interarrival times of customers are arbitrarily distributed and the interarrival rates of servers are exponentially distributed, and that the servers ‘serve’ the customers in batches. Using the supplementary variable technique we develop a recursive method to compute the steady-state probabilities of the number of units in the system at an arbitrary and arrival instants. Besides obtaining the state probabilities at various instants, other practically interesting performance measures have also been discussed. Some numerical results are presented.

Keywords: Steady-state analysis; finite-capacity; bulk-service; double-ended queue; supplementary variable technique.

\textsuperscript{1}Corresponding author. E-mail address: jainsu@utstat.toronto.edu (Sudha Jain)

\textsuperscript{*}This research project has been supported by Natural Sciences and Engineering Research Council of Canada to Sudha Jain.

\textsuperscript{2}This research work was carried out when this author was a Post-Doctoral Fellow at the University of Toronto, Canada.
1. Introduction

In recent years there has been a considerable interest in computing state probabilities of various queueing models. Since performance measures are the function of state probabilities then their exact evaluation is important. Mostly a system designer/practitioner is interested in probability distributions of the number of units in the system at various instants which provide more information than few moments. In this paper, we consider a finite-capacity bulk-service double ended queue in which customers and servers wait for each other, with the later serving customers in batches. The arrivals of customers and servers are considered as general and Poisson, respectively. A practical example of this type of queueing system is a taxi-stand (queueing point) where either passengers (customers) are waiting for taxis (servers) or taxis are waiting for passengers. This type of situations can be realised in many areas such as transportation systems, inventory processes, production and flexible manufacturing systems. Using generating function method, Srivastava and Kashyap [1] have obtained the transient solution of this bulk-service double-ended queueing model. However, the analytical solution given by them in terms of summation of the integrals of Bessel functions is very complex and numerical results are difficult to compute. The similar remark has also been made by Chaudhry and Templeton [2]. Hlynka and Sheahan [3] gave a control policy on the Poisson arrival mechanism for the double-ended queue and derived the time-dependent solution. Sharma [4] studied the transient behaviour of a Markovian double-ended queue with single-service in which the arrivals of customers and servers are distributed exponentially having parameters $\lambda$ and $\mu$, respectively. To the authors' best knowledge, the steady-state analysis of a finite-capacity bulk-service double-ended queueing model has never been treated.

The purpose of this research article is to provide a simple and easily implementable algorithm for the steady-state analysis of a more general finite-capacity bulk-service double-ended queueing model. The approach we take here for the analysis of the problem is the
supplementary variable technique using which a recursive method has been developed. The method is straightforward to obtain the state probabilities of the number of units in the system at the instant of an arrival and at an arbitrary instant, and practically interesting performance measures such as probabilities of queue of customers, and of servers; average numbers of customers and of servers in the system; and probabilities of blocking of customers and of servers, etc.

2. Description of the model

Consider a finite-capacity bulk-service double-ended queueing system in which the arrivals of customers and servers are taken as general and Poisson, respectively, and that the servers 'serve' the customers in batches. The maximum numbers of customers and servers in the system including the ones being served are $N$ and $K$, respectively. For linguistic convenience we will describe the queueing system in terms of a taxi-stand. Let passengers (customers) arrive singly at a taxi-stand (queueing point) and form a queue if taxis (servers) are not available. The interarrival times of passengers being arbitrarily distributed having a probability density function $a(x)$, $(x \geq 0)$. Taxis arrive at the stand in a exponentially distributed manner with mean rate $\mu$. The maximum number of seats in each taxi is, $B$. An arriving taxi, on finding a queue of customers presents, departs taking a batch of maximum size $B$ customers from the queue; if it finds less than $B$ customers are waiting, then it takes all of them in a batch for service; otherwise, it waits at the stand if there are no passengers present in the queue. Thus, at any time there is either a queue of passengers or one of taxis or of neither. Assume that the numbers on the negative-axis stand for the taxis waiting for passengers whereas the numbers on the positive-axis denote the passengers waiting for the taxis, and the system is at the position $i$ ($-K \leq i \leq N$) where $K$ and $N$ are the positive integers. We allow $i$ to assume the following values so that

(i) when $i > 0$ it denotes the number ($i$) of customers are waiting at time $t$,  
(ii) when \( i = 0 \), neither customers nor servers are waiting at time \( t \).

(iii) when \( i < 0 \) it signifies that number \((-i)\) of servers are waiting at time \( t \).

3. Time-dependent system equations and definitions

Let \( P_i(x,t) \Delta x \) \((-K \leq i \leq N)\) be the probability that at time \( t \), there are \( i \) customers in the system and the remaining interarrival time for the next arrival lies in the interval \((x,x + \Delta x)\). Therefore, the state of the system at time \( t \) is defined by \( \{\xi(t), X(t); \; t \geq 0\} \) which is Markovian in continuous time. Define

\[
P_i(x,t) \Delta x = \Pr[\xi(t) = i, x < X(t) \leq x + \Delta x],
\]
\[
x \geq 0, \quad -K \leq i \leq N.
\]

