Asymptotic Expansions For The Joint Distribution Of Correlated Hotelling's T^2 Statistics Under Normality

by

Yasunori Fujikoshi
Department of Mathematics
Hiroshima University

and

Takashi Seo*
Department of Statistics
University of Toronto

Technical Report No. 9804, February 9, (1998)

TECHNICAL REPORT SERIES

University of Toronto

Department of Statistics

ASYMPTOTIC EXPANSIONS FOR THE JOINT DISTRIBUTION OF CORRELATED HOTELLING'S T^2 STATISTICS UNDER NORMALITY

Yasunori Fujikoshi and Takashi Seo*

Department of Mathematics
Hiroshima University
Higashi-Hiroshima 739
Japan

Department of Statistics
University of Toronto
Toronto, Ontario M5S 3G3
Canada

Key Words and Phrases: Asymptotic expansions; Characteristic function; Correlated Hotelling's T^2 statistics; Distribution function; $\max(T_1^2, \cdots, T_k^2)$.

ABSTRACT

Let $T_i^2 = \boldsymbol{z}_i' \boldsymbol{S}^{-1} \boldsymbol{z}_i$, $i = 1, \cdots, k$ be correlated Hotelling's T^2 statistics under normality, where $\boldsymbol{z} = (\boldsymbol{z}_1', \cdots, \boldsymbol{z}_k')'$ and $n\boldsymbol{S}$ are independently distributed as $N_{kp}(\boldsymbol{0}, \boldsymbol{\Gamma} \otimes \boldsymbol{\Sigma})$ and Wishart distribution $W_p(\boldsymbol{\Sigma}, n)$, respectively. The purpose of this paper is to study the distribution function $F(x_1, \cdots, x_k)$ of $(T_1^2, \cdots, T_k^2)$ when n is large. First we derive an asymptotic expansion of the characteristic function of $(T_1^2, \cdots, T_k^2)$ up to the order n^{-2}. Next we give asymptotic expansions for $F(x_1, \cdots, x_k)$ in two cases (i) $\boldsymbol{\Gamma} = \boldsymbol{I}_k$ and (ii) $k = 2$ by inverting the expanded characteristic function up to the orders n^{-2} and n^{-1}, respectively. Our results can be applied to the distribution function of $\max(T_1^2, \cdots, T_k^2)$ as a special case.

*On leave from Department of Information Sciences, Faculty of Science & Technology, Science University of Tokyo, Noda City, Chiba 278, Japan.

1. INTRODUCTION

This paper deals with the joint distribution of k correlated Hotelling's T^2 statistics

$$T_i^2 = \boldsymbol{z}_i' \boldsymbol{S}^{-1} \boldsymbol{z}_i, \ i = 1, \cdots, k \tag{1.1}$$

under normality, where $\boldsymbol{z} = (\boldsymbol{z}_1', \cdots, \boldsymbol{z}_k')'$ and $n\boldsymbol{S}$ are independently distributed as $N_{kp}(\boldsymbol{0}, \boldsymbol{\Gamma} \otimes \boldsymbol{\Sigma})$ and Wishart distribution $W_p(\boldsymbol{\Sigma}, n)$, respectively. Without loss of generality we may assume $\boldsymbol{\Sigma} = \boldsymbol{I}_p$. In the following we shall do that. Let F be the distribution function of $(T_1^2, \cdots, T_k^2)$, i.e.,

$$\Pr(T_1^2 \le x_1, \cdots, T_k^2 \le x_k) = F(x_1, \cdots, x_k), \tag{1.2}$$

which involves the distribution function of $T_{\max}^2 = \max(T_1^2, \cdots, T_k^2)$ as a special case. More precisely it holds that

$$\begin{aligned} \Pr(T_{\max}^2 \le x) &= F_{\max}(x) \\ &= F(x, \cdots, x). \end{aligned} \tag{1.3}$$

The statistic $T_{\max}^2$ has appeared in some simultaneous and multiple inference procedures(see, Siotani(1959), Siotani, Hayakawa and Fujikoshi(1985), etc).

The purpose of this paper is to study asymptotic expansion of the characteristic function

$$C(t_1, \cdots, t_k) = \mathrm{E}[\exp(it_1 T_1^2 + \cdots + it_k T_k^2)] \tag{1.4}$$

and the distribution function $F(x_1, \cdots, x_k)$ of $(T_1^2, \cdots, T_k^2)$. In Section 2 we obtain an asymptotic expansion for $C(t_1, \cdots, t_k)$ up to the order n^{-2}. In general, it is difficult to carry out inversion of the expanded characteristic function except for some special cases. We consider two cases (i) $\boldsymbol{\Gamma} = \boldsymbol{I}_p$ and (ii) $k = 2$, which are discussed in Sections 3 and 4, respectively. For the cases

(i) and (ii), we derive asymptotic expansions for $F(x_1, \cdots, x_k)$ up to the orders n^{-2} and n^{-1}, respectively. The results yield asymptotic expansions for the distribution function of $T^2_{\max}$ under the two cases. The distribution of $T^2_{\max}$ in the case (i) is closely related to a conservative evaluation of that in a general case(see, e.g., Seo, Mano and Fujikoshi(1994)). Related to approximation for the distribution of $T^2_{\max}$, Siotani(1959) and Seo and Siotani(1992) obtained asymptotic expansions of the distribution of $\min(T_1^2, T_2^2)$.

2. EXPANSION OF THE CHARACTERISTIC FUNCTION

Let $C(t_1,\cdots,t_k|\boldsymbol{S})$ be the conditional characteristic function of $(T_1^2, \cdots, T_k^2)$ given $\boldsymbol{S}$. Then we can write

$$\begin{aligned} C(t_1,\cdots,t_k|\boldsymbol{S}) &= \mathrm{E}_{\boldsymbol{z}}\left[\exp(it_1T_1^2 + \cdots + it_kT_k^2)\right] \\ &= \int\cdots\int(2\pi)^{-kp/2}|\Gamma\otimes\boldsymbol{I}_p|^{-1/2}\exp\left\{-\frac{1}{2}\boldsymbol{z}'(\Gamma\otimes\boldsymbol{I}_p)^{-1}\}\boldsymbol{z}\right\} \\ &\qquad\times\exp\{i\boldsymbol{z}'(\boldsymbol{D}_t\otimes\boldsymbol{S}^{-1})\boldsymbol{z}\}d\boldsymbol{z} \\ &= |\boldsymbol{I}_{kp} - 2i\Gamma\boldsymbol{D}_t\otimes\boldsymbol{S}^{-1}|^{-1/2}, \end{aligned} \tag{2.1}$$

where $\boldsymbol{D}_t = \mathrm{diag}(t_1, \cdots, t_k)$. Let

$$\boldsymbol{M} = (\boldsymbol{I}_k - 2i\Gamma\boldsymbol{D}_t)^{-1}, \quad \boldsymbol{B} = -2i\boldsymbol{M}\Gamma\boldsymbol{D}_t, \quad \boldsymbol{U} = \boldsymbol{B}\otimes(\boldsymbol{I}_p - \boldsymbol{S}^{-1}). \tag{2.2}$$

Note that

$$\boldsymbol{I}_{kp} - 2i\Gamma\boldsymbol{D}_t\otimes\boldsymbol{S}^{-1} = (\boldsymbol{M}^{-1}\otimes\boldsymbol{I}_p)(\boldsymbol{I}_{kp} - \boldsymbol{U}). \tag{2.3}$$

Letting $\boldsymbol{S} = \boldsymbol{I}_p + (1/\sqrt{n})\boldsymbol{V}$, it holds that

$$\boldsymbol{S}^{-1} = \boldsymbol{I}_p + \sum_{j=1}^{5}(-1)^j n^{-j/2}\boldsymbol{V}^j + \mathrm{O}_p(n^{-2}). \tag{2.4}$$

Therefore from (2.1), (2.3) and (2.4) we obtain

$$\begin{aligned}\log C(t_1,\cdots,t_k|\boldsymbol{S}) &= \log|\boldsymbol{M}|^{p/2}+\sum_{j=1}^{5}\frac{1}{j}(\mathrm{tr}\boldsymbol{B}^j)\mathrm{tr}(\boldsymbol{I}_p-\boldsymbol{S}^{-1})^j+\mathrm{O}_p(n^{-2}) \\ &= \log|\boldsymbol{M}|^{p/2}+\sum_{j=1}^{5}n^{-j/2}a_j+\mathrm{O}_p(n^{-2}),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}a_j &= \frac{1}{2j}\mathrm{tr}\{(\boldsymbol{B}-\boldsymbol{I}_k)^j-(-1)^j\boldsymbol{I}_k\} \\ &= \frac{1}{2j}(-1)^j\{\mathrm{tr}\boldsymbol{M}^j-k\}.\end{aligned}$$

Theorem 2.1. Let $C(t_1,\cdots,t_k)$ be the characteristic function of $(T_1^2,\cdots,T_k^2)$, where T_i^2's are defined by (1.1). Then we can expand $C(t_1,\cdots,t_k)$ as

$$\begin{aligned}C(t_1,\cdots,t_k) = &|\boldsymbol{M}|^{p/2}\Big[\,1 \\ &+\frac{p}{4n}\{\,k(k-p-1)-2k\,\mathrm{tr}\boldsymbol{M}+(\mathrm{tr}\boldsymbol{M})^2+(p+1)\mathrm{tr}\boldsymbol{M}^2\} \\ &+\frac{p}{96n^2}\Big\{b_0+b_1\mathrm{tr}\boldsymbol{M}+b_2(\mathrm{tr}\boldsymbol{M})^2+b_3\mathrm{tr}\boldsymbol{M}^2 \\ &\quad+b_4(\mathrm{tr}\boldsymbol{M})^3+b_5(\mathrm{tr}\boldsymbol{M})\mathrm{tr}\boldsymbol{M}^2+b_6\mathrm{tr}\boldsymbol{M}^3 \\ &\quad+b_7(\mathrm{tr}\boldsymbol{M})^4+b_8(\mathrm{tr}\boldsymbol{M})^2\mathrm{tr}\boldsymbol{M}^2+b_9(\mathrm{tr}\boldsymbol{M}^2)^2 \\ &\quad+b_{10}(\mathrm{tr}\boldsymbol{M})\mathrm{tr}\boldsymbol{M}^3+b_{11}\mathrm{tr}\boldsymbol{M}^4\Big\}\Big]+\mathrm{O}(n^{-2}),\end{aligned} \tag{2.6}$$

where

$b_0 = k\{3pk^3-2(3p^2+3p+4)k^2+3(p^3+2p^2+5p+4)k-4(2p^2+3p-1)\}$,

$b_1 = -12pk^2(k-p-1)$, $\quad b_2 = 6k(3pk-p^2-p+4)$,

$b_3 = 6k\{(p^2+p+4)k-p^3-2p^2+3p+4\}$, $\quad b_4 = -4(3pk+4)$,

$b_5 = -12\{(p^2+p+4)k+4p+4\}$, $\quad b_6 = -16\{3(p+1)k+p^2+3p+4\}$,

$b_7 = 3p$, $\quad b_8 = 6(p^2+p+4)$, $\quad b_9 = 48(p+1)$,

$b_{10} = 3(p^3+2p^2+5p+4)$, $\quad b_{11} = 12(p^2+5p+5)$.

Proof. From (2.5) we can write

$$
\begin{aligned}
C(t_1,\cdots,t_k) =&|\boldsymbol{M}|^{\frac{p}{2}}\mathrm{E}\Big[\ 1+\frac{1}{\sqrt{n}}a_1 \\
&+\frac{1}{n}\left(\frac{1}{2}a_1^2+a_2\right)+\frac{1}{n\sqrt{n}}\left(\frac{1}{6}a_1^3+a_1a_2+a_3\right) \\
&+\frac{1}{n^2}\left(\frac{1}{24}a_1^4+\frac{1}{2}a_1^2a_2+\frac{1}{2}a_2^2+a_1a_3+a_4\right)+\frac{1}{n^2\sqrt{n}}a_5\Big]+\mathrm{O}(n^{-3}),
\end{aligned}
$$

where a_5 is a homogeneous polynominal of degree 5 in the elements of $\boldsymbol{V}$. The expectation with respect to $\boldsymbol{V}$ or $\boldsymbol{S}$ can be carried out by the use of the following results:

$$
\begin{aligned}
&\mathrm{E}[\mathrm{tr}\boldsymbol{V}]=0,\quad \mathrm{E}[(\mathrm{tr}\boldsymbol{V})^2]=2p,\quad \mathrm{E}[\mathrm{tr}\boldsymbol{V}^2]=p(p+1),\\
&\mathrm{E}[(\mathrm{tr}\boldsymbol{V})^3]=8p/\sqrt{n},\quad \mathrm{E}[(\mathrm{tr}\boldsymbol{V})\mathrm{tr}\boldsymbol{V}^2]=4p(p+1)/\sqrt{n},\\
&\mathrm{E}[\mathrm{tr}\boldsymbol{V}^3]=p(p^2+3p+4)/\sqrt{n},\quad \mathrm{E}[(\mathrm{tr}\boldsymbol{V})^4]=12p^2,\\
&\mathrm{E}[(\mathrm{tr}\boldsymbol{V})^2\mathrm{tr}\boldsymbol{V}^2]=2p(p^2+p+4),\quad \mathrm{E}[(\mathrm{tr}\boldsymbol{V}^2)^2]=p(p^3+2p^2+5p+4),\\
&\mathrm{E}[(\mathrm{tr}\boldsymbol{V})\mathrm{tr}\boldsymbol{V}^3]=6p(p+1),\quad \mathrm{E}[\mathrm{tr}\boldsymbol{V}^4]=p(2p^2+5p+5),\\
&\mathrm{E}[a_5]=\mathrm{O}(1/\sqrt{n}).
\end{aligned}
$$

A simplification of the resultant expression yields the final result (2.6).

As a special case we obtain

$$
\lim_{n\to\infty}C(t_1,\cdots,t_k)=|\boldsymbol{I}_k-2i\boldsymbol{\Gamma}\boldsymbol{D}_t|^{-p/2} \tag{2.7}
$$

which is known as the characteristic function of the multivariate chi-square distribution. Inversion of (2.7) has been studied by Krishnamoorthy and Parthasarathy(1951), Royen(1991, 1994), etc. However, apart from some special cases no "simple formulas" are known even for the limiting distribution function.

3. THE CASE $\Gamma = I_p$

When $\Gamma = I_p$, from Theorem 2.1 we obtain

$$
\begin{aligned}
C(t_1, \cdots, t_k) = \prod_{j=1}^{k} \phi_j^{-p/2} \Big[\, 1 \\
+ \frac{p}{4n}\Big\{k(k-p-1) - 2kL_1 + (p+2)L_2 + L_3\Big\} \\
+ \frac{p}{96n^2}\Big\{d_0 + \sum_{j=1}^{11} d_j L_j\Big\}\Big] + \mathrm{O}(n^{-3}),
\end{aligned}
\tag{3.1}
$$

where $\phi_j = 1 - 2it_j$, $L_1 = \sum_{\alpha=1}^{k} \phi_\alpha^{-1}$, $L_2 = \sum_{\alpha=1}^{k} \phi_\alpha^{-2}$, $L_3 = \sum_{\alpha \neq \beta} \phi_\alpha^{-1}\phi_\beta^{-1}$, $L_4 = \sum_{\alpha=1}^{k} \phi_\alpha^{-3}$, $L_5 = \sum_{\alpha \neq \beta} \phi_\alpha^{-2}\phi_\beta^{-1}$, $L_6 = \sum_{\alpha \neq \beta \neq \xi} \phi_\alpha^{-1}\phi_\beta^{-1}\phi_\xi^{-1}$, $L_7 = \sum_{\alpha=1}^{k} \phi_\alpha^{-4}$, $L_8 = \sum_{\alpha \neq \beta} \phi_\alpha^{-3}\phi_\beta^{-1}$, $L_9 = \sum_{\alpha \neq \beta} \phi_\alpha^{-2}\phi_\beta^{-2}$, $L_{10} = \sum_{\alpha \neq \beta \neq \xi} \phi_\alpha^{-2}\phi_\beta^{-1}\phi_\xi^{-1}$, $L_{11} = \sum_{\alpha \neq \beta \neq \xi \neq \eta} \phi_\alpha^{-1}\phi_\beta^{-1}\phi_\xi^{-1}\phi_\eta^{-1}$, and

$$
\begin{aligned}
&d_0 = b_0, \; d_1 = b_1, \\
&d_2 = 6k\{(p^2+4p+4)k - p^3 - 3p^2 + 2p + 8\}, \; d_3 = b_2, \\
&d_4 = -4(p^2+6p+8)(3k+4), \\
&d_5 = -12\{(p^2+4p+4)k + 4p + 8\}, \; d_6 = b_4, \\
&d_7 = 3(p^3 + 12p^2 + 44p + 48), \\
&d_8 = 3(p^3 + 6p^2 + 13p + 20), \\
&d_9 = 3(2p^2 + 21p + 24), \; d_{10} = 6(p^2 + 4p + 4), \; d_{11} = b_7.
\end{aligned}
$$

The notations of summation, for example, $\sum_{\alpha \neq \beta}$ means $\sum_{\alpha=1}^{k}\sum_{\beta=1,\beta\neq\alpha}^{k}$. Let G_f and g_f be the distribution function and the density function of a chi-square variate with f degrees of freedom. It is easy to invert the expanded characteristic function termwisely and integrate out the resultant density over

$(0, x_1) \times \cdots \times (0, x_k)$. For example,

$$
\begin{aligned}
&I[\prod_{\alpha=1}^{k} \phi_j^{-p/2}] = \prod_{\alpha=1}^{k} G_p(x_\alpha), \\
(3.2) \qquad &I[\prod_{\alpha=1}^{k} \phi_j^{-p/2} \cdot L_1] = \prod_{\alpha=1}^{k} G_p(x_\alpha) \cdot \sum_{\alpha=1}^{k} G_{p+2}(x_\alpha) G_p(x_\alpha)^{-1}, \\
&I[\prod_{\alpha=1}^{k} \phi_j^{-p/2} \cdot L_2] = \prod_{\alpha=1}^{k} G_p(x_\alpha) \cdot \sum_{\alpha=1}^{k} G_{p+4}(x_\alpha) G_p(x_\alpha)^{-1}.
\end{aligned}
$$

Here the operation $I[\{\cdot\}]$ is defined as

$$
\begin{aligned}
(3.3) \qquad I[\{\cdot\}] = \int_0^{x_1} \cdots \int_0^{x_k} \Big[(2\pi)^{-k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\{i(t_1 x_1 + \cdots + t_k x_k)\} \\
\times \{\cdot\} dt_1 \cdots dt_k \Big] dx_1 \cdots dx_k.
\end{aligned}
$$

Inverting of (3.1) yields

$$
\begin{aligned}
F(x_1, \cdots, x_k) = \prod_{\alpha=1}^{k} G_p(x_\alpha) \Big[1 \\
&+ \frac{p}{4n} \Big\{ k(k-p-1) + \sum_{\alpha=1}^{k} \{-2k G_{p+2}(x_\alpha) + (p+2) G_{p+4}(x_\alpha)\} G_p(x_\alpha)^{-1} \\
&\quad + \sum_{\alpha \neq \beta} G_{p+2}(x_\alpha) G_{p+2}(x_\beta) \{G_p(x_\alpha) G_p(x_\beta)\}^{-1} \Big\} \\
&+ \frac{p}{96n^2} \Big\{ d_0 + \sum_{\alpha=1}^{k} \{ d_1 G_{p+2}(x_\alpha) + d_2 G_{p+4}(x_\alpha) \\
&\quad + d_4 G_{p+6}(x_\alpha) + d_7 G_{p+8}(x_\alpha)\} G_p(x_\alpha)^{-1} \\
(3.4) \qquad &+ \sum_{\alpha \neq \beta} \{ d_3 G_{p+2}(x_\alpha) G_{p+2}(x_\beta) + d_5 G_{p+4}(x_\alpha) G_{p+2}(x_\beta) \\
&+ d_8 G_{p+6}(x_\alpha) G_{p+2}(x_\beta) + d_9 G_{p+4}(x_\alpha) G_{p+4}(x_\beta) \} \{G_p(x_\alpha) G_p(x_\beta)\}^{-1} \\
&+ \sum_{\alpha \neq \beta \neq \xi} \{ d_6 G_{p+2}(x_\alpha) G_{p+2}(x_\beta) G_{p+2}(x_\xi) \\
&\quad + d_{10} G_{p+4}(x_\alpha) G_{p+2}(x_\beta) G_{p+2}(x_\xi)\} \{G_p(x_\alpha) G_p(x_\beta) G_p(x_\xi)\}^{-1} \\
&+ \sum_{\alpha \neq \beta \neq \xi \neq \eta} d_{11} G_{p+2}(x_\alpha) G_{p+2}(x_\beta) G_{p+2}(x_\xi) G_{p+2}(x_\eta) \\
&\quad \times \{G_p(x_\alpha) G_p(x_\beta) G_p(x_\xi) G_p(x_\eta)\}^{-1} \Big\} \Big] + O(n^{-3}).
\end{aligned}
$$

