Robustness of Control Procedures For Integrated Moving Average Process of Order One

by

M.S. Srivastava
Department of Statistics
University of Toronto

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Dept of Statistics, University of Toronto, Toronto, Ontario, Canada. M5S 1A1

ABSTRACT

A commonly used model in process adjustment using feedback is the integrated moving average (IMA) process of order one in which the disturbances are assumed to be normally distributed. Box and Luceño recently studied the robustness of the control procedures for departure from normality using mixture normal and uniform distributions for the disturbances as an alternative. These studies are, however, based on simulations and apply only to the random walk model since they assume the non-stationarity parameter is one. In this paper, the average cost rate and the optimum action limit are evaluated for any disturbance that is symmetrically distributed around zero having six moments and for any non-stationary parameter lying between zero and one. These results not only help to study the robustness of control procedures for IMA process but also can be used for control procedures should the disturbances not be normally distributed.

Key Words and Phrases: Average adjustment interval, Average cost rate, Mean lower case deviation, Optimum control limit.
1. INTRODUCTION

Shewhart, CUSUM, EWMA, and Shirayayev-Roberts procedures are used to monitor the statistical process in any experiment to distinguish between common causes and assignable causes. If the variation in the process can be explained by only common causes, the process is said to be in a state of control and the observations are considered to be independent and identically distributed (iid); the common causes represent the random fluctuations in the experiment. However, it is possible, due to malfunction of the machine etc. that after some time, the observations may deviate from the given iid model to another model either through a shift in the mean, variance or distribution. The above control charts help determine this situation and thus lead to finding assignable causes. This aspect of quality control is known as statistical process control (SPC) in the literature.

In many situations, however, the process may wander away from the target value due to imperfect feed material, variation in temperature and humidity, etc. In such a process, an on-line control procedure is needed so that an adjustment can be made whenever the deviations from the target value exceeds a specified control limit. The adjustment, can be affected automatically or manually, and it can be applied directly to the process or by compensating an input variable employing feedback control to produce the desired adjustment.

In order to obtain an optimum value of the control limit or action limit, a reasonable model for the deviation from the target value is desired. Since the deviations are of a wandering kind, a nonstationary model such as a random walk model or an integrated moving average process of order one will be more appropriate. In this paper, we consider the latter model, in which the deviation from the largest value, represented by $Z_t$, can be expressed as

$$Z_t = \hat{Z}_t + a_t,$$  \hspace{1cm} (1.1)

where $\hat{Z}_t$ is independent of $a_t$ and is an exponentially weighted moving average (EWMA) of the past data defined by

$$\hat{Z}_t = \lambda[Z_{t-1} + \theta Z_{t-2} + \theta^2 Z_{t-3} + \ldots]$$
$$= \lambda Z_{t-1} + \theta \hat{Z}_{t-1},$$  \hspace{1cm} (1.2)

$\theta = 1 - \lambda$, $0 < \lambda \leq 1$ and is the predicted or forecasted value of the $t^{th}$ observation having observed
\[ Z_{t+1} = \lambda Z_t + \theta \hat{Z}_t. \]

From (1.1), we can also write

\[ \hat{Z}_{t+1} = \lambda [\hat{Z}_t + a_t] + (1 - \lambda) \hat{Z}_t, \]

giving

\[ \hat{Z}_{t+1} - \hat{Z}_t = \lambda a_t. \quad (1.3) \]

Thus, the first difference of \( Z_t \) is the first order moving average model

\[ Z_{t+1} - Z_t = \hat{Z}_{t+1} + a_{t+1} - \hat{Z}_t - a_t \]
\[ = a_{t+1} - a_t + \lambda a_t. \quad (1.4) \]

which, by summing and noting that \( Z_1 - a_1 = \hat{Z}_1 \) from (1.1), gives

\[ Z_t = \hat{Z}_1 + a_t + \lambda \sum_{i=1}^{t-1} a_i, \quad (1.5) \]

and

\[ \hat{Z}_{t+1} = \hat{Z}_1 + \lambda \sum_{j=1}^{t} a_j \quad (1.6) \]

If at time \( t = 1 \), the adjustment is perfect, then \( \hat{Z}_1 = 0 \) and from (1.1), \( Z_1 = a_1 \). We shall assume that the adjustment is perfect at \( t = 1 \), and the integrated moving average process of order one, denoted by \( IMA_1(\lambda, \sigma^2_a) \), is given by

\[ Z_t = a_t + \lambda \sum_{i=1}^{t-1} a_i, \quad Z_1 = a_1, \quad 0 < \lambda \leq 1 \quad (1.7) \]
where \( a_i \)'s are iid with mean 0 and variance \( \sigma_a^2 \). The parameter \( \lambda \) is called the nonstationarity parameter. Most of the derivations of the optimum action limit is based on the assumption that \( a_i \)'s are iid \( N(0, \sigma_a^2) \), see Box and Jenkins (1963) and, Srivastava and Wu (1991) for \( \lambda = 1 \). The objective of this paper is to derive the optimum action limit for any symmetric distribution where there is a cost of adjustment and thus the adjustment is made not at every inspection but when the forecasted value of the process exceeds a control limit. There is also a cost of being off target.

It is assumed that the cost of being off target is proportional to \( Z_i^2 \), the square of the deviation from the target value and \( C_T \) is the constant of proportionality that represents the cost of being off target for one time interval by an amount \( \sigma_a^2 \). Thus, when the inspection cost is zero and every item is inspected, the overall cost, denoted by \( C(L) \), is given by

\[
C(L) = \frac{C_A}{(AAI)} + C_o[MSD],
\]

where

\[
MSD = \frac{E[\sum_{j=1}^{N} Z_j^2]}{(AAI)},
\]

as defined by Box and Luceño (1994),

\[
C_o = \frac{C_T}{\sigma_a^2},
\]

and AAI is the average adjustment interval, where an adjustment is made as soon as the forecasted value \( \hat{Z}_{n+1} \) of the \( (n+1)^{st} \) observation exceeds a control limit \( L \), that is, at

\[
N = \min\{ n \geq 1 : |\hat{Z}_{n+1}| > L \}
\]

Thus,

\[
AAI = E(N).
\]

When \( a_i \)'s are iid \( N(0, \sigma_a^2) \), the cost function \( C(L) \) was evaluated by Box and Jenkins (1963) and for \( \lambda = 1 \) by Srivastava and Wu (1991). They also obtained the optimum value of \( L \), the one that minimizes the average cost rate \( C(L) \). It may be noted that from (1.6)

\[
\hat{Z}_{n+1} = \lambda \sum_{i=1}^{n} a_i, \quad \text{since } \hat{Z}_1 = 0.
\]
Thus, $\hat{Z}_{n+1}$ is a random walk which is a sum of $n$ iid random variables with mean 0 and variance $\lambda^2\sigma^2_a$.

2. **EVALUATION OF THE AVERAGE COST RATE UNDER A SYMMETRIC ERROR DISTRIBUTION.**

In this section, we shall assume that the error terms $a_j$'s are iid with mean 0, variance $\sigma^2_a$, third moment zero, the fourth moment $b\sigma^4_a$ and have a symmetric probability density function (pdf). For the normal distribution $b = 3$. Since $\hat{Z}_{n+1}$ is a random walk, we can write it as

$$\hat{Z}_{n+1} = \lambda \sum_{j=1}^{n} a_j = \lambda\sigma_a \sum_{j=1}^{n} w_j = \lambda\sigma_a X_n$$  \hspace{1cm} (2.1)

where $w_j$'s are iid with mean 0, variance 1 and symmetric pdf $f(w)$ and $X_n$ is a standard (mean 0, variance 1) random walk with $X_0 = 0$, since $\hat{Z}_1 = 0$. Thus, the random variable $N$ can be rewritten as

$$N = \min\{n \geq 1 : |X_n| > u\} ,$$  \hspace{1cm} (2.2)

where

$$u = \left(\frac{L}{\lambda \sigma_a}\right) \hspace{1cm} (2.3)$$

It is shown in the appendix that

$$AAI \equiv E(N) = E(X^2_N). \hspace{1cm} (2.4)$$

To evaluate $E(X^2_N)$, we shall consider the case of $u \geq 1$ separately from the case of $u < 1$.

2.1 **Evaluation of AAI.**

We shall first consider the case of $u \geq 1$. In this case it is shown in the appendix that

$$E(X^2_N) = u^2 + 2\rho_1 u + \rho_2, \hspace{0.5cm} u \geq 1, \hspace{1cm} (2.5)$$

where $\rho_1$ and $\rho_2$ are the first and second moment of the overshoot of the random walk $X_N$ from the boundry. Exact evaluations of $\rho_1$ and $\rho_2$ are not yet available in the literature. However, an approximation of $\rho_1$ and $\rho_2$ can be obtained using renewal theory as in Siegmund (1985) and
Srivastava and Wu (1991). For example if \(X_n\)'s are standard normal random walk (mean 0 and variance 1), then

\[
\rho_{1\phi} \simeq 0.583, \quad \rho_{2\phi} \simeq 0.59
\]  

(2.6)

Similarly, if \(w_j\)'s are iid uniformly distributed over \((-\sqrt{3}, \sqrt{3})\), then it has mean 0 and variance 1. In this case

\[
\rho_{1u} \simeq 0.5161, \quad \rho_{2u} \simeq 0.4164
\]

(2.7)

In the case of mixture normal \((1 - \alpha)\phi(0, \sigma_1^2) + \alpha\phi(0, \sigma_2^2)\), where \(\sigma_1^2 = 1 - \epsilon\) and \(\sigma_2^2 = 1 + \frac{\epsilon(1 - \alpha)}{\alpha}\), \(\epsilon \geq 0, \quad 0 < \alpha \leq 1\), and \(\phi(0, \sigma^2)\) denotes the normal density with mean 0 and variance \(\sigma^2\), these quantