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Sudha Jain
Department of Statistics
University of Toronto

and

T. S. S. Srinivasa Rao
Institute for Systems Studies
and Analysis
Defence R & D Organization
Metcalfe House

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Department of Statistics

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Sudha Jain[a,*,1] , **T. S. S. Srinivasa Rao**[b,2]

[a]*Department of Statistics, University of Tornto, Toronto, Ontario, M5S 3G3, Canada*

[b]*Institute for Systems Studies and Analyses, Defence R & D Organisation, Metcalfe House, Delhi-110 054, India.*

Abstract

This paper studies the steady-state analysis of a finite-capacity bulk-service double-ended queueing system in which either customers wait in queue for service or idle servers queue up for customers. The interarrival times of customers are arbitrarily distributed and the interarrival rates of servers are exponentially distributed, and that the servers 'serve' the customers in batches. Using the supplementary variable technique we develop a recursive method to compute the steady-state probabilities of the number of units in the system at an arbitrary and arrival instants. Besides obtaining the state probabilities at various instants, other practically interesting performance measures have also been discussed. Some numerical results are presented.

Keywords: Steady-state analysis; finite-capacity; bulk-service; double-ended queue; supplementary variable technique.

[1]Corresponding author. E-mail address: jainsu@utstat.toronto.edu (Sudha Jain)

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1. Introduction

In recent years there has been a considerable interest in computing state probabilities of various queueing models. Since performance measures are the function of state probabilities then their exact evaluation is important. Mostly a system designer/practitioner is interested in probability distributions of the number of units in the system at various instants which provide more information than few moments. In this paper, we consider a finite-capacity bulk-service double ended queue in which customers and servers wait for each other, with the later serving customers in batches. The arrivals of customers and servers are considered as general and Poisson, respectively. A practicle example of this type of queueing system is a taxi-stand (queueing point) where either passengers (customers) are waiting for taxis (servers) or taxis are waiting for passengers. This type of situations can be realised in many areas such as transportation systems, inventory processes, production and flexible manufacturing systems. Using generating function method, Srivastava and Kashyap [1] have obtained the transient solution of this bulk-service double-ended queueing model. However, the analytical solution given by them in terms of summation of the integrals of Bessel functions is very complex and numerical results are difficult to compute. The similar remark has also been made by Chaudhry and Templeton [2]. Hlynka and Sheahan [3] gave a control policy on the Poisson arrival mechanism for the double-ended queue and derived the time-dependent solution. Sharma [4] studied the transient behaviour of a Markovian double-ended queue with single-service in which the arrivals of customers and servers are distributed exponentially having parameters λ and μ, respectively. To the authors' best knowledge, the steady-state analysis of a finite-capacity bulk-service double-ended queueing model has never been treated.

The purpose of this research article is to provide a simple and easily implementable algorithm for the steady-state analysis of a more general finite-capacity bulk-service double-ended queueing model. The approach we take here for the analysis of the problem is the supplementary variable technique using which a recursive method has been developed. The method is straightforward to obtain the state probabilities of the number of units in the system at the instant of an arrival and at an arbitrary instant, and practically interesting performance measures such as probabilities of queue of customers, and of servers; average numbers of customers and of servers in the system; and probabilities of blocking of customers and of servers, etc.

2. Description of the model

Consider a finite-capacity bulk-service double-ended queueing system in which the arrivals of customers and servers are taken as general and Poisson, respectively, and that the servers 'serve' the customers in batches. The maximum numbers of customers and servers in the system including the ones being served are N and K, respectively. For linguistic convenience we will describe the queueing system in terms of a taxi-stand. Let passengers (customers) arrive singly at a taxi-stand (queueing point) and form a queue if taxis (servers) are not available. The interarrival times of passengers being arbitrarily distributed having a probability density function $a(x)$, $(x \geq 0)$. Taxis arrive at the stand in a exponentially distributed manner with mean rate μ. The maximum number of seats in each taxi is, B. An arriving taxi, on finding a queue of customers presents, departs taking a batch of maximum size B customers from the queue; if it finds less than B customers are waiting, then it takes all of them in a batch for service; otherwise, it waits at the stand if there are no passengers present in the queue. Thus, at any time there is either a queue of passengers or one of taxis or of neither. Assume that the numbers on the negative-axis stand for the taxis waiting for passengers whereas the numbers on the positive-axis denote the passengers waiting for the taxis, and the system is at the position i $(-K \leq i \leq N)$ where K and N are the positive integers. We allow i to assume the following values so that

(i) when $i > 0$ it denotes the number (i) of customers are waiting at time t,

(ii) when $i = 0$, neither customers nor servers are waiting at time t.

(iii) when $i < 0$ it signifies that number $(-i)$ of servers are waiting at time t.

3. Time-dependent system equations and definitions

Let $P_i(x,t)\,\Delta x$ $(-K \le i \le N)$ be the probability that at time t, there are i customers in the system and the remaining interarrival time for the next arrival lies in the interval $(x, x+\Delta x)$. Therefore, the state of the system at time t is defined by $\{\xi(t), X(t);\ t \ge 0\}$ which is Markovian in continuous time. Define

$$P_i(x,t)\Delta x = Pr[\xi(t) = i,\ x < X(t) \le x + \Delta x], \qquad (1)$$
$$x \ge 0,\ -K \le i \le N.$$

Relating the states of the system at time t and $t+\Delta t$, we obtain the following Chapman-Kolmogorov forward equations

$$P_{-K}(x-\Delta t, t+\Delta t) = P_{-K}(x,t) + P_{-K+1}(x,t)\mu\Delta t + O(\Delta t), \qquad (2)$$

$$P_i(x-\Delta t, t+\Delta t) = [1-\mu\Delta t]P_i(x,t) + a(x)\ P_{i-1}(0,t)\Delta t + P_{i+1}(x,t)\mu\Delta t + O(\Delta t), \qquad -K+1 \le i \le -1, \qquad (3)$$

$$P_0(x-\Delta t, t+\Delta t) = [1-\mu\Delta t]P_0(x,t) + P_{-1}(0,t)\ a(x)\Delta t + \sum_{j=1}^{B} P_j(x,t)\mu\Delta t + O(\Delta t), \qquad (4)$$

$$P_i(x-\Delta t, t+\Delta t) = [1-\mu\Delta t]P_i(x,t) + P_{i-1}(0,t)\ a(x)\Delta t + P_{i+B}(x,t)\mu\Delta t + O(\Delta t), \qquad 1 \le i \le N-B, \qquad (5)$$

$$P_i(x-\Delta t, t+\Delta t) = [1-\mu\Delta t]P_i(x,t) + P_{i-1}(0,t)\ a(x)\Delta t + O(\Delta t), \qquad N-B+1 \le i \le N-1, \qquad (6)$$

$$P_N(x-\Delta t, t+\Delta t) = [1-\mu\Delta t]P_N(x,t) + P_{N-1}(0,t)\ a(x)\Delta t + P_N(0,t)\ a(x)\Delta t + O(\Delta t). \qquad (7)$$

Expand the left hand side of (2) to (7) in powers of Δt and let $\Delta t \to 0$, we have the following partial differential equations of the system

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right) P_{-K}(x,t) = \mu P_{-K+1}(x,t), \quad (8)$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right) P_i(x,t) = -\mu P_i(x,t)+a(x)\, P_{i-1}(0,t)+\mu\, P_{i+1}(x,t), \quad (9)$$

$$-K+1 \leq i \leq -1,$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right) P_0(x,t) = -\mu P_0(x,t)+a(x)\, P_{-1}(0,t)+\mu \sum_{j=1}^{B} P_j(x,t), \quad (10)$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right) P_i(x,t) = -\mu P_i(x,t)+a(x)\, P_{i-1}(0,t)+\mu\, P_{i+B}(x,t), \quad (11)$$

$$1 \leq i \leq N-B,$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right) P_i(x,t) = -\mu P_i(x,t)+a(x)\, P_{i-1}(0,t), \quad (12)$$

$$N-B+1 \leq i \leq N-1,$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right) P_N(x,t) = -\mu P_N(x,t)+a(x)\, [P_{N-1}(0,t)+P_N(0,t)]. \quad (13)$$