Note that

$$G_{f+2}(x) = -\frac{2x}{f} g_f(x) + G_f(x),$$

$$G_{f+4}(x) = -\frac{2x}{f}\left(\frac{x}{f+2}+1\right) g_f(x) + G_f(x),$$

$$G_{f+6}(x) = -\frac{2x}{f}\left(\frac{x^2}{(f+4)(f+2)}+\frac{x}{f+2}+1\right) g_f(x) + G_f(x),$$

$$G_{f+8}(x) = -\frac{2x}{f}\left(\frac{x^3}{(f+6)(f+4)(f+2)}+\frac{x^2}{(f+4)(f+2)}+\frac{x}{f+2}+1\right) g_f(x) + G_f(x).$$

Using the above relations, we can simplify (3.4) as in the following theorem.

Theorem 3.1. When $\boldsymbol{\Gamma} = \boldsymbol{I}_k$, the distribution function of $(T_1^2, \cdots, T_k^2)$ can be expanded as

$$\begin{aligned} F(x_1,\cdots,x_k) = \prod_{\alpha=1}^{k} G_p(x_\alpha)\Bigg[\ 1 &+ \frac{1}{n}\sum_{\alpha=1}^{k} x_\alpha h_p(x_\alpha)\Big\{-\frac{1}{2}(x_\alpha+p)+\frac{1}{p}\sum_{\beta\neq\alpha}^{k} x_\beta h_p(x_\beta)\Big\} \\ &+ \frac{1}{n^2}\Big[\sum_{\alpha=1}^{k} x_\alpha h_p(x_\alpha)\Big\{c_1+\sum_{\beta\neq\alpha}^{k} x_\beta h_p(x_\beta) \\ &\quad \Big\{c_2+\sum_{\xi\neq\alpha\neq\beta}^{k} x_\xi h_p(x_\xi)\{c_3+c_4\sum_{\eta\neq\alpha\neq\beta\neq\xi}^{k} x_\eta h_p(x_\eta)\}\Big\}\Big\}\Big]\Bigg] \\ &+ O(n^{-3}), \end{aligned} \tag{3.5}$$

where $h_p(x_i) = g_p(x_i)/G_p(x_i)$ for $i = 1, \cdots, k$, and

$$c_1 = \frac{1}{48}\Big[3p^3-8p^2+8+\frac{1}{p+2}\Big\{3(p^3+2p^2-11p-12)k-2p^2+33p+44\Big\}x_\alpha - \frac{1}{p+2}\Big\{3(p^2-2p-3)k+2p-11\Big\}x_\alpha^2-3x_\alpha^3\Big],$$

$$c_2 = \frac{1}{8p}\Big[p^3-4p^2+6p-4+\frac{1}{p+2}(p^3+2p^2+7p+4)x_\alpha+\frac{1}{p+2}(p^2+2p+5)x_\alpha^2 + \frac{1}{(p+2)^2}(2p^2+21p+24)x_\alpha x_\beta,\Big]$$

$$c_3 = -\frac{1}{6p^2}\{3p^2-6p+4+3(p+2)x_\alpha\}, \quad c_4 = \frac{1}{2p^2}.$$

Here, $c_j = 0$ if $j > k$.

Letting $x_1 = \cdots = x_k = x$ in Theorem 3.1, we obtain

$$
\begin{aligned}
F_{\max}(x) &= \Pr(T^2_{\max} \le x) \\
&= \{G_p(x)\}^k \Big[1 + \frac{1}{n} kxh_p(x)\Big\{ -\frac{1}{2}(x+p) + \frac{1}{p}(k-1)xh_p(x)\Big\} \\
&\quad + \frac{1}{n^2} kxh_p(x)\Big\{c_1 + (k-1)xh_p(x)\Big\{c_2 + (k-2)xh_p(x) \\
&\quad \times \{c_3 + c_4(k-3)xh_p(x)\}\Big\}\Big\}\Big] + \mathrm{O}(n^{-3}),
\end{aligned} \tag{3.6}
$$

where $h_p(x) = g_p(x)/G_p(x)$, and the coefficients c_1, c_2, c_3 and c_4 are given by the ones in Theorem 3.1 with $x_1 = \cdots = x_k = x$.

Further, the upper $100\alpha\%$ point u^* of $F_{\max}$ can be expanded in terms of the upper $100\alpha\%$ point u of $\{G_p(u)\}^k$, i.e., u is a solution of

$$\{G_p(u)\}^k = 1 - \alpha.$$

In fact, letting $u^* = u\{1 + n^{-1}u^{(1)} + n^{-2}u^{(2)} \cdots\}$ and substituting it to (3.6), we can determine u^* as

$$u^* = u\left[1 + \frac{1}{n}u^{(1)} + \frac{1}{n^2}u^{(2)} + \mathrm{O}(n^{-3})\right], \tag{3.7}$$

where

$$
\begin{aligned}
u^{(1)} &= -\frac{1}{2}(u+p) + \frac{1}{p}(k-1)uh_p(u), \\
u^{(2)} &= \frac{1}{4}\{u - p + 2 + 2(k-1)uh_p(u)\}(u^{(1)})^2 \\
&\quad + \frac{1}{4}\{p^2 - u^2 + 2u - \frac{4}{p}(k-1)u(p - u - uh_p(u))h_p(u)\}u^{(1)} \\
&\quad - \{c_1^* + c_2^*(k-1)uh_p(u) + c_3^*(k-1)(k-2)(uh_p(u))^2 \\
&\qquad + c_4^*(k-1)(k-2)(k-3)(uh_p(u))^3\}.
\end{aligned}
$$

Here c_j^* is defined from the c_j in (3.6) by substituting u to x.

TABLE I

Values of asymptotic expansions for $F(x, \cdots, x)$ by the simulated upper α percentiles x

$p = 2$	$\alpha = 0.1$					
	$k = 3$			$k = 6$		
n	L.T.	A.E.(n^{-1})	A.E.(n^{-2})	L.T.	A.E.(n^{-1})	A.E.(n^{-2})
10	0.990	0.953	0.913	0.996	0.974	0.934
20	0.958	0.913	0.903	0.971	0.923	0.906
40	0.932	0.903	0.901	0.941	0.905	0.901
	$\alpha = 0.05$					
10	0.999	0.990	0.971	1.000	0.996	0.984
20	0.987	0.964	0.953	0.991	0.970	0.957
40	0.972	0.953	0.951	0.976	0.956	0.951
	$\alpha = 0.01$					
10	1.000	1.000	0.999	1.000	1.000	1.000
20	0.999	0.997	0.993	1.000	0.998	0.995
40	0.997	0.992	0.991	0.997	0.993	0.991

Note. L.T. is the limiting term. A.E.'s are the asymptotic expansion up to the orders n^{-1} and n^{-2}.

Using the result of Theorem 3.1, some values of asymptotic expansions for $F(x_1, \cdots, x_k)$ with given $x_1 = \cdots = x_k = x$ are listed Table I. The values of x are given by computing the upper α percentiles of $T^2_{\max}$ by Monte Carlo simulation. The Monte Carlo simulation was based on 100 replications of 10,000 simulations for selected values p, k, n and α. The average for 100 estimates of the upper α percentiles is the value of given x. Table I gives the values of the limiting term (L.T.), the asymptotic expansion up to the order n^{-1}(A.E.(n^{-1})), and the one up to the order n^{-2}(A.E.(n^{-2})) for $p = 2$, $k = 3$, 6, $n = 10$, 20, 40 and $\alpha = 0.10$, 0.05, 0.01.

TABLE II

Values of asymptotic expansions and simulation results for the upper percentiles of $F(x, \cdots, x)$

$p=2,\ k=3$		Level α		
n		0.10	0.05	0.01
	P.A.E.(n^{-1})	9.592	12.238	19.018
10	P.A.E.(n^{-2})	10.932	14.255	22.424
	Simulation	11.469	15.516	27.677
	P.A.E.(n^{-1})	8.162	10.196	15.210
20	P.A.E.(n^{-2})	8.497	10.700	16.061
	Simulation	8.530	10.815	16.733
	P.A.E.(n^{-1})	7.448	9.175	13.305
40	P.A.E.(n^{-2})	7.531	9.302	13.518
	Simulation	7.538	9.312	13.621
∞	P.L.T.	6.733	8.155	11.401

Note. P.A.E.'s are the upper percentiles by asymptotic expansions up to the orders n^{-1} and n^{-2}. P.L.T. is the upper percentile by the limiting term.

In addition, we compute the upper $100\alpha\%$ point u^* of $F_{\max}$ based on the expansion of (3.7). Table II gives the upper percentiles by using the limiting term, the asymptotic expansion up to the order n^{-1}, and the one up to the order n^{-2}, respectively, for $p=2,\ k=3,\ n=10,\ 20,\ 40$ and $\alpha=0.10,\ 0.05,\ 0.01$. Also, in Table II, we add to the simulation result corresponding to the upper percentiles of $F_{\max}$ with the same as previous simulation process.

The results from the numerical examination show that the values of the asymptotic expansion are good approximation. Hence, it may be noted that the asymptotic expansion for distribution function of $(T_1^2, \cdots, T_k^2)$ in Theorem 3.1 can be used effectively when n is large.

4. THE CASE $k = 2$

Siotani(1959)(see, also Seo and Siotani(1992)) proposed to approximate the distribution of $T^2_{\max}$, based on Bonferroni's inequalities

$$\sum_{j=1}^{k} \Pr(T_j^2 > x) \; > \; \Pr(T^2_{\max} > x) \; > \; \sum_{j=1}^{k} \Pr(T_j^2 > x) - \beta, \tag{4.1}$$

for $x > 0$, where

$$\beta = \sum_{i<j} \Pr(T_i^2 > x, T_j^2 > x).$$

In the approximation method, they obtained an asymptotic expansion for $\Pr(T_i^2 > x, T_j^2 > x)$. In this section we shall obtain an asymptotic expansion of the distribution function of (T_i^2, T_j^2), i.e., $\Pr(T_i^2 \leq x_i, T_j^2 \leq x_j)$, by inverting the characteristic function (2.6). Without loss of generality we treat the case

$$k = 2 \quad \text{and} \quad \boldsymbol{\Gamma} = \begin{bmatrix} 1 & \gamma \\ \gamma & 1 \end{bmatrix}. \tag{4.2}$$

Then from (2.6) we obtain

$$\begin{aligned} C(t_1, t_2) = &|\boldsymbol{M}|^{p/2}\Big[\, 1 + \frac{p}{4n}\big\{-2(p-1) - 4\mathrm{tr}\boldsymbol{M} + (\mathrm{tr}\boldsymbol{M})^2 + (p+1)\mathrm{tr}\boldsymbol{M}^2\big\}\Big] \\ &+ \mathrm{O}(n^{-2}), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \boldsymbol{M} &= (1-\gamma^2)\begin{bmatrix} \varphi_1 - \gamma^2 & \gamma(\varphi_2 - 1) \\ \gamma(\varphi_1 - 1) & \varphi_2 - \gamma^2 \end{bmatrix}^{-1} \\ &= \frac{1}{\varphi_1\varphi_2}\left(1 - \frac{\gamma^2}{\varphi_1\varphi_2}\right)^{-1}\begin{bmatrix} \varphi_2 - \gamma^2 & -\gamma(\varphi_2 - 1) \\ -\gamma(\varphi_1 - 1) & \varphi_1 - \gamma^2 \end{bmatrix} \end{aligned} \tag{4.4}$$

and

$$\varphi_1 = 1 - 2i(1-\gamma^2)t_1, \quad \varphi_2 = 1 - 2i(1-\gamma^2)t_2. \tag{4.5}$$

Using (4.4) and (4.5), we can write

$$
\begin{aligned}
C(t_1,t_2) &= (1-\gamma^2)^{p/2}(\varphi_1\varphi_2)^{-p/2}\varphi^{-p/2}\Big[\,1\\
&\quad+\frac{p}{4n}\Big\{2\varphi^{-1}(p+1)\Big[2(\varphi_1^{-1}+\varphi_2^{-1})-(\varphi_1\varphi_2)^{-1}-3\Big]\\
&\quad+\varphi^{-2}(p+2)[\varphi_1^{-2}+\varphi_2^{-2}+2(\varphi_1\varphi_2)^{-1}-4(\varphi_1^{-1}+\varphi_2^{-1})+4]\Big\}\Big]\\
&\quad+\mathrm{O}(n^{-2}),
\end{aligned}
\tag{4.6}
$$

where $\varphi = 1-\gamma^2(\varphi_1\varphi_2)^{-1}$. For inversion of (4.3), we use

$$
\varphi^{-(p+2\ell)/2} = \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}p+\ell\right)_j}{j!}\gamma^{2j}(\varphi_1\varphi_2)^{-j},
\tag{4.7}
$$

where $(a)_j = a(a+1)\cdots(a+j-1)$. Note that the characteristic function of $\widetilde{T}_j^2 = T_j^2/(1-\gamma^2)$, $j=1,2$ can be expanded as (4.3) with $\varphi_j = 1-2it_j$. From these results we can invert (4.3) as

$$
\begin{aligned}
F(x_1,x_2) &= (1-\gamma^2)^{p/2}\sum_{j=0}^{\infty}\frac{1}{j!}\gamma^{2j}\Big[\left(\frac{1}{2}p\right)_j G_{p+2j}(\widetilde{x}_1)G_{p+2j}(\widetilde{x}_2)\\
&\quad+\frac{p}{4n}\Big\{2(p+1)\left(\frac{1}{2}p+1\right)_j\Big(2G_{p+2j+2}(\widetilde{x}_1)G_{p+2j}(\widetilde{x}_2)\\
&\quad+2G_{p+2j}(\widetilde{x}_1)G_{p+2j+2}(\widetilde{x}_2)-G_{p+2j+2}(\widetilde{x}_1)G_{p+2j+2}(\widetilde{x}_2)\\
&\quad-3G_p(\widetilde{x}_1)G_p(\widetilde{x}_2)\Big)+(p+2)\left(\frac{1}{2}p+2\right)_j\Big(G_{p+2j+4}(\widetilde{x}_1)G_{p+2j}(\widetilde{x}_2)\\
&\quad+G_{p+2j}(\widetilde{x}_1)G_{p+2j+4}(\widetilde{x}_2)-4G_{p+2j+2}(\widetilde{x}_1)G_{p+2j}(\widetilde{x}_2)\\
&\quad-4G_{p+2j}(\widetilde{x}_1)G_{p+2j+2}(\widetilde{x}_2)+2G_{p+2j+2}(\widetilde{x}_1)G_{p+2j+2}(\widetilde{x}_2)\\
&\quad+4G_{p+2j}(\widetilde{x}_1)G_{p+2j}(\widetilde{x}_2)\Big)\Big\}\Big]+\mathrm{O}(n^{-2}),
\end{aligned}
\tag{4.8}
$$

where $\widetilde{x}_j = x_j/(1-\gamma^2)$, $j=1,2$. After much simplification, we obtain the following theorem.

