Asymptotic Approximations for EPMC's of the Linear and the Quadratic Discriminant Functions
When the Sample Sizes and the Dimension are Large

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Abstract

In this paper we obtain asymptotic approximations for the expected probability of misclassification (EPMC) of a classification statistic including the quadratic discriminant function as well as the linear discriminant function when both the sample sizes and the dimension are large. We also make a comparison for the accuracy of our asymptotic approximations with other asymptotic expansions when the sample sizes are large. It is shown that the approximations proposed in this paper are considerably good for low-dimensional cases as well as high-dimensional cases. Further, among them, we propose to use an extension of Raudys (1972) as a better approximation.

Key Words and Phrases: Accuracy, Asymptotic approximations, Discriminant functions, Expected probability of misclassification, Simulation.


1 On leave from Science University of Tokyo
1. INTRODUCTION

Let $\Pi_i : N_p(\mu_i, \Sigma)$, $i = 1, 2$ be the two p-variate normal populations into one of which we wish to classify a $p \times 1$ vector observation $\mathbf{x}$. Suppose that $\mu_1 \neq \mu_2$, $\Sigma$ is positive definite and all the parameters are unknown. Let $\bar{\mathbf{x}}_i$ and $\mathbf{S}$ be the sample mean vectors and the sample covariance matrix, based on a sample of $N_1$ observations from $\Pi_1$ and $N_2$ observations from $\Pi_2$.

The observation $\mathbf{x}$ may be classified by means of the classification statistic $W$ or $Z$ defined by

$$W = (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)'S^{-1}[\mathbf{x} - \frac{1}{2}(\bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_2)]$$

and

$$Z = \frac{N_2}{N_2 + 1}(\mathbf{x} - \bar{\mathbf{x}}_2)'S^{-1}(\mathbf{x} - \bar{\mathbf{x}}_2) - \frac{N_1}{N_1 + 1}(\mathbf{x} - \bar{\mathbf{x}}_1)'S^{-1}(\mathbf{x} - \bar{\mathbf{x}}_1),$$

respectively. The rule is usually to classify $\mathbf{x}$ as coming from $\Pi_1$ if $W > 0$ ( or $Z > 0$) and from $\Pi_2$ if $W \leq 0$ ( or $Z \leq 0$). One of the important problems on the classification procedures is to evaluate the expected (or unconditional) probabilities of misclassification (EPMC), i.e.,

$$e_T(2 \mid 1) = \Pr(T \leq 0 \mid \mathbf{x} \in \Pi_1),$$

$$e_T(1 \mid 2) = \Pr(T > 0 \mid \mathbf{x} \in \Pi_2),$$

where $T = W$ or $Z$.

In general, it is hard to obtain the exact evaluation of the EPMC's, but there are considerable works for their asymptotic approximations including asymptotic expansions. It may be noted that there are two types (type-I, type-II) of their asymptotic approximations. Type-I approximations are the ones under a framework such that $N_1$ and $N_2$ are large and $p$ is fixed. For a review of these results, see, e.g., Siotani (1982). Naturally, the accuracy of these approximations will become bad as the dimension is large. On the other hand, type-II approximations are the ones under a framework such that $N_1$, $N_2$ and $p$ are large. For type-II approximations, Deev (1970) gave an asymptotic expansion of the distribution of $W$. Raudys (1972) proposed an approximation for the EPMC of $W$ in the case $N_1 = N_2$. Wyman et al. (1990) compared the accuracy of several approximations for $W$ in the case $N_1 = N_2$, and pointed that the approximation due to Raudys (1972) has overall the best accuracy for the combinations of the parameters considered in the study.
The purpose of this paper is to obtain type-II approximations for the EPMC of a classification statistic including the quadratic discriminant function $Z$ as well as the linear discriminant function $W$ and make a comparison of these approximations with type-I approximations. Among them, we propose to use an extension of Raudys(1972) as a better approximation.

The present paper is organized in the following way. Some distributional reductions are given in section 2. In section 3 we derive type-II approximations. In section 4 we compared the accuracy of these approximations and type-I approximations. It is shown that our approximations are considerably good also for the case $N_1 \neq N_2$ and low-dimensional cases as well as high-dimensional cases.

2. DISTRIBUTIONAL REDUCTIONS

In order to treat the two discriminant functions $W$ and $Z$ in a unified way, we consider the discriminant function defined by

$$T = \frac{1}{2} \left[ (x - \bar{x}_2)'S^{-1}(x - \bar{x}_2) - b(x - \bar{x}_1)'S^{-1}(x - \bar{x}_1) \right], \quad (2.1)$$

where $b$ is a constant. The rule is to classify $x$ as coming from $\Pi_1$ if $T > c$ and from $\Pi_2$ if $T \leq c$, where $c$ is a constant, particularly $c = 0$. The important special cases are

(i) $T = W$ when $b = 1$, and

(ii) $T = \frac{1}{2}(1 + \frac{1}{N_2})Z$ when $b = (1 + \frac{1}{N_2})^{-1}(1 + \frac{1}{N_1})$.

The distribution of $T$ depends on the parameters $\mu_1$, $\mu_2$ and $\Sigma$ through the Mahalanobis squared distance $\Delta^2 = (\mu_1 - \mu_2)'\Sigma^{-1}(\mu_1 - \mu_2)$. The sample Mahalanobis squared distance is denoted by $D^2$, i.e., $D^2 = (\bar{x}_1 - \bar{x}_2)'S^{-1}(\bar{x}_1 - \bar{x}_2)$. We denote the distribution function of $T$ for $x$ coming from $\Pi_1$ by

$$\Pr(T \leq t \mid x \in \Pi_1) = G(t, b, N_1, N_2, \Delta^2). \quad (2.2)$$

Then, it is easily seen that

$$\Pr(T \leq t \mid x \in \Pi_2) = 1 - G(-b^{-1}t; b^{-1}, N_2, N_1, \Delta^2). \quad (2.3)$$

The EPMC's of the classification rule with $c = 0$ are given by

$$e_T(2|1) = G(0; b, N_1, N_2, \Delta^2),$$

$$e_T(1|2) = G(0; b^{-1}, N_2, N_1, \Delta^2). \quad (2.4)$$
That is, \( e_T(1|2) \) is obtained from \( e_T(2|1) \) by substituting \((b, N_1, N_2)\) into \((b^{-1}, N_2, N_1)\). Therefore, it is sufficient to study the distribution of \( T \) for \( x \) coming from \( \Pi_1 \). In the following we assume that \( x \) is distributed as \( N_p(\mu, \Sigma) \).

Let \( \bar{x}_1 = \{1/(N_1 + 1)\}(x + N_1 x_1) \). Then we have

\[
T = \frac{1}{2} \{ 1 - b(1 + \frac{1}{N_1}) \} (x - \bar{x}_1)' S^{-1} (x - \bar{x}_1) \\
+ (x - \bar{x}_1)' S^{-1} (\bar{x}_1 - \bar{x}_2) + \frac{1}{2} (\bar{x}_1 - \bar{x}_2)' S^{-1} (\bar{x}_1 - \bar{x}_2).
\]

Now we define

\[
u_1 = \left(1 + \frac{1}{N_1}\right)^{1/2} \Sigma^{-1/2}(x - \bar{x}_1), \quad \nu_2 = \left\{ \frac{(N_1 + 1)N_2}{(N + 1)} \right\}^{1/2} \Sigma^{-1/2}(\bar{x}_1 - \bar{x}_2)
\]

and \( V = n \Sigma^{-1/2} \Sigma^{-1/2} \), where \( N = N_1 + N_2 \) and \( n = N - 2 \). Then

\[
u_1 \sim N_p(0, I_p), \quad \nu_2 \sim N_p\left( \left\{ \frac{(N_1 + 1)N_2}{(N + 1)} \right\}^{1/2} (\mu_1 - \mu_2), \ I_p \right), \tag{2.5}
\]

\[
V \sim W_p(n, I_p),
\]

and they are independent. We can express \( T(\text{see, Srivastava and Khatri}(1979)) \) as

\[
T = a_1 T_1 + a_2 T_2 + a_3 T_3, \tag{2.6}
\]

where

\[
T_1 = \nu_1 V^{-1} \nu_1, \quad T_2 = \sqrt{N_1} \nu_1 V^{-1} \nu_2, \quad T_3 = \nu_2 V^{-1} \nu_2;
\]

\[
a_1 = \frac{1}{2} \left( \frac{nN_1}{N_1 + 1} \right) \left\{ 1 - b \left( 1 + \frac{1}{N_1} \right)^2 \right\},
\]

\[
a_2 = \frac{n}{N_1 + 1} \left\{ \frac{N_1(N + 1)}{N_2} \right\}^{1/2}, \quad a_3 = \frac{1}{2} \frac{n(N + 1)}{N_1 + 1}.
\]

Let \( \nu_1 \) and \( V \) be partitioned as

\[
\nu_1 = \begin{bmatrix} \nu_{11} \\ \nu_{21} \end{bmatrix}, \quad V = \begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{bmatrix},
\]

where \( \nu_{11} : 1 \times 1 \) and \( \nu_{11} : 1 \times 1 \).

**Lemma 2.1.** The distribution of \((T_1, T_2, T_3)\) in (2.6) is the same as that of \((T_1, T_2, T_3)\) with the following expressions:

\[
T_1 = \frac{1}{\nu_{11,2}} (\nu_{11} - \nu_{21} V^{-1} \nu_{21})^2 + \nu_{21} V^{-1} \nu_{21},
\]

\[
T_2 = \frac{\sqrt{N_1}}{\nu_{11,2}} (\nu_{11} - \nu_{21} V^{-1} \nu_{21})(\nu_{21} \nu_{22})^{1/2}, \tag{2.8}
\]

\[
T_3 = \frac{1}{\nu_{11,2}} \nu_{21} \nu_{22},
\]

- 3 -
where \( v_{11.2} = v_{11} - v_{21}'V_{22}^{-1}v_{21} \). Here \( u_1 \), \( u_2 \) and \( V \) are independently distributed as in (2.5).

**Proof.** Let \( H \) be a \( p \times p \) random orthogonal matrix with its first column proportional to \( u_2 \). Define

\[
\overline{u}_1 = H'u_1, \quad \overline{V} = H'VH.
\]

Then the distribution of \((\overline{u}_1, \overline{V}, u_2)\) is the same as that of \((u_1, V, u_2)\). Furthermore, we have

\[
T_1 = \overline{u}_1'\overline{V}^{-1}\overline{u}_1, \quad T_2 = \overline{u}_1'\overline{V}^{-1}e_1(u_2'u_2)^{1/2}, \quad T_3 = \overline{v}_{11}u_2'u_2,
\]

where \( e_1 = (1, 0, \cdots, 0)' \). Without loss of generality, we may replace \( \overline{u}_1 \) and \( \overline{V} \) in the above expression by \( u_1 \) and \( V \), respectively, and so we shall do that. Then the resultant expressions can be simplified by using an inverse formula of a partitioned matrix, which yields the conclusion of the Theorem.

**Lemma 2.2.** Suppose that \( n - p + 1 > 0 \). Then the statistics \((T_1, T_2, T_3)\) in (2.6) can be expressed in terms of independent standard normal variates \( z_i, i = 1, 2, 3 \) and chi-squared variates \( y_i, (i = 1, \cdots, 6) \) with \( f_i \) degrees of freedom as follows:

\[
T_1 = \frac{1}{y_2} \left[ z_1 + z_2 \left( \frac{y_3y_5}{y_4(y_6 + z_2^2)} \right)^{1/2} \right]^2 + \frac{y_3}{y_4},
\]

\[
T_2 = \frac{\sqrt{N_2}}{y_2} \left[ z_1 + z_2 \left( \frac{y_3y_5}{y_4(y_6 + z_2^2)} \right)^{1/2} \right] \{(z_3 + \xi)^2 + y_1\}^{1/2},
\]

\[
T_3 = \frac{1}{y_2} \{(z_3 + \xi)^2 + y_1\},
\]

where \( f_1 = f_3 = f_5 = p - 1, f_2 = n - p + 1, f_4 = n - p + 2, f_6 = p - 2, \) and \( \xi = ((N_1 + 1)N_2/(N + 1))^{1/2} \Delta \). Here, for the case \( p = 2, y_6 \) should be regarded as zero. Similarly, for the case \( p = 1, y_1, y_3, y_5 \) and \( y_6 \) should be regarded as zero.

**Proof.** Our proof is based on Lemma 2.1, and the fact (see, e.g., Siotani et al. (1985)) that \( v_{11.2} \sim \chi_{n-p+1}^2, \) \( V_{22} \sim W_{p-1}(n, I_{p-1}), \) \( \ell = V^{-1/2}v_{21} \sim N_{p-1}(0, I_{p-1}) \) and they are independent. From these results we can write as

\[
z_1 = u_2'V_{22}^{-1}v_{21}, \quad y_2 = v_{11.2}, \quad \frac{y_3}{y_4} = u_2'V_{22}^{-1}u_{21}.
\]

Note that

\[
 u_2'V_{22}^{-1}v_{21} = (u_2'V_{22}^{-1}u_{21})^{1/2} (\ell'\ell)^{1/2},
\]
where $\gamma = g'\ell(\ell'\ell)^{1/2}$ and $g = (u_{21}V^{-1}_{22}u_{21})^{-1/2}V^{-1}_{22}u_{21}$. Using Theorem 1.5.7 in Muirhead(1982) we can express

$$\gamma = z_2(y_6 + z_2^2)^{-1/2}.$$ 

Furthermore, $\gamma$ is independent of $u_{21}'V^{-1}_{22}u_{21}$ and $\ell'\ell$. The remainder of the proof is to show that $u_2'u_2$ can be express as $(z_2 + \xi)^2 + y_1$. This follows from that $u_2'u_2$ is distributed as a noncentral chi-square distribution with $p$ degrees of freedom and noncentrality parameter $\xi^2$.

For the linear discriminant function, Deev(1970) has given a similar representation as in Lemma 2.2.

### 3. Asymptotic Approximations

We consider asymptotic behaviour of $T$ in the situation where $N_1, N_2$ and $p$ are large. For this, let

$$f_i = \rho_i m, \quad i = 1, \cdots, 6; \quad N_i = \lambda_i m, \quad i = 1, 2,$$

where $m$ may be taken as $p$ or $N$. It is assumed that

$$\rho_i = O(1) \quad (i = 1, \cdots, 6), \quad \text{and} \quad \lambda_i = O(1) \quad (i = 1, 2)$$

as $N_1, N_2$ and $p$ tend to infinity$^*$. In general, we denote the term of the $j$th order with respect to $(N_1^{-1}, N_2^{-1}, p^{-1})$ by $O_j$.

**Theorem 3.1.** Let $T$ be the discriminant function defined by (2.1) such that $b = 1 + O_1$. Suppose that $\varpi$ comes from $\Pi : N_p(\mu, \Sigma)$. Then, under (3.2) $T$ is asymptotically distributed as $N(\zeta, \sigma^2)$, where

$$\zeta = \frac{1}{2} \left( \frac{N}{N-p} \right)^{p(N_1 - N_2)} \left( \frac{N_1 N_2}{p(1 + b)(1 + N_1^{-1})} \right),$$

$$\sigma^2 = \left( \frac{N}{N-p} \right)^3 \left( \frac{pN}{N_1 N_2} \right).$$

**Proof.** The theorem can be proved with the help of Lemma 2.2. It is well known that

$$\nu_j = \sqrt{\frac{f_j}{2}} \left( \frac{y_j}{f_j} - 1 \right)$$
is asymptotically distributed as \( N(0, 1) \) when \( f_j \) tends to infinity. Using this property and the assumption (3.2), we can see that \( T \) is asymptotically equivalent to

\[
\tilde{T} = a_1 \frac{\rho_3}{\rho_4} + a_2 \frac{1}{\rho_2} \left\{ \lambda_1 (\rho_1 + \frac{1}{m} \xi^2) \right\}^{1/2} \left[ z_1 + \left\{ \frac{\rho_3 \rho_5}{\rho_4 \rho_6} \right\}^{1/2} z_2 \right] + a_3 \frac{1}{\rho_2} (\rho_1 + \frac{1}{m} \xi^2).
\]

This implies that \( \tilde{T} \) (and hence \( T \)) is asymptotically distributed as \( N(\bar{\xi}, \bar{\sigma}^2) \), where

\[
\bar{\xi} = a_1 \frac{\rho_3}{\rho_4} + a_3 \frac{1}{\rho_2} (\rho_1 + \frac{1}{m} \xi^2),
\]

\[
\bar{\sigma}^2 = \left( \frac{a_2}{\rho_2} \right)^2 \left\{ \lambda_1 (\rho_1 + \frac{1}{m} \xi^2) \right\} \left[ 1 + \left\{ \frac{\rho_3 \rho_5}{\rho_4 \rho_6} \right\}^2 \right].
\]

Our conclusion can be obtained by simplifying \( \bar{\xi} \) and \( \bar{\sigma}^2 \) in an asymptotic sense.

From Theorem 3.1 and (2.3) we can obtain asymptotic distributions for \( W \) and \( Z \).

**Theorem 3.2.** Suppose that \( \mathbf{z} \) comes from \( \Pi_i : N_p(\mu_i, \Sigma), \ i = 1, 2 \). Then under (3.2) \( W \) and \( \frac{1}{2}Z \) are asymptotically distributed as \( N(\zeta_W^{(i)}, \sigma^2) \) and \( N(\zeta_Z^{(i)}, \sigma^2) \), respectively, where

\[
\zeta_W^{(i)} = \frac{1}{2} \left( \frac{N}{N - p} \right) \left\{ \Delta^2 + \frac{p}{N_1 N_2} (-1)^{i+1} (N_1 - N_2) \right\},
\]

\[
\zeta_Z^{(i)} = \zeta_Z^{(2)} = \frac{1}{2} \left( \frac{N}{N - p} \right) \Delta^2,
\]

\[
\sigma^2 = \left( \frac{N}{N - p} \right)^3 \left\{ \Delta^2 + \frac{p N}{N_1 N_2} \right\}.
\]

**Corollary 3.1.** Under (3.2) the EPMC’s for \( W \) and \( Z \) are asymptotically evaluated as follows:

\[
e_{W}(2 \mid 1) \simeq \Phi(\gamma_W^{(1)}), \quad e_{W}(1 \mid 2) \simeq \Phi(\gamma_W^{(2)}),
\]

\[
e_{Z}(2 \mid 1) \simeq \Phi(\gamma_Z^{(1)}), \quad e_{Z}(1 \mid 2) \simeq \Phi(\gamma_Z^{(2)}),
\]

where \( \Phi \) is the distribution function of \( N(0, 1) \),

\[
\gamma_W^{(i)} = -\frac{1}{2} \left( \frac{N - p}{N} \right)^{1/2} \left\{ \Delta^2 + \frac{p}{N_1 N_2} (-1)^{i+1} (N_1 - N_2) \right\} \left\{ \Delta^2 + \frac{p N}{N_1 N_2} \right\}^{-1/2},
\]

\[
\gamma_Z^{(1)} = \gamma_Z^{(2)} = -\frac{1}{2} \left( \frac{N - p}{N} \right)^{1/2} \Delta^2 \left\{ \Delta^2 + \frac{p N}{N_1 N_2} \right\}^{-1/2}.
\]
In a special case \( N_1 = N_2 \) we obtain

\[
\gamma_w^{(1)} = \gamma_w^{(2)} = -\frac{1}{2} \left( \frac{N - p}{N} \right)^{1/2} \Delta^2 \left\{ \Delta^2 + \frac{pN}{N_1N_2} \right\}^{-1/2}.
\]

which was proposed by Raudys(1972).

On the other hand, Lachenbruch(1968) has proposed the approximations for \( e_w(2 | 1) \) and \( e_w(1 | 2) \), which are expressed as

\[
e_w(2 \mid 1) \simeq \Phi(\tilde{\gamma}_w^{(1)}), \quad e_w(1 \mid 2) \simeq \Phi(\tilde{\gamma}_w^{(2)}), \tag{3.7}
\]

where

\[
\tilde{\gamma}_w^{(i)} = -\frac{1}{2} \left\{ \frac{(N - p - 2)(N - p - 5)}{(N - 3)(N - p - 3)} \right\}^{1/2} \times \left\{ \Delta^2 + \frac{p}{N_1N_2}(-1)^{i+1}(N_1 - N_2) \right\} \left\{ \Delta^2 + \frac{pN}{N_1N_2} \right\}^{-1/2}, \quad i = 1, 2. \tag{3.8}
\]

This approximations were proposed, without considering their asymptotic distributions. In fact, he proposed it by replacing the mean and variance in the conditional normality by their expectations. From Corollary 3.1 we can point that the approximation in (3.7) can be justified as an asymptotic result when \( N_1, N_2 \) and \( p \) are large. For further results including error bounds, see Fujikoshi(1997).

In general, asymptotic approximations in (3.5) or (3.7) are useful in estimating the conditional probabilities of misclassification (CPMC) for \( W \)- and \( Z \)-discrimination rule, given \( \bar{x}_1, \bar{x}_2 \) and \( S \). In fact, such estimators can be constructed by using the unbiased estimator

\[
\frac{n - p - 1}{n} D^2 - \frac{pN}{N_1N_2}
\]

of \( \Delta^2 \) based on \( D^2 \), in the asymptotic approximations.

4. A COMPARISON OF ACCURACY

We study a comparison of the accuracy of asymptotic approximations in (3.5) and (3.7) with the other approximations based on asymptotic expansions when the sample sizes \( N_1 \) and \( N_2 \) tend to infinity and \( N_2/N_1 \) tends to a finite positive constant. For the latter asymptotic expansions, we used the approximations up to terms of the first order with respect to \((N_1^{-1}, N_2^{-1}, n^{-1})\), due to Okamoto(1963) for \( W \) and Memon and Okamoto(1971) for \( Z \). These approximations are denoted by \( AE_W^{(i)} \) and \( AE_Z^{(i)} \), \( i = 1, 2 \). For the linear discriminant function \( W \), Wyman et al.(1990)
compared the accuracy of these approximations except for \( \Phi(\gamma_L^{(1)}) \) with some other approximations. However, their comparison has been done only for the case \( N_1 = N_2 \).

Here we attempt to add a comparison of the accuracy of asymptotic approximations for \( W \) in the case \( N_1 \neq N_2 \) and for the quadratic discriminant function \( Z \). First we note that there are some relations between the EPMC's. For the case \( N_1 = N_2 \), the two discriminant procedures based on \( W \) and \( Z \) are equivalent, and so \( e_W(2 \mid 1) = e_Z(2 \mid 1) \) and \( e_W(1 \mid 2) = e_Z(1 \mid 2) \). Furthermore, there is a certain symmetric property such that \( e_W(2 \mid 1) \) (or \( e_Z(2 \mid 1) \)) is obtained from \( e_W(1 \mid 2) \) (or \( e_Z(1 \mid 2) \)) by interchanging \( N_1 \) ans \( N_2 \). Our comparison was done for the following approximations:

(i) For \( W \); \( \Phi(\gamma_W^{(1)}), \Phi(\gamma_W^{(1)}), AE_W^{(1)} \),

(ii) For \( Z \); \( \Phi(\gamma_Z^{(1)}), AE_Z^{(1)} \).

The values of \( p, N_1, N_2 \) and \( \Delta \) were chosen as follows:

\( p; 2, 5, 10, \)

\( (N_1, N_2); (10,10), (10,20), (10,30), (20,20), (20,40), (20,10), (30,10), (40,20), \)

\( \Delta; 1.05, 1.68, 2.56, 3.29. \)

Here the values of \( \Delta \) were chosen as in Wyman et al.(1990).

Although there are some relations between EPMC's as being mentioned above, we lists the approximations and simulation results for all combinations of \( p, N_1, N_2 \) and \( \Delta \). From Tables 1, 2 and 3 it is seen that the type-II approximations are considerably good even for low-dimentional cases. The approximation \( \Phi(\gamma_W^{(1)}) \) is better than the approximation \( \Phi(\gamma_W^{(1)}) \) in all the cases. Further, it may be noted that the approximations \( AE_W^{(1)} \) and \( AE_Z^{(1)} \) get bad as \( p \) is comparable with \( N_1 \) and \( N_2 \).
Table 1. Values of approximations and simulation.

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<th>$\Phi(\gamma_{W}^{(1)})$</th>
<th>$\Phi(\gamma_{Z}^{(1)})$</th>
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<td>$AE_{Z}^{(1)}$</td>
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| 2.56 | 0.2067 | 0.1919 | 0.1793 | 0.1926 | 0.1729 | 0.1610 | 0.1713 |
| 3.29 | 0.1297 | 0.1150 | 0.09805 | 0.1172 | 0.1041 | 0.0888 | 0.1036 |

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| 2.56 | 0.1410 | 0.1328 | 0.1307 | 0.1394 | 0.1561 | 0.1550 | 0.1615 |
| 3.29 | 0.08390 | 0.07679 | 0.07290 | 0.07944 | 0.08944 | 0.08513 | 0.09517 |

| (40,20) | 1.05 | 0.3277 | 0.3251 | 0.3343 | 0.3266 | 0.3558 | 0.3727 | 0.3570 |
| 1.68 | 0.2328 | 0.2291 | 0.2303 | 0.2294 | 0.2478 | 0.2506 | 0.2485 |
| 2.56 | 0.1292 | 0.1253 | 0.1237 | 0.1259 | 0.1342 | 0.1328 | 0.1360 |
| 3.29 | 0.07195 | 0.06867 | 0.06650 | 0.06934 | 0.07322 | 0.07108 | 0.07484 |
REFERENCES