Relating the states of the system at time \( t \) and \( t + \Delta t \), we obtain the following Chapman-Kolmogorov forward equations

\[
P_{-K}(x - \Delta t, i + \Delta t) = P_{-K}(x,t) + P_{-K+1}(x,t) \mu \Delta t + O(\Delta t),
\]
\[
P_i(x - \Delta t, i + \Delta t) = \left[1 - \mu \Delta t\right] P_i(x,t) + a(x) P_{i-1}(0,t) \Delta t +
\]
\[
\quad + P_{i-1}(x,t) \mu \Delta t + O(\Delta t), \quad -K + 1 \leq i \leq -1,
\]
\[
P_0(x - \Delta t, i + \Delta t) = \left[1 - \mu \Delta t\right] P_0(x,t) + P_{-1}(0,t) a(x) \Delta t
\]
\[
\quad + \sum_{j=1}^{B} P_j(x,t) \mu \Delta t + O(\Delta t),
\]
\[
P_i(x - \Delta t, i + \Delta t) = \left[1 - \mu \Delta t\right] P_i(x,t) + P_{i-1}(0,t) a(x) \Delta t + P_{i+B}(x,t) \mu \Delta t
\]
\[
\quad + O(\Delta t), \quad 1 \leq i \leq N - B,
\]
\[
P_i(x - \Delta t, i + \Delta t) = \left[1 - \mu \Delta t\right] P_i(x,t) + P_{i-1}(0,t) a(x) \Delta t + O(\Delta t),
\]
\[
\quad N - B + 1 \leq i \leq N - 1,
\]
\[
P_N(x - \Delta t, i + \Delta t) = \left[1 - \mu \Delta t\right] P_N(x,t) + P_{N-1}(0,t) a(x) \Delta t
\]
\[
\quad + P_N(0,t) a(x) \Delta t + O(\Delta t).
\]

Expand the left hand side of (2) to (7) in powers of \( \Delta t \) and let \( \Delta t \to 0 \), we have the
following partial differential equations of the system

\[
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) P_{-K}(x, t) = \mu P_{-K+1}(x, t),
\]

\[
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) P_{i}(x, t) = -\mu P_{i}(x, t) + \alpha(x) P_{i-1}(0, t) + \mu P_{i+1}(x, t), \quad -K + 1 \leq i \leq -1,
\]

\[
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) P_0(x, t) = -\mu P_0(x, t) + \alpha(x) P_{-1}(0, t) + \mu \sum_{j=1}^{B} P_j(x, t),
\]

\[
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) P_i(x, t) = -\mu P_i(x, t) + \alpha(x) P_{i-1}(0, t) + \mu P_{i+B}(x, t), \quad 1 \leq i \leq N - B,
\]

\[
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) P_{N}(x, t) = -\mu P_{N}(x, t) + \alpha(x) [P_{N-1}(0, t) + P_{N}(0, t)].
\]

4. Steady-state system equations

As we will restrict ourselves now to study the steady-state, we let \( t \to \infty \) and thus the derivatives with respect to \( t \) tend to zero in (8) to (13). Let \( P_i(x) = \lim_{t \to \infty} P_i(x, t) \), \(-K \leq i \leq N, x \geq 0\) which gives the steady-state distribution of \{\( \xi(t), X(t); t \geq 0 \}\).

\[
-\frac{d}{dx} P_{-K}(x) = \mu P_{-K+1}(x),
\]

\[
-\frac{d}{dx} P_i(x) = -\mu P_i(x) + \alpha(x) P_{i-1}(0) + \mu P_{i+1}(x), \quad -K + 1 \leq i \leq -1,
\]

\[
-\frac{d}{dx} P_0(x) = -\mu P_0(x) + \alpha(x) P_{-1}(0) + \mu \sum_{j=1}^{B} P_j(x),
\]

\[
-\frac{d}{dx} P_i(x) = -\mu P_i(x) + \alpha(x) P_{i-1}(0) + \mu P_{i+B}(x), \quad 1 \leq i \leq N - B,
\]

\[
-\frac{d}{dx} P_{N}(x) = -\mu P_{N}(x) + \alpha(x) P_{N-1}(0),
\]

\[
N - B + 1 \leq i \leq N - 1,
\]
\[-\frac{d}{dx} P_N(x) = -\mu P_N(x) + a(x) \left[ P_{N-1}(0) + P_N(0) \right]. \quad (19)\]

Define the Laplace transforms of $a(x)$, $P_i(x)$ and $\frac{dP_i(x)}{dx}$ are given by

\[ A^*(\phi) = \int_0^\infty e^{-\phi x} a(x) dx, \quad (20) \]
\[ P_i^*(\phi) = \int_0^\infty e^{-\phi x} P_i(x) dx, \quad -K \leq i \leq N, \quad (21) \]
\[ \int_0^\infty e^{-\phi x} \frac{dP_i(x)}{dx} dx = \phi P_i^*(\phi) - P_i(0). \quad (22) \]

Multiplying (14) to (19) by $e^{-\phi x}$ and integrating with respect to $x$ from 0 to $\infty$, it then follows that

\[ \phi P_{-K}^*(\phi) = P_{-K}(0) - \mu P_{-K+1}^*(\phi), \quad (23) \]
\[ (\phi - \mu)P_i^*(\phi) = P_i(0) - A^*(\phi)P_{i-1}(0) - \mu P_{i+1}^*(\phi), \quad -K + 1 \leq i \leq -1, \quad (24) \]
\[ (\phi - \mu)P_0^*(\phi) = P_0(0) - A^*(\phi)P_{-1}(0) - \mu \sum_{j=1}^B P_j^*(\phi), \quad (25) \]
\[ (\phi - \mu)P_i^*(\phi) = P_i(0) - A^*(\phi)P_{i-1}(0) - \mu P_{i+1}^*(\phi), \quad 1 \leq i \leq N - B, \quad (26) \]
\[ (\phi - \mu)P_i^*(\phi) = P_i(0) - A^*(\phi)P_{i-1}(0), \quad N - B + 1 \leq i \leq N - 1, \quad (27) \]
\[ (\phi - \mu)P_N^*(\phi) = P_N(0) - A^*(\phi)[P_{N-1}(0) + P_N(0)]. \quad (28) \]

Lemma 4.1.

\[ \sum_{i=-K}^N P_i^*(\phi) - \frac{1 - A^*(\phi)}{\phi} \sum_{i=-K}^N P_i(0). \quad (29) \]

**Proof.** Adding (23) to (28), we get (29). \qed

Lemma 4.2.

\[ \sum_{i=-K}^N P_i(0) = \frac{1}{a_1}, \quad (30) \]
where \( a_1 = -A^{*}(1)(0) = -\left[ \frac{d}{d\phi} A^{*}(\phi) \right]_{\phi=0} \) is the mean interarrival time of the distribution. Relation (30) can be used as a check for the numerical calculations.

**Proof.** Letting \( \phi \to 0 \) in (29) and using the normalizing condition \( \sum_{i=-K}^{N} P_i^*(0) = 1 \), we get (30). \( \square \)

5. **Recursive method to compute steady-state probabilities**

Our objective is to compute the steady-state distribution of the number of units in the system at an arbitrary \( \{ P_i \equiv P_i^*(0); -K \leq i \leq N \} \), and arrival \( \{ Q_i; -K \leq i \leq N \} \) instants. To achieve our goal we have to solve the equations (23) to (28) recursively starting from the last equation and working backward.

Inserting \( \phi = 0 \) and \( \phi = \mu \), respectively in (28), we get

\[
P_{N-1}(0) = \mu P_N^*(0), \tag{31}
\]

\[
P_N(0) = \frac{A^*(\mu) P_{N-1}(0)}{[1 - A^*(\mu)]}. \tag{32}
\]

Next setting \( \phi = \mu \), respectively in (27), (26), (25), and (24). Then we have

\[
P_{i-1}(0) = \frac{P_i(0)}{A^*(\mu)}, \quad N - 1 \leq i \leq N - B + 1, \tag{33}
\]

\[
P_{i-1}(0) = \frac{1}{A^*(\mu)} \left[ P_i(0) - \mu P_{i+B}^*(\mu) \right], \quad N - B \leq i \leq 1. \tag{34}
\]

\[
P_{i-1}(0) = \frac{1}{A^*(\mu)} \left[ P_0(0) - \mu \sum_{j=1}^{B} P_j^*(\mu) \right], \tag{35}
\]

\[
P_{i-1}(0) = \frac{1}{A^*(\mu)} \left[ P_i(0) - \mu P_{i+1}^*(\mu) \right], \quad -1 \leq i \leq -K + 1. \tag{36}
\]

Thus, we need \( \{ P_n^*(\mu); -K+1 \leq n \leq N-B \} \) in order to get the explicit expressions for \( \{ P_{i-1}(0); -K+1 \leq i \leq N-B \} \). Differentiate (28), (27), (26), (25), and (24), respectively \( m+1 \) times with respect to \( \phi \) and setting \( \phi = \mu \). It follows that

\[
P_N^{(m)}(\mu) = -\frac{A^{*(m+1)}(\mu)}{m+1} [P_{N-1}(0) + P_N(0)], \quad 0 \leq m \leq K + N - B - 1, \tag{37}
\]

\[
P_i^{(m)}(\mu) = -\frac{A^{*(m+1)}(\mu)}{m+1} P_{i-1}(0), \tag{38}
\]
\[ N - B + 1 \leq i \leq N - 1, \quad 0 \leq m \leq i + N - 2, \]

\[
P_i^{(m)}(\mu) = -\frac{1}{m+1} \left[ A^{(m+1)}(\mu) P_{i-1}(0) + \mu P_{i+B}^{(m+1)}(\mu) \right], \tag{39} \]

\[ N - B \leq i \leq 1, \quad 0 \leq m \leq i + N - 2, \]

\[
P_0^{(m)}(\mu) = -\frac{1}{m+1} \left[ A^{(m+1)}(\mu) P_{-1}(0) + \mu \sum_{j=1}^{B} P_j^{(m+1)}(\mu) \right], \tag{40} \]

\[ 0 \leq m \leq K - 2, \]

\[
P_i^{(m)}(\mu) = -\frac{1}{m+1} \left[ A^{(m+1)}(\mu) P_{i-1}(0) + \mu P_{i+1}^{(m+1)}(\mu) \right], \tag{41} \]

\[ -1 \leq i \leq -K + 2, \quad 0 \leq m \leq i + K - 2, \]

where \( A^{(m)}(\mu) = \left[ \frac{d^m}{d\phi^m} A^*(\phi) \right]_{\phi = \mu} \) and \( P_n^{(m)}(\mu) \) is also defined in the same way. Hence \( \{P_i(0); -K \leq i \leq N\} \) can be obtained in terms of \( P_N = P_N^*(0) \) recursively from (31) to (41).