4. Steady-state system equations

As we will restrict ourselves now to study the steady-state, we let $t \to \infty$ and thus the derivatives with respect to t tend to zero in (8) to (13). Let $P_i(x) = \lim_{t\to\infty} P_i(x,t)$, $-K \leq i \leq N$, $x \geq 0$ which gives the steady-state distribution of $\{\xi(t), X(t);\ t \geq 0\}$.

$$-\frac{d}{dx}P_{-K}(x) = \mu P_{-K+1}(x), \quad (14)$$

$$-\frac{d}{dx}P_i(x) = -\mu P_i(x)+a(x)\, P_{i-1}(0)+\mu\, P_{i+1}(x), \quad (15)$$

$$-K+1 \leq i \leq -1,$$

$$-\frac{d}{dx}P_0(x) = -\mu P_0(x)+a(x)\, P_{-1}(0)+\mu \sum_{j=1}^{B} P_j(x), \quad (16)$$

$$-\frac{d}{dx}P_i(x) = -\mu P_i(x)+a(x)\, P_{i-1}(0)+\mu\, P_{i+B}(x), \quad (17)$$

$$1 \leq i \leq N-B,$$

$$-\frac{d}{dx}P_i(x) = -\mu P_i(x)+a(x)\, P_{i-1}(0), \quad (18)$$

$$N-B+1 \leq i \leq N-1,$$

$$-\frac{d}{dx}P_N(x) = -\mu P_N(x) + a(x)\,[P_{N-1}(0) + P_N(0)]. \tag{19}$$

Define the Laplace transforms of $a(x)$, $P_i(x)$ and $\frac{dP_i(x)}{dx}$ are given by

$$A^*(\phi) = \int_0^\infty e^{-\phi x}\, a(x)dx, \tag{20}$$

$$P_i^*(\phi) = \int_0^\infty e^{-\phi x}\, P_i(x)dx, \quad -K \le i \le N, \tag{21}$$

$$\int_0^\infty e^{-\phi x}\, \frac{dP_i(x)}{dx}\, dx = \phi P_i^*(\phi) - P_i(0). \tag{22}$$

Multiplying (14) to (19) by $e^{-\phi x}$ and integrating with respect to x from 0 to ∞, it then follows that

$$\phi P_{-K}^*(\phi) = P_{-K}(0) - \mu P_{-K+1}^*(\phi), \tag{23}$$

$$(\phi - \mu)P_i^*(\phi) = P_i(0) - A^*(\phi)P_{i-1}(0) - \mu P_{i+1}^*(\phi), \tag{24}$$

$$-K + 1 \le i \le -1,$$

$$(\phi - \mu)P_0^*(\phi) = P_0(0) - A^*(\phi)P_{-1}(0) - \mu \sum_{j=1}^{B} P_j^*(\phi), \tag{25}$$

$$(\phi - \mu)P_i^*(\phi) = P_i(0) - A^*(\phi)P_{i-1}(0) - \mu P_{i+B}^*(\phi), \tag{26}$$

$$1 \le i \le N - B,$$

$$(\phi - \mu)P_i^*(\phi) = P_i(0) - A^*(\phi)P_{i-1}(0), \tag{27}$$

$$N - B + 1 \le i \le N - 1,$$

$$(\phi - \mu)P_N^*(\phi) = P_N(0) - A^*(\phi)[P_{N-1}(0) + P_N(0)]. \tag{28}$$

Lemma 4.1.

$$\sum_{i=-K}^{N} P_i^*(\phi) = \frac{1 - A^*(\phi)}{\phi} \sum_{i=-K}^{N} P_i(0). \tag{29}$$

Proof. Adding (23) to (28), we get (29). □

Lemma 4.2.

$$\sum_{i=-K}^{N} P_i(0) = \frac{1}{a_1}, \tag{30}$$

where $a_1 = -A^{*(1)}(0) = -\left[\frac{d}{d\phi}A^*(\phi)\right]_{\phi=0}$ is the mean interarrival time of the distribution. Relation (30) can be used as a check for the numerical calculations.

Proof. Letting $\phi \to 0$ in (29) and using the normalizing condition $\sum_{i=-K}^{N} P_i^*(0) = 1$, we get (30). □

5. Recursive method to compute steady-state probabilities

Our objective is to compute the steady-state distribution of the number of units in the system at an arbitrary $\{P_i \equiv P_i^*(0);\ -K \le i \le N\}$, and arrival $\{Q_i;\ -K \le i \le N\}$ instants. To achieve our goal we have to solve the equations (23) to (28) recursively starting from the last equation and working backward.

Inserting $\phi = 0$ and $\phi = \mu$, respectively in (28) , we get

$$P_{N-1}(0) = \mu P_N^*(0), \tag{31}$$

$$P_N(0) = \frac{A^*(\mu) P_{N-1}(0)}{[1 - A^*(\mu)]}. \tag{32}$$

Next setting $\phi = \mu$, respectively in (27), (26), (25), and (24). Then we have

$$P_{i-1}(0) = \frac{P_i(0)}{A^*(\mu)}, \qquad N-1 \le i \le N-B+1, \tag{33}$$

$$P_{i-1}(0) = \frac{1}{A^*(\mu)}\left[P_i(0) - \mu P_{i+B}^*(\mu)\right], \qquad N-B \le i \le 1. \tag{34}$$

$$P_{-1}(0) = \frac{1}{A^*(\mu)}\left[P_0(0) - \mu \sum_{j=1}^{B} P_j^*(\mu)\right], \tag{35}$$

$$P_{i-1}(0) = \frac{1}{A^*(\mu)}\left[P_i(0) - \mu P_{i+1}^*(\mu)\right], \qquad -1 \le i \le -K+1. \tag{36}$$

Thus, we need $\{P_n^*(\mu);\ -K+1 \le n \le N-B\}$ in order to get the explicit expressions for $\{P_{i-1}(0);\ -K+1 \le i \le N-B\}$. Differentiate (28), (27), (26), (25), and (24), respectively $m+1$ times with respect to ϕ and setting $\phi = \mu$. It follows rhat

$$P_N^{*(m)}(\mu) = -\frac{A^{*(m+1)}(\mu)}{m+1}\left[P_{N-1}(0) + P_N(0)\right], \quad 0 \le m \le K+N-B-1, \tag{37}$$

$$P_i^{*(m)}(\mu) = -\frac{A^{*(m+1)}(\mu)}{m+1} P_{i-1}(0), \tag{38}$$

$$N-B+1 \le i \le N-1, \quad 0 \le m \le i+N-2,$$

$$P_i^{*(m)}(\mu) = -\frac{1}{m+1}\left[A^{*(m+1)}(\mu)P_{i-1}(0) + \mu P_{i+B}^{*(m+1)}(\mu)\right], \tag{39}$$

$$N-B \le i \le 1, \quad 0 \le m \le i+N-2,$$

$$P_0^{*(m)}(\mu) = -\frac{1}{m+1}\left[A^{*(m+1)}(\mu)P_{-1}(0) + \mu \sum_{j=1}^{B} P_j^{*(m+1)}(\mu)\right], \tag{40}$$

$$0 \le m \le K-2,$$

$$P_i^{*(m)}(\mu) = -\frac{1}{m+1}\left[A^{*(m+1)}(\mu)P_{i-1}(0) + \mu P_{i+1}^{*(m+1)}(\mu)\right], \tag{41}$$

$$-1 \le i \le -K+2, \quad 0 \le m \le i+K-2,$$

where $A^{*(m)}(\mu) = \left[\frac{d^m}{d\phi^m}A^*(\phi)\right]_{\phi=\mu}$ and $P_n^{*(m)}(\mu)$ is also defined in the same way. Hence $\{P_i(0);\ -K \le i \le N\}$ can be obtained in terms of $P_N \equiv P_N^*(0)$ recursively from (31) to (41).

Setting $\phi = 0$ respectively in (27), (26), (25), and (24), we have

$$P_i^*(0) = \frac{1}{\mu}\left[P_{i-1}(0) - P_i(0)\right], \qquad N-1 \le i \le N-B+1, \tag{42}$$

$$P_i^*(0) = \frac{1}{\mu}\left[\mu P_{i+B}^*(0) + P_{i-1}(0) - P_i(0)\right], \qquad N-B \le i \le 1, \tag{43}$$

$$P_0^*(0) = \frac{1}{\mu}\left[\mu \sum_{j=1}^{B} P_j^*(0) + P_{-1}(0) - P_0(0)\right], \tag{44}$$

$$P_i^*(0) = \frac{1}{\mu}\left[\mu P_{i+1}^*(0) + P_{i-1}(0) - P_i(0)\right], \qquad -1 \le i \le -K+1. \tag{45}$$

