Autoregressive process and its Applications To Queueing Model

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AUTOREGRESSIVE PROCESS AND ITS APPLICATION TO QUEUEING MODEL

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ABSTRACT

A first order autoregressive process is proposed to express the number of customers Q_n observed after the *nth* departure for M/G/1 queue. This approach is then applied to develop a sequential Probability Ratio Test (SPRT) to detect changes in Q_n. Numerical procedures are employed to discuss the dependence of the Average Sample Number on the autoregressive parameter.

Keywords: Sequential Probability Ratio Test, Autoregressive Process, Operating Characteristic Function, Average Sample number.

1. Introduction

Very little literature exists on sequential detection of parameter changes in queueing models. Bhat and SubbaRao [3], Bhat [4], SubbaRao, Bhat and Harishchandra [13], Bhat and SubbaRao [5] discussed statistical quality control and Sequential Probability Ratio Test (SPRT) procedures for the control of traffic intensity. Their approach was based on sequential analysis of dependent observations given by Phatarfod [11]. SubbaRao et al. [13] considered a SPRT to detect changes in traffic intensity by observing only the number of customers in the system at successive departure epoch Q_n, which are embedded Markov points. Recently, a SPRT based on the number of arrivals during the *nth* service periods for $M/E_k/1$ queue was proposed to regulate the traffic intensity by Jain and Templeton [9].

In this paper, the number of customers, Q_n, observed at the departure point is expressed as a first-order autoregressive process, which is stationary Markov process. A SPRT is then developed to detect changes in Q_n for M/G/1 queue. It is shown that the ASN approaches minimum value when autoregressive parameter decreases to zero, implying the independence of the sequence of observations Q_n.

Sequential Probability Ratio Test for Q_n

Let a sequence of observations $Q_0, Q_1, ..., Q_n$ observed at the departure point can be expressed as a first-order autoregressive process given by

$$Q_n - L^D = \phi(Q_{n-1} - L^D) + a_n \ , \tag{2.1}$$

where $|\phi| < 1$ for stationary process, and a_n is white noise or random shock, assumed to be independent normal random variable with mean zero and variance σ^2. The mean and variance of Q_n are given by (Anderson [1], page 15).

$$E(Q_n) = L^D \ , \tag{2.2}$$

and

$$Var(Q_n) = \frac{\sigma^2}{1-\phi^2} \, . \tag{2.3}$$

Denote

$$X_r = (Q_r - \phi Q_{r-1}), \;\; r = 1,2,...,n. \tag{2.4}$$

It can easily be seen from (2.1) that $X_1, X_2, ..., X_n$ are independent, identically and normally distributed with mean $L^D(1-\phi)$ and variance σ^2. Therefore, they are jointly normally distributed. The random variables $Q_1, Q_2, ..., Q_n$ can be written as a linear combination of $X_1, X_2, ..., X_n$. Therefore, $Q_1, Q_2, ..., Q_n$ are jointly normally distributed (Parzen [10], Theorem A, page 90), and thus, they form a normal Markov sequence.

For an effective operation of the systems, a certain level of traffic intensity is to be maintained. And for maintaining the desired level of traffic intensity, service rate is required to be changed with the arrival rate. Thus, the number of customers left behind by a departing customer is an important quantity in a queueing system to regulate the traffic intensity.

Consider the null hypothesis $H_o : L^D = L_o^D$ against an alternative hypothesis $H_1 : L^D = L_1^D$. The joint probability distributions of $X_1, X_2, ..., X_n$ under H_o and H_1 are given by

$$L_{i,n} = f\left[X_1, X_2, ..., X_n \mid L_i^D\right], \;\; i=0,1. \tag{2.5}$$

Therefore, the log-likelihood ratio is

$$\ln \frac{L_{1,n}}{L_{o,n}} = \ln \left[\frac{f\left[x_1, x_2, ..., x_n \mid L_1^D\right]}{f\left[x_1, x_2, ..., x_n \mid L_o^D\right]} \right] . \tag{2.6}$$

Wald's [14] SPRT to test $H_o : L^D = L_o^D$ vs $H_1 : L^D = L_1^D$ becomes:

Accept H_1 if

$$\ln\left[\frac{f\left(x_1, x_2, \ldots, x_n \mid L_1^D\right)}{f\left(x_1, x_2, \ldots, x_n \mid L_o^D\right)}\right] \geq lnA \ . \tag{2.7}$$

Accept H_o if

$$\ln\left[\frac{f\left(x_1, x_2, \ldots, x_n \mid L_1^D\right)}{f\left(x_1, x_2, \ldots, x_n \mid L_o^D\right)}\right] \leq lnB \ . \tag{2.8}$$

Continue sampling if

$$lnB < \ln\left[\frac{f\left(x_1, x_2, \ldots, x_n \mid L_1^D\right)}{f\left(x_1, x_2, \ldots, x_n \mid L_o^D\right)}\right] < lnA \ . \tag{2.9}$$

Wald [14] determined the bounds for constants A and B in terms of the probabilities of type I and type II errors, given by the following relations:

$$A \simeq \frac{1-\beta}{\alpha}, \tag{2.10}$$

$$B \simeq \frac{\beta}{1-\alpha}. \tag{2.11}$$

Since $x_1, x_2, \ldots, x_n$ are independent, therefore we can write

$$\ln\left[\frac{f\left(x_1, x_2, \ldots, x_n \mid L_1^D\right)}{f\left(x_1, x_2, \ldots, x_n \mid L_o^D\right)}\right] = \ln\left[\frac{f\left(x_1 \mid L_1^D\right) f\left(x_2 \mid L_1^D\right) \ldots f\left(x_n \mid L_1^D\right)}{f\left(x_1 \mid L_o^D\right) f\left(x_2 \mid L_o^D\right) \ldots f\left(x_n \mid L_o^D\right)}\right]. \tag{2.12}$$

Let

$$\ln\left[\prod_{i=1}^{n} \frac{f\left(x_i \mid L_1^D\right)}{f\left(x_i \mid L_o^D\right)}\right] = \sum_{i=1}^{n} Z_i \ , \tag{2.13}$$

where

$$Z_i = \frac{\left[(L_o^D)^2 - (L_1^D)^2\right](1-\phi)^2}{2\sigma^2} + \frac{(L_1^D - L_o^D)\,(1-\phi)\,(Q_i - \phi Q_{i-1})}{\sigma^2} \,. \tag{2.14}$$

To continue sampling, equation (2.9) can be rewritten as follows:

$$\ln\frac{\beta}{1-\alpha} < \sum_{i=1}^{n} Z_i < \ln\frac{1-\beta}{\alpha} \,. \tag{2.15}$$

Simplification of (2.15) yields

$$a_1 + bn < \sum_{i=1}^{n}\left[Q_i - \phi Q_{i-1}\right] < a_2 + bn \tag{2.16}$$

where

$$a_1 = \frac{\sigma^2}{(1-\phi)\,(L_1^D - L_o^D)} \ln\frac{\beta}{1-\alpha} \,, \tag{2.17}$$

$$a_2 = \frac{\sigma^2}{(1-\phi)\,(L_1^D - L_o^D)} \ln\frac{1-\beta}{\alpha} \,, \tag{2.18}$$

and

$$b = \frac{L_o^D + L_1^D\,(1-\phi)}{2} \,. \tag{2.19}$$

The procedure is terminated with acceptance of H_o if

$$\sum_{i=1}^{n}\left[Q_i - \phi Q_{i-1}\right] < a_1 + bn \tag{2.20}$$

or with rejection of H_o if

$$\sum_{i=1}^{n}\left[Q_i - \phi Q_{i-1}\right] > a_2 + bn \,. \tag{2.21}$$

Operating Characteristic Function.

The approximate formula for the OC function is given by

$$\Pi(L^D) = \frac{A^{h(L^D)} - 1}{A^{h(L^D)} - B^{h(L^D)}} \,, \tag{2.22}$$

where

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