Setting \( \phi = 0 \) respectively in (27), (26), (25), and (24), we have

\[
P_i^*(0) = \frac{1}{\mu} [P_{i-1}(0) - P_i(0)], \quad N - 1 \leq i \leq N - B + 1, \tag{42} \]

\[
P_i^*(0) = \frac{1}{\mu} [\mu P_{i+B}^*(0) + P_{i-1}(0) - P_i(0)], \quad N - B \leq i \leq 1, \tag{43} \]

\[
P_0^*(0) = \frac{1}{\mu} \left[ \mu \sum_{j=1}^{B} P_j^*(0) + P_{-1}(0) - P_0(0) \right], \tag{44} \]

\[
P_i^*(0) = \frac{1}{\mu} [\mu P_{i+1}^*(0) + P_{i-1}(0) - P_i(0)], \quad -1 \leq i \leq -K + 1. \tag{45} \]

Therefore, \( \{P_i = P_i^*(0); -K \leq i \leq N\} \) are calculated from (42) to (45) and the only unknown quantity is \( P_N^*(0) \) which can be determined from (23). We differentiate (23) with respect to \( \phi \) and set \( \phi = 0 \). Then we get

\[
P_{-K}^*(0) = -\mu P_{-K+1}^*(0). \tag{46} \]

To get \( P_{-K+1}^*(0) \), differentiate (24) to (28), respectively with respect to \( \phi \) and setting \( \phi = 0 \). Then we obtain

\[
P_i^{(1)}(0) = \frac{1}{\mu} \left[ P_i^*(0) + A^{(1)}(0) P_{i-1}(0) + \mu P_{i+1}^{(1)}(0) \right], \tag{47} \]
\[ P_0^{*}(0) = \frac{1}{\mu} \left[ P_0^{*}(0) + A^{*}(0)P_{-1}(0) + \mu \sum_{j=1}^{B} P_j^{*}(0) \right], \quad -1 \leq i \leq -K + 1, \tag{48} \]
\[ P_i^{*}(0) = \frac{1}{\mu} \left[ P_i^{*}(0) + A^{*}(0)P_{i-1}(0) + \mu P_{i+B}(0) \right], \quad N - B \leq i \leq 1, \tag{49} \]
\[ P_i^{*}(0) = \frac{1}{\mu} \left[ P_i^{*}(0) + A^{*}(0)P_{i-1}(0) \right], \quad N - 1 \leq i \leq N - B + 1, \tag{50} \]
\[ P_N^{*}(0) = \frac{1}{\mu} \left[ P_N^{*}(0) + A^{*}(0)(P_{N-1}(0) + P_N(0)) \right]. \tag{51} \]

As \( \{ P_i^{*}(0); -K + 1 \leq i \leq N \} \) can be obtained recursively from (47) to (51), and hence \( P_{-K}^{*}(0) \) can be calculated from (46). Since the probability of \( i \) units present in the system at an arbitrary instant is \( P_i^{*}(0) \) \( (-K \leq i \leq N) \), we can completely determine the steady-state probabilities at an arbitrary instant with the normalizing condition
\[ \sum_{i=-K}^{N} P_i^{*}(0) = 1. \tag{52} \]

Furthermore, the probability that \( i \) units in the system at an arrival instant \( Q_i \) \( (-K \leq i \leq N) \), is given by
\[ Q_i = \frac{P_i(0)}{\sum_{i=-K}^{N} P_i(0)}, \quad -K \leq i \leq N. \tag{53} \]

This quantity is obtained by using (31) to (36).

6. Some further results

In this section, we shall derive a few important results which will be further used to develop a relations between \( P_i \) \( (-K \leq i \leq N) \) and \( Q_i \) \( (-K \leq i \leq N) \).

**Lemma 6.1.**

\[ \mu P_N = P_{N-1}(0), \tag{54} \]
\[ \mu P_i = P_{i-1}(0) - P_i(0), \quad N - 1 \leq i \leq N - B + 1. \tag{55} \]
\[
\begin{align*}
\mu P_i &= \mu P_{i+B} + P_{i-1}(0) - P_i(0), & N - B \leq i \leq 1, \\
\mu P_0 &= \mu \sum_{j=1}^{B} P_j + P_{-1}(0) - P_0(0), \\
\mu P_i &= \mu P_{i+1} + P_{i-1}(0) - P_i(0), & -1 \leq i \leq -K + 1.
\end{align*}
\]

**Proof.** The above results can easily be obtained by using (31) and (42) to (45). \(\Box\)

**Lemma 6.2.** The relations between arbitrary instant, \(P_i\) \((-K \leq i \leq N)\) and arrival instant, \(Q_i\) \((-K \leq i \leq N)\) are given by

\[
\begin{align*}
P_N &= \rho B Q_{N-1}, \\
P_i &= \rho B [Q_{i-1} - Q_i], & N - 1 \leq i \leq N - B + 1, \\
P_i &= P_{i+B} + \rho B [Q_{i-1} - Q_i], & N - B \leq i \leq 1, \\
P_0 &= \sum_{j=1}^{B} P_j + \rho B [Q_{-1} - Q_0], \\
P_i &= P_{i+1} + \rho B [Q_{i-1} - Q_i], & -1 \leq i \leq -K + 1, \\
P_{-K} &= 1 - \sum_{i=-K+1}^{N} P_i.
\end{align*}
\]

**Proof.** Using (53) and \(\rho = 1/a_1 B \mu\) in (54) to (58), we obtain (59) to (63). Also adding (59) to (63), we get (64). \(\Box\)

7. An algorithm

In this section, we summarize the major steps necessary in the computation of the state probabilities at an arbitrary \(P_i \equiv P^*_i(0)\) \((-K \leq i \leq N)\) and arrival \(Q_i\) \((-K \leq i \leq N)\) instants.

**Step 1:** Set \(P_N = 1.0\).

**Step 2:** Compute \(P_{N-1}(0)\) and \(P_N(0)\) using (31) and (32).

**Step 3:** Compute \(P^*_N(\mu)\), \((0 \leq m \leq K + N - B - 1)\) using (37).

**Step 4:** Compute \(P_{i-1}(0)\), \((N - 1 \leq i \leq N - B + 1)\) using (33).

**Step 5:** Compute \(P^*_i(\mu)\), \((N - B + 1 \leq i \leq N - 1, 0 \leq m \leq i + N - 2)\) using (38).

**Step 6:** Compute \(P_{i-1}(0)\), \((N - B \leq i \leq 1)\) using (34).
Step 7: Compute $P_i^{(m)}(\mu)$, ($N - B \leq i \leq 1, 0 \leq m \leq i + N - 2$) using (39).

Step 8: Compute $P_{-1}(0)$ using (35).

Step 9: Compute $P_0^{(m)}(\mu)$, ($0 \leq m \leq K - 2$) using (40).

Step 10: Compute $P_{i-1}(0)$ and $P_i^{(m)}(\mu)$, ($-1 \leq i \leq -K + 2, 0 \leq m \leq i + K - 2$) using (36) and (41), respectively.

Step 11: Compute $P_{i-1}(0)$, ($-1 \leq i \leq -K + 1$) using (36).

Step 12: Compute $P_i^*(0)$, ($N - 1 \leq i \leq N - B + 1, N - B \leq i \leq 1$) using (42) and (43), respectively.

Step 13: Compute $P_0^*(0)$ using (44).

Step 14: Compute $P_i^*(0)$, ($-1 \leq i \leq -K + 1$) using (45).

Step 15: Compute $P_N^*(1)(0)$ using (51).

Step 16: Compute $P_i^*(1)(0)$; ($N - 1 \leq i \leq N - B + 1, N - B \leq i \leq 1$) using (50) and (51), respectively.

Step 17: Compute $P_0^*(1)(0)$ using (48).

Step 18: Compute $P_i^*(1)(0)$; ($-1 \leq i \leq -K + 1$) using (47).

Step 19: Compute $P_{-K}^*(0)$ using (46).

Step 20: Compute $SUM = \sum_{i=-K}^{N} P_i$.

Step 21: Compute $P_i = P_i/SUM$, ($-K \leq i \leq N$).

Step 22: Compute $Q_i$, ($-K \leq i \leq N$) using (53).

8. Performance measures

Since performance measures are the function of the distributions of the numbers in the system at various instants (arbitrary and arrival), they can now be easily be obtained and are defined below.

The probabilities of queue of customers (passengers), and of servers (taxis), and the queueing point is empty are given, respectively, by $P_{C_q} = \sum_{i=1}^{N} P_i$; $P_{S_q} = \sum_{i=-K}^{-1} P_i$; $P_0 = 1 - \sum_{i=-K}^{-1} P_i - \sum_{i=1}^{N} P_i$. 
The average numbers of customers and of servers in the system are given by, \( E[L_C] = \sum_{i=1}^{N} i \cdot P_i \); \( E[L_S] = \sum_{i=1}^{N} (-i) \cdot P_i \), respectively.

The probabilities of blocking of customers and of servers are given by, \( PBL_C = Q_N \); \( PBL_S = Q_K \), respectively, and the probability that the queueing point is busy is given by, \( \eta = 1 - P_0 \).

9. Numerical computations

We have implemented the recursive algorithm of section 7 for the variety of interarrival time distributions viz., exponential (\( M \)), Erlangian (\( E_r \)), deterministic (\( D \)), hyperexponential (\( HE_r \)) with balanced mean, uniform \( U(\alpha, \beta) \), generalized Erlangian (\( GE_r \)), Coxian (\( C_r \)), and generalized hyperexponential (\( GH_r \)), etc. The algorithm has been tested for various values of traffic intensity \( \rho = \frac{1}{a_1 B} \) where \( a_1 \) is the mean interarrival time of the distribution and \( B \) is the maximum batch size. In the \( HE_2 \) distribution used, we assume balanced mean (as in Allen [5]) with squared coefficient of variation (where \( C^2 > 1 \)).

While testing algorithm we have produced numerous tables but only a few are appended here. The validity of the numerical results was checked using the relation (30). The results for the distribution of the number of units in the system at an arbitrary \( (P_i) \) and arrival \( (Q_i) \) instants are given in self-explanatory Tables 1 and 2, respectively by taking \( K = 10 \), \( N = 10 \) with varying \( a_1 \), \( \mu \) and \( B \). In bottom rows of these tables we included some performance measures such as probabilities of queue of customers \( (P_{C_q}) \), and of servers \( (P_{S_q}) \); and average numbers of customers \( E[L_C] \), and of servers \( E[L_S] \); and probabilities of blocking of customers \( (PBL_C) \) and of servers \( (PBL_S) \), etc. The algorithm derived in this paper for the more general finite-capacity bulk-service double-ended queueing model is simple, elegant, and computationally efficient, and also works for high and low values of the queueing parameters.

Insert Tables 1 & 2
Table 1

Distribution of the number of units in the system at an arbitrary instant and performance measures for $K = 10$, $N = 10$

<table>
<thead>
<tr>
<th>$P_n$</th>
<th>$M$</th>
<th>$P_n$</th>
<th>$E_0$</th>
<th>$P_n$</th>
<th>$D$</th>
<th>$P_n$</th>
<th>$HE_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B = 3$</td>
<td></td>
<td>$B = 6$</td>
<td></td>
<td>$B = 5$</td>
<td></td>
<td>$B = 8$</td>
<td></td>
</tr>
<tr>
<td>$a_1 = 1.0$</td>
<td></td>
<td>$a_1 = 1.0$</td>
<td></td>
<td>$a_1 = 0.5$</td>
<td></td>
<td>$a_1 = 0.67$</td>
<td></td>
</tr>
<tr>
<td>$\mu = 1.45$</td>
<td></td>
<td>$\mu = 1.11$</td>
<td></td>
<td>$\mu = 1.33$</td>
<td></td>
<td>$\mu = 1.88$</td>
<td></td>
</tr>
<tr>
<td>$\beta_1 = 0.23, \beta_2 = 0.77$</td>
<td></td>
<td>$C^2_v = 1.8$</td>
<td></td>
<td>$\lambda_1 = 0.7, \lambda_2 = 2.3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P_{-10}$</td>
<td>0.313093</td>
<td>$P_{-10}$</td>
<td>0.114978</td>
<td>$P_{-10}$</td>
<td>0.000087</td>
<td>$P_{-10}$</td>
<td>0.228268</td>
</tr>
<tr>
<td>$P_{-9}$</td>
<td>0.217267</td>
<td>$P_{-9}$</td>
<td>0.168478</td>
<td>$P_{-9}$</td>
<td>0.000296</td>
<td>$P_{-9}$</td>
<td>0.138782</td>
</tr>
<tr>
<td>$P_{-8}$</td>
<td>0.149815</td>
<td>$P_{-8}$</td>
<td>0.141048</td>
<td>$P_{-8}$</td>
<td>0.000635</td>
<td>$P_{-8}$</td>
<td>0.117701</td>
</tr>
<tr>
<td>$P_{-7}$</td>
<td>0.103274</td>
<td>$P_{-7}$</td>
<td>0.118083</td>
<td>$P_{-7}$</td>
<td>0.001361</td>
<td>$P_{-7}$</td>
<td>0.099831</td>
</tr>
<tr>
<td>$P_{-6}$</td>
<td>0.071160</td>
<td>$P_{-6}$</td>
<td>0.098858</td>
<td>$P_{-6}$</td>
<td>0.002919</td>
<td>$P_{-6}$</td>
<td>0.084692</td>
</tr>
<tr>
<td>$P_{-5}$</td>
<td>0.049002</td>
<td>$P_{-5}$</td>
<td>0.082763</td>
<td>$P_{-5}$</td>
<td>0.006258</td>
<td>$P_{-5}$</td>
<td>0.071878</td>
</tr>
<tr>
<td>$P_{-4}$</td>
<td>0.033713</td>
<td>$P_{-4}$</td>
<td>0.069288</td>
<td>$P_{-4}$</td>
<td>0.013418</td>
<td>$P_{-4}$</td>
<td>0.061055</td>
</tr>
<tr>
<td>$P_{-3}$</td>
<td>0.023163</td>
<td>$P_{-3}$</td>
<td>0.058066</td>
<td>$P_{-3}$</td>
<td>0.028774</td>
<td>$P_{-3}$</td>
<td>0.051952</td>
</tr>
<tr>
<td>$P_{-2}$</td>
<td>0.015884</td>
<td>$P_{-2}$</td>
<td>0.048547</td>
<td>$P_{-2}$</td>
<td>0.061686</td>
<td>$P_{-2}$</td>
<td>0.044365</td>
</tr>
<tr>
<td>$P_{-1}$</td>
<td>0.010861</td>
<td>$P_{-1}$</td>
<td>0.040549</td>
<td>$P_{-1}$</td>
<td>0.131349</td>
<td>$P_{-1}$</td>
<td>0.038163</td>
</tr>
<tr>
<td>$P_0$</td>
<td>0.007396</td>
<td>$P_0$</td>
<td>0.033447</td>
<td>$P_0$</td>
<td>0.268056</td>
<td>$P_0$</td>
<td>0.033311</td>
</tr>
<tr>
<td>$P_1$</td>
<td>0.003179</td>
<td>$P_1$</td>
<td>0.016418</td>
<td>$P_1$</td>
<td>0.229427</td>
<td>$P_1$</td>
<td>0.015400</td>
</tr>
<tr>
<td>$P_2$</td>
<td>0.001295</td>
<td>$P_2$</td>
<td>0.006033</td>
<td>$P_2$</td>
<td>0.121025</td>
<td>$P_2$</td>
<td>0.007507</td>
</tr>
<tr>
<td>$P_3$</td>
<td>0.000530</td>
<td>$P_3$</td>
<td>0.002217</td>
<td>$P_3$</td>
<td>0.063796</td>
<td>$P_3$</td>
<td>0.003647</td>
</tr>
<tr>
<td>$P_4$</td>
<td>0.000218</td>
<td>$P_4$</td>
<td>0.000815</td>
<td>$P_4$</td>
<td>0.033606</td>
<td>$P_4$</td>
<td>0.001772</td>
</tr>
<tr>
<td>$P_5$</td>
<td>0.000086</td>
<td>$P_5$</td>
<td>0.000299</td>
<td>$P_5$</td>
<td>0.017897</td>
<td>$P_5$</td>
<td>0.000861</td>
</tr>
<tr>
<td>$P_6$</td>
<td>0.000037</td>
<td>$P_6$</td>
<td>0.000110</td>
<td>$P_6$</td>
<td>0.009445</td>
<td>$P_6$</td>
<td>0.000419</td>
</tr>
<tr>
<td>$P_7$</td>
<td>0.000016</td>
<td>$P_7$</td>
<td>0.000040</td>
<td>$P_7$</td>
<td>0.004849</td>
<td>$P_7$</td>
<td>0.000203</td>
</tr>
<tr>
<td>$P_8$</td>
<td>0.000007</td>
<td>$P_8$</td>
<td>0.000015</td>
<td>$P_8$</td>
<td>0.002490</td>
<td>$P_8$</td>
<td>0.000099</td>
</tr>
<tr>
<td>$P_9$</td>
<td>0.000003</td>
<td>$P_9$</td>
<td>0.000005</td>
<td>$P_9$</td>
<td>0.001278</td>
<td>$P_9$</td>
<td>0.000048</td>
</tr>
<tr>
<td>$P_{10}$</td>
<td>0.000002</td>
<td>$P_{10}$</td>
<td>0.000003</td>
<td>$P_{10}$</td>
<td>0.001349</td>
<td>$P_{10}$</td>
<td>0.000045</td>
</tr>
<tr>
<td>SUM</td>
<td>1.000000</td>
<td>SUM</td>
<td>1.000000</td>
<td>SUM</td>
<td>1.000000</td>
<td>SUM</td>
<td>1.000000</td>
</tr>
<tr>
<td>$P_{C_1}$</td>
<td>0.005372</td>
<td>$P_{C_4}$</td>
<td>0.025956</td>
<td>$P_{C_4}$</td>
<td>0.485161</td>
<td>$P_{C_4}$</td>
<td>0.030003</td>
</tr>
<tr>
<td>$P_{S_4}$</td>
<td>0.987232</td>
<td>$P_{S_5}$</td>
<td>0.940597</td>
<td>$P_{S_5}$</td>
<td>0.246782</td>
<td>$P_{S_5}$</td>
<td>0.936687</td>
</tr>
<tr>
<td>$E[L_{C_1}]$</td>
<td>0.009090</td>
<td>$E[L_{C_4}]$</td>
<td>0.041030</td>
<td>$E[L_{C_4}]$</td>
<td>1.022293</td>
<td>$E[L_{C_4}]$</td>
<td>0.058366</td>
</tr>
<tr>
<td>$E[L_{S_4}]$</td>
<td>7.926710</td>
<td>$E[L_{S_5}]$</td>
<td>6.216819</td>
<td>$E[L_{S_5}]$</td>
<td>0.461656</td>
<td>$E[L_{S_5}]$</td>
<td>6.566651</td>
</tr>
<tr>
<td>$PBL_{C_1}$</td>
<td>0.000002</td>
<td>$PBL_{C_4}$</td>
<td>0.000002</td>
<td>$PBL_{C_4}$</td>
<td>0.000949</td>
<td>$PBL_{C_4}$</td>
<td>0.000054</td>
</tr>
<tr>
<td>$PBL_{S_4}$</td>
<td>0.314879</td>
<td>$PBL_{S_5}$</td>
<td>0.187197</td>
<td>$PBL_{S_5}$</td>
<td>0.000197</td>
<td>$PBL_{S_5}$</td>
<td>0.173478</td>
</tr>
</tbody>
</table>
Table 2
Distribution of the number of units in the system at an arrival instant for
$K = 10, N = 10$

<table>
<thead>
<tr>
<th>$Q_0$</th>
<th>$E_0$</th>
<th>$Q_0$</th>
<th>$E_0$</th>
<th>$Q_0$</th>
<th>$E_0$</th>
<th>$Q_0$</th>
<th>$E_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{B=3}$</td>
<td>$\mu_{B=3}$</td>
<td>$M_{B=6}$</td>
<td>$\mu_{B=6}$</td>
<td>$M_{B=5}$</td>
<td>$\mu_{B=5}$</td>
<td>$M_{B=8}$</td>
<td>$\mu_{B=8}$</td>
</tr>
<tr>
<td>$a_1 = 1.0$</td>
<td>$\mu = 1.45$</td>
<td>$a_1 = 1.0$</td>
<td>$\mu = 1.11$</td>
<td>$a_1 = 0.5$</td>
<td>$\mu = 1.33$</td>
<td>$a_1 = 0.67$</td>
<td>$\mu = 1.88$</td>
</tr>
<tr>
<td>$C_0^2 = 1.8$</td>
<td>$\beta_1 = 0.23, \beta_2 = 0.77$</td>
<td>$\lambda_1 = 0.7, \lambda_2 = 2.3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q_{-10}$</td>
<td>0.314879</td>
<td>$Q_{-10}$</td>
<td>0.187197</td>
<td>$Q_{-10}$</td>
<td>0.000197</td>
<td>$Q_{-10}$</td>
<td>0.173478</td>
</tr>
<tr>
<td>$Q_{-9}$</td>
<td>0.217124</td>
<td>$Q_{-9}$</td>
<td>0.156719</td>
<td>$Q_{-9}$</td>
<td>0.000423</td>
<td>$Q_{-9}$</td>
<td>0.147126</td>
</tr>
<tr>
<td>$Q_{-8}$</td>
<td>0.149672</td>
<td>$Q_{-8}$</td>
<td>0.131204</td>
<td>$Q_{-8}$</td>
<td>0.000908</td>
<td>$Q_{-8}$</td>
<td>0.124789</td>
</tr>
<tr>
<td>$Q_{-7}$</td>
<td>0.103131</td>
<td>$Q_{-7}$</td>
<td>0.109842</td>
<td>$Q_{-7}$</td>
<td>0.001946</td>
<td>$Q_{-7}$</td>
<td>0.105865</td>
</tr>
<tr>
<td>$Q_{-6}$</td>
<td>0.071017</td>
<td>$Q_{-6}$</td>
<td>0.091959</td>
<td>$Q_{-6}$</td>
<td>0.004172</td>
<td>$Q_{-6}$</td>
<td>0.089847</td>
</tr>
<tr>
<td>$Q_{-5}$</td>
<td>0.048859</td>
<td>$Q_{-5}$</td>
<td>0.076987</td>
<td>$Q_{-5}$</td>
<td>0.008945</td>
<td>$Q_{-5}$</td>
<td>0.076318</td>
</tr>
<tr>
<td>$Q_{-4}$</td>
<td>0.033570</td>
<td>$Q_{-4}$</td>
<td>0.064451</td>
<td>$Q_{-4}$</td>
<td>0.019182</td>
<td>$Q_{-4}$</td>
<td>0.064940</td>
</tr>
<tr>
<td>$Q_{-3}$</td>
<td>0.023020</td>
<td>$Q_{-3}$</td>
<td>0.053942</td>
<td>$Q_{-3}$</td>
<td>0.041124</td>
<td>$Q_{-3}$</td>
<td>0.055456</td>
</tr>
<tr>
<td>$Q_{-2}$</td>
<td>0.015741</td>
<td>$Q_{-2}$</td>
<td>0.045054</td>
<td>$Q_{-2}$</td>
<td>0.087566</td>
<td>$Q_{-2}$</td>
<td>0.047704</td>
</tr>
<tr>
<td>$Q_{-1}$</td>
<td>0.010718</td>
<td>$Q_{-1}$</td>
<td>0.037163</td>
<td>$Q_{-1}$</td>
<td>0.178704</td>
<td>$Q_{-1}$</td>
<td>0.041639</td>
</tr>
<tr>
<td>$Q_0$</td>
<td>0.007253</td>
<td>$Q_0$</td>
<td>0.028769</td>
<td>$Q_0$</td>
<td>0.310501</td>
<td>$Q_0$</td>
<td>0.037366</td>
</tr>
<tr>
<td>$Q_1$</td>
<td>0.002961</td>
<td>$Q_1$</td>
<td>0.010572</td>
<td>$Q_1$</td>
<td>0.163845</td>
<td>$Q_1$</td>
<td>0.018197</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>0.001209</td>
<td>$Q_2$</td>
<td>0.003885</td>
<td>$Q_2$</td>
<td>0.086395</td>
<td>$Q_2$</td>
<td>0.008869</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>0.000494</td>
<td>$Q_3$</td>
<td>0.001427</td>
<td>$Q_3$</td>
<td>0.045524</td>
<td>$Q_3$</td>
<td>0.004310</td>
</tr>
<tr>
<td>$Q_4$</td>
<td>0.000202</td>
<td>$Q_4$</td>
<td>0.000525</td>
<td>$Q_4$</td>
<td>0.023972</td>
<td>$Q_4$</td>
<td>0.002095</td>
</tr>
<tr>
<td>$Q_5$</td>
<td>0.000086</td>
<td>$Q_5$</td>
<td>0.000193</td>
<td>$Q_5$</td>
<td>0.012940</td>
<td>$Q_5$</td>
<td>0.001018</td>
</tr>
<tr>
<td>$Q_6$</td>
<td>0.000037</td>
<td>$Q_6$</td>
<td>0.000071</td>
<td>$Q_6$</td>
<td>0.006644</td>
<td>$Q_6$</td>
<td>0.000495</td>
</tr>
<tr>
<td>$Q_7$</td>
<td>0.000016</td>
<td>$Q_7$</td>
<td>0.000026</td>
<td>$Q_7$</td>
<td>0.003411</td>
<td>$Q_7$</td>
<td>0.000240</td>
</tr>
<tr>
<td>$Q_8$</td>
<td>0.000007</td>
<td>$Q_8$</td>
<td>0.000009</td>
<td>$Q_8$</td>
<td>0.001751</td>
<td>$Q_8$</td>
<td>0.000117</td>
</tr>
<tr>
<td>$Q_9$</td>
<td>0.000003</td>
<td>$Q_9$</td>
<td>0.000003</td>
<td>$Q_9$</td>
<td>0.000899</td>
<td>$Q_9$</td>
<td>0.000057</td>
</tr>
<tr>
<td>$Q_{10}$</td>
<td>0.000002</td>
<td>$Q_{10}$</td>
<td>0.000002</td>
<td>$Q_{10}$</td>
<td>0.000949</td>
<td>$Q_{10}$</td>
<td>0.000054</td>
</tr>
</tbody>
</table>

SUM 1.000000 1.000000 SUM 1.000000 SUM 1.000000
References