Therefore, $\{P_i \equiv P_i^*(0);\ -K \le i \le N\}$ are calculated from (42) to (45) and the only unknown quantity is $P_N^*(0)$ which can be determined from (23). We differentiate (23) with respect to ϕ and set $\phi = 0$. Then we get

$$P_{-K}^*(0) = -\mu P_{-K+1}^{*(1)}(0). \tag{46}$$

To get $P_{-K+1}^{*(1)}(0)$, differentiate (24) to (28), respectively with respect to ϕ and setting $\phi = 0$. Then we obtain

$$P_i^{*(1)}(0) = \frac{1}{\mu}\left[P_i^*(0) + A^{*(1)}(0)P_{i-1}(0) + \mu P_{i+1}^{*(1)}(0)\right], \tag{47}$$

$$-1 \leq i \leq -K+1,$$

$$P_0^{*(1)}(0) = \frac{1}{\mu}\left[P_0^*(0) + A^{*(1)}(0)P_{-1}(0) + \mu\sum_{j=1}^{B} P_j^{*(1)}(0)\right], \qquad (48)$$

$$P_i^{*(1)}(0) = \frac{1}{\mu}\left[P_i^*(0) + A^{*(1)}(0)P_{i-1}(0) + \mu P_{i+B}^{*(1)}(0)\right], \qquad (49)$$

$$N-B \leq i \leq 1,$$

$$P_i^{*(1)}(0) = \frac{1}{\mu}\left[P_i^*(0) + A^{*(1)}(0)P_{i-1}(0)\right], \qquad (50)$$

$$N-1 \leq i \leq N-B+1,$$

$$P_N^{*(1)}(0) = \frac{1}{\mu}\left[P_i^*(0) + A^{*(1)}(0)(P_{N-1}(0) + P_N(0))\right]. \qquad (51)$$

As $\{P_i^{*(1)}(0);\ -K+1 \leq i \leq N\}$ can be obtained recursively from (47) to (51), and hence $P^*_{-K}(0)$ can be calculated from (46). Since the probability of i units present in the system at an arbitrary instant is $P_i \equiv P_i^*(0)$ $(-K \leq i \leq N)$, we can completely determine the steady-state probabilities at an arbitrary instant with the normalizing condition

$$\sum_{i=-K}^{N} P_i^*(0) = 1. \qquad (52)$$

Furthermore, the probability that i units in the system at an arrival instant Q_i $(-K \leq i \leq N)$, is given by

$$Q_i = \frac{P_i(0)}{\sum_{r=-K}^{N} P_r(0)}, \qquad -K \leq i \leq N. \qquad (53)$$

This quantity is obtained by using (31) to (36).

6. Some further results

In this section, we shall derive a few important results which will be further used to develop a relations between P_i $(-K \leq i \leq N)$ and Q_i $(-K \leq i \leq N)$.

Lemma 6.1.

$$\mu P_N = P_{N-1}(0), \qquad (54)$$

$$\mu P_i = P_{i-1}(0) - P_i(0), \qquad N-1 \leq i \leq N-B+1, \qquad (55)$$

$$\mu P_i = \mu P_{i+B} + P_{i-1}(0) - P_i(0), \qquad N-B \le i \le 1, \tag{56}$$

$$\mu P_0 = \mu \sum_{j=1}^{B} P_j + P_{-1}(0) - P_0(0), \tag{57}$$

$$\mu P_i = \mu P_{i+1} + P_{i-1}(0) - P_i(0), \qquad -1 \le i \le -K+1. \tag{58}$$

Proof. The above results can easily be obtained by using (31) and (42) to (45). □

Lemma 6.2. The relations between arbitrary instant, P_i $(-K \le i \le N)$ and arrival instant, Q_i $(-K \le i \le N)$ are given by

$$P_N = \rho B\, Q_{N-1}, \tag{59}$$

$$P_i = \rho B\,[Q_{i-1} - Q_i], \qquad N-1 \le i \le N-B+1, \tag{60}$$

$$P_i = P_{i+B} + \rho B\,[Q_{i-1} - Q_i], \qquad N-B \le i \le 1, \tag{61}$$

$$P_0 = \sum_{j=1}^{B} P_j + \rho B\,[Q_{-1} - Q_0], \tag{62}$$

$$P_i = P_{i+1} + \rho B\,[Q_{i-1} - Q_i], \qquad -1 \le i \le -K+1, \tag{63}$$

$$P_{-K} = 1 - \sum_{i=-K+1}^{N} P_i. \tag{64}$$

Proof. Using (53) and $\rho = 1/a_1 B\mu$ in (54) to (58), we obtain (59) to (63). Also adding (59) to (63), we get (64). □

7. An algorithm

In this section, we summarize the major steps necessary in the computation of the state probabilities at an arbitrary $P_i \equiv P_i^*(0)$ $(-K \le i \le N)$ and arrival Q_i $(-K \le i \le N)$ instants.

Step 1: Set $P_N = 1.0$.

Step 2: Compute $P_{N-1}(0)$ and $P_N(0)$ using (31) and (32).

Step 3: Compute $P_N^{*(m)}(\mu)$, $(0 \le m \le K+N-B-1)$ using (37).

Step 4: Compute $P_{i-1}(0)$, $(N-1 \le i \le N-B+1)$ using (33).

Step 5: Compute $P_i^{*(m)}(\mu)$, $(N-B+1 \le i \le N-1,\ 0 \le m \le i+N-2)$ using (38).

Stpe 6: Compute $P_{i-1}(0)$, $(N-B \le i \le 1)$ using (34).

Step 7: Compute $P_i^{*(m)}(\mu)$, $(N-B \leq i \leq 1,\ 0 \leq m \leq i+N-2)$ using (39).

Stpe 8: Compute $P_{-1}(0)$ using (35).

Step 9: Compute $P_0^{*(m)}(\mu)$, $(0 \leq m \leq K-2)$ using (40).

Step 10: Compute $P_{i-1}(0)$ and $P_i^{*(m)}(\mu)$, $(-1 \leq i \leq -K+2,\ 0 \leq m \leq i+K-2)$ using (36) and (41), respectively.

Stpe 11: Compute $P_{i-1}(0)$, $(-1 \leq i \leq -K+1)$ using (36).

Step 12: Compute $P_i^*(0)$, $(N-1 \leq i \leq N-B+1,\ N-B \leq i \leq 1)$ using (42) and (43), respectively.

Step 13: Compute $P_0^*(0)$ using (44).

Step 14: Compute $P_i^*(0)$, $(-1 \leq i \leq -K+1)$ using (45).

Step 15: Compute $P_N^{*(1)}(0)$ using (51).

Step 16: Compute $P_i^{*(1)}(0)$; $(N-1 \leq i \leq N-B+1,\ N-B \leq i \leq 1)$ using (50) and (51), respectively.

Step 17: Compute $P_0^{*(1)}(0)$ using (48).

Step 18: Compute $P_i^{*(1)}(0)$; $(-1 \leq i \leq -K+1)$ using (47).

Step 19: Compute $P_{-K}^*(0)$ using (46).

Step 20: Compute $SUM = \sum_{i=-K}^{N} P_i$.

Step 21: Compute $P_i = P_i/SUM$, $(-K \leq i \leq N)$.

Step 22: Compute Q_i, $(-K \leq i \leq N)$ using (53).

8. Performance measures

Since performance measures are the function of the distributions of the numbers in the system at various instants (arbitrary and arrival), they can now be easily be obtained and are defined below.

The probabilities of queue of customers (passengers), and of servers (taxis), and the queueing point is empty are given, respectively, by $P_{C_q} = \sum_{i=1}^{N} P_i$; $P_{S_q} = \sum_{i=-K}^{-1} P_i$; $P_0 = 1 - \sum_{i=-K}^{-1} P_i - \sum_{1=1}^{N} P_i$.

The average numbers of customers and of servers in the system are given by, $E[L_C] = \sum_{i=1}^{N} i.P_i$; $E[L_S] = \sum_{i=-K}^{-1} (-i).P_i$, respectively.

The Probabilities of blocking of customers and of servers are given by, $PBL_C = Q_N$; $PBL_S = Q_{-K}$, respectively, and the probability that the queueing point is busy is given by, $\eta = 1 - P_0$.