Theorem 4.1. Under (4.2) the distribution function of (T_1^2, T_2^2) can be expand as

$$
\begin{aligned}
F(x_1,x_2) = (1-\gamma^2)^{p/2}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}p\right)_j}{j!}\gamma^{2j}G_{p+2j}(\tilde{x}_1)G_{p+2j}(\tilde{x}_2)\Big[\,1 \\
+\frac{1}{4n}\big\{-2\tilde{x}_1(\tilde{x}_1+p-2j)h_{p+2j}(\tilde{x}_1)-2\tilde{x}_2(\tilde{x}_2+p-2j)h_{p+2j}(\tilde{x}_1) \\
+8\tilde{x}_1\tilde{x}_2\frac{2j+1}{p+2j}h_{p+2j}(\tilde{x}_1)h_{p+2j}(\tilde{x}_2)\big\}\Big]+\mathrm{O}(n^{-2}),
\end{aligned} \tag{4.9}
$$

where $h_{p+2j}(\tilde{x}_i)=g_{p+2j}(\tilde{x}_i)/G_{p+2j}(\tilde{x}_i)$, $\tilde{x}_i=x_i/(1-\gamma^2)$, $i=1,2$.

Theorem 4.1 implies that

$$
\begin{aligned}
F(x_1,x_2) &= \Pr\Big(\max(T_1^2,T_2^2)\le x\Big) \\
&= (1-\gamma^2)^{p/2}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}p\right)_j}{j!}\gamma^{2j}G_{p+2j}^2(\tilde{x})\Big[\,1 \\
&\quad+\frac{1}{n}\Big\{-\tilde{x}(\tilde{x}+p-2j)h_{p+2j}(\tilde{x})+2\tilde{x}^2\frac{2j+1}{p+2j}h_{p+2j}^2(\tilde{x})\Big\}\Big] \\
&\quad+\mathrm{O}(n^{-2}),
\end{aligned} \tag{4.10}
$$

where $h_{p+2j}(\tilde{x})=g_{p+2j}(\tilde{x})/G_{p+2j}(\tilde{x})$.

On the other hand, it is possible to obtain an expansion for $\Pr\{T_1^2>x_1, T_2^2>x_2\}$ by integrating an expansion of the density function of (T_1^2,T_2^2) in a region of $(x_1,\infty)\times(x_2,\infty)$. The resulting expansion is the one obtained from (4.8) by substituting $1-G_f(x_i)$ for $G_f(x_i)$, $i=1,2$. Therefore if we put $G_f^*(x)$ as $G_f^*(x)=1-G_f(x)$. Noting the relations,

$$
\begin{aligned}
G_{f+2}^*(x) &= \frac{2x}{f}g_f(x)+G_f^*(x), \\
G_{f+4}^*(x) &= \frac{2x}{f}\left(\frac{x}{f+2}+1\right)g_f(x)+G_f^*(x),
\end{aligned}
$$

we obtain

$$\begin{aligned}\Pr\{T_1^2 > x,\, T_2^2 > x\} = (1-\gamma^2)^{p/2}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}p\right)_j}{j!}\gamma^{2j}G^{*2}_{p+2j}(\tilde{x})\Big[\,1 \\ + \frac{1}{n}\Big\{\tilde{x}(\tilde{x}+p-2j)h^{*}_{p+2j}(\tilde{x}) + 2\tilde{x}^2\frac{2j+1}{p+2j}h^{*2}_{p+2j}(\tilde{x})\Big\}\Big] \\ + \mathrm{O}(n^{-2}),\end{aligned} \tag{4.11}$$

where $h^*_{p+2j}(\tilde{x}) = g_{p+2j}(\tilde{x})/G^*_{p+2j}(\tilde{x})$. We note that the result (4.11) is essentially the same as that of Seo and Siotani(1992), and coincides with Theorem 4.3 with the case of $p = q$ in Seo(1995) since the relations to the notations of Seo(1995) are $\eta = \tilde{x}/2$, $g_{p/2+j}(\eta) = 2g_{p+2j}(\tilde{x})$ and $G_{p/2+j}(\eta) = G^*_{p+2j}(\tilde{x})$. The accuracy of an asymptotic expansion for $\Pr\{T_i^2 > x,\ T_j^2 > x\}$ has been investigated in Siotani(1959) and Seo and Siotani(1992) through the approximation to the upper percentiles of $T^2_{\max}$.

BIBLIOGRAPHY

Krishnamoorthy, A. S. and Parthasarathy, M. (1951). A multivariate gamma type distribution. *Annals of Mathematical Statistics,* **22**, 549–577. Correction(1960). *Annals of Mathematical Statistics,* **31**, 229.

Royen, T. (1991). Expansions for the multivariate chi-square distribution. *Journal of Multivariate Analysis,* **38**, 213–232.

Royen, T. (1994). On some multivariate gamma-distributions connected with spanning trees. *Annals of the Institute of Statistical Mathematics,* **46**, 361–371.

Seo, T. (1995). Simultaneous confidence procedures for multiple comparisons of mean vectors in multivariate normal populations. *Hiroshima Mathematical Journal,* **25**, 387–422.

Seo, T., Mano, S. and Fujikoshi, Y. (1994). A generalized Tukey conjecture for multiple comparisons among mean vectors. *Journal of the American Statistical Association,* **89**, 676–679.

Seo, T. and Siotani, M. (1992). The multivariate Studentized range and its upper percentiles. *Journal of the Japan Statistical Society,* **22**, 123–137.

Siotani, M. (1959). The extreme value of the generalized distances of the individual points in the multivariate normal sample. *Annals of the Institute of Statistical Mathematics,* **10**, 183–208.

Siotani, M., Hayakawa, T. and Fujikoshi, Y. (1985). *Modern Multivariate Statistical Analysis: A Graduate Course and Handbook,* American Science Press, Ohio.