9. Numerical computations

We have implemented the recursive algorithm of section 7 for the variety of interarrival time distributions viz., exponential (M), Erlangian (E_r), deterministic (D), hyperexponential (HE_r) with balanced mean, uniform $U(\alpha, \beta)$, generalized Erlangian (GE_r), Coxian (C_r), and generalized hyperexponential (GH_r), etc. The algorithm has been tested for various values of traffic intensity ($\rho = \frac{1}{a_1 B \mu}$) where a_1 is the mean interarrival time of the distribution and B is the maximum batch size. In the HE_2 distribution used, we assume balanced mean (as in Allen [5]) with squared coefficient of variation (where $C_v^2 > 1$). While testing algorithm we have produced numerous tables but only a few are appended here. The validity of the numerical results was checked using the relation (30). The results for the distribution of the number of units in the system at an arbitrary (P_i) and arrival (Q_i) instants are given in self-explanatory Tables 1 and 2, respectively by taking $K = 10$, $N = 10$ with varying a_1, μ and B. In bottom rows of these tables we included some performance measures such as probabilities of queue of customers (P_{C_q}), and of servers (P_{S_q}); and average numbers of customers $E[L_C]$, and of servers $E[L_S]$; and probabilities of blocking of customers (PBL_C) and of servers (PBL_S), etc. The algorithm derived in this paper for the more general finite-capacity bulk-service double-ended queueing model is simple, elegant, and computationally efficient, and also works for high and low values of the queueing parameters.

Insert Tables 1 & 2

Table 1

Distribution of the number of units in the system at an arbitrary instant and performance measures for $K = 10$, $N = 10$

P_n	M $B=3$ $a_1=1.0$ $\mu=1.45$	P_n	E_5 $B=6$ $a_1=1.0$ $\mu=1.11$	P_n	D $B=5$ $a_1=0.5$ $\mu=1.33$	P_n	HE_2 $B=8$ $a_1=0.67$ $\mu=1.88$ $C_v^2=1.8$ $\beta_1=0.23$, $\beta_2=0.77$ $\lambda_1=0.7$, $\lambda_2=2.3$
P_{-10}	0.313093	P_{-10}	0.114978	P_{-10}	0.000087	P_{-10}	0.228268
P_{-9}	0.217267	P_{-9}	0.168478	P_{-9}	0.000296	P_{-9}	0.138782
P_{-8}	0.149815	P_{-8}	0.141048	P_{-8}	0.000635	P_{-8}	0.117701
P_{-7}	0.103274	P_{-7}	0.118083	P_{-7}	0.001361	P_{-7}	0.099831
P_{-6}	0.071160	P_{-6}	0.098858	P_{-6}	0.002919	P_{-6}	0.084692
P_{-5}	0.049002	P_{-5}	0.082763	P_{-5}	0.006258	P_{-5}	0.071878
P_{-4}	0.033713	P_{-4}	0.069288	P_{-4}	0.013418	P_{-4}	0.061055
P_{-3}	0.023163	P_{-3}	0.058006	P_{-3}	0.028774	P_{-3}	0.051952
P_{-2}	0.015884	P_{-2}	0.048547	P_{-2}	0.061686	P_{-2}	0.044365
P_{-1}	0.010861	P_{-1}	0.040549	P_{-1}	0.131349	P_{-1}	0.038163
P_0	0.007396	P_0	0.033447	P_0	0.268056	P_0	0.033311
P_1	0.003179	P_1	0.016418	P_1	0.229427	P_1	0.015400
P_2	0.001295	P_2	0.006033	P_2	0.121025	P_2	0.007507
P_3	0.000530	P_3	0.002217	P_3	0.063796	P_3	0.003647
P_4	0.000218	P_4	0.000815	P_4	0.033606	P_4	0.001772
P_5	0.000086	P_5	0.000299	P_5	0.017897	P_5	0.000861
P_6	0.000037	P_6	0.000110	P_6	0.009445	P_6	0.000419
P_7	0.000016	P_7	0.000040	P_7	0.004849	P_7	0.000203
P_8	0.000007	P_8	0.000015	P_8	0.002490	P_8	0.000099
P_9	0.000003	P_9	0.000005	P_9	0.001278	P_9	0.000048
P_{10}	0.000002	P_{10}	0.000003	P_{10}	0.001349	P_{10}	0.000045
SUM	1.000000		1.000000	SUM	1.000000	SUM	1.000000
P_{C_q}	0.005372	P_{C_q}	0.025956	P_{C_q}	0.485161	P_{C_q}	0.030003
P_{S_q}	0.987232	P_{S_q}	0.940597	P_{S_q}	0.246782	P_{S_q}	0.936687
$E[L_C]$	0.009090	$E[L_c]$	0.041030	$E[L_C]$	1.022293	$E[L_C]$	0.058366
$E[L_S]$	7.926710	$E[L_s]$	6.216819	$E[L_S]$	0.461656	$E[L_S]$	6.566651
PBL_C	0.000002	PBL_C	0.000002	PBL_C	0.000949	PBL_C	0.000054
PBL_S	0.314879	PBL_S	0.187197	PBL_S	0.000197	PBL_S	0.173478

Table 2

Distribution of the number of units in the system at an arrival instant for

$K = 10,\ N = 10$

Q_n	M $B = 3$ $a_1 = 1.0$ $\mu = 1.45$	Q_n	E_5 $B = 6$ $a_1 = 1.0$ $\mu = 1.11$	Q_n	D $B = 5$ $a_1 = 0.5$ $\mu = 1.33$	Q_n	HE_2 $B = 8$ $a_1 = 0.67$ $\mu = 1.88$ $C_v^2 = 1.8$ $\beta_1 = 0.23,\ \beta_2 = 0.77$ $\lambda_1 = 0.7,\ \lambda_2 = 2.3$
Q_{-10}	0.314879	Q_{-10}	0.187197	Q_{-10}	0.000197	Q_{-10}	0.173478
Q_{-9}	0.217124	Q_{-9}	0.156719	Q_{-9}	0.000423	Q_{-9}	0.147126
Q_{-8}	0.149672	Q_{-8}	0.131204	Q_{-8}	0.000908	Q_{-8}	0.124789
Q_{-7}	0.103131	Q_{-7}	0.109842	Q_{-7}	0.001946	Q_{-7}	0.105865
Q_{-6}	0.071017	Q_{-6}	0.091959	Q_{-6}	0.004172	Q_{-6}	0.089847
Q_{-5}	0.048859	Q_{-5}	0.076987	Q_{-5}	0.008945	Q_{-5}	0.076318
Q_{-4}	0.033570	Q_{-4}	0.064451	Q_{-4}	0.019182	Q_{-4}	0.064940
Q_{-3}	0.023020	Q_{-3}	0.053942	Q_{-3}	0.041124	Q_{-3}	0.055456
Q_{-2}	0.015741	Q_{-2}	0.045054	Q_{-2}	0.087566	Q_{-2}	0.047704
Q_{-1}	0.010718	Q_{-1}	0.037163	Q_{-1}	0.178704	Q_{-1}	0.041639
Q_0	0.007253	Q_0	0.028769	Q_0	0.310501	Q_0	0.037386
Q_1	0.002961	Q_1	0.010572	Q_1	0.163845	Q_1	0.018197
Q_2	0.001209	Q_2	0.003885	Q_2	0.086395	Q_2	0.008869
Q_3	0.000494	Q_3	0.001427	Q_3	0.045524	Q_3	0.004310
Q_4	0.000202	Q_4	0.000525	Q_4	0.023972	Q_4	0.002095
Q_5	0.000086	Q_5	0.000193	Q_5	0.012940	Q_5	0.001018
Q_6	0.000037	Q_6	0.000071	Q_6	0.006644	Q_6	0.000495
Q_7	0.000016	Q_7	0.000026	Q_7	0.003411	Q_7	0.000240
Q_8	0.000007	Q_8	0.000009	Q_8	0.001751	Q_8	0.000117
Q_9	0.000003	Q_9	0.000003	Q_0	0.000899	Q_9	0.000057
Q_{10}	0.000002	Q_{10}	0.000002	Q_{10}	0.000949	Q_{10}	0.000054
SUM	1.000000		1.000000	SUM	1.000000	SUM	1.000000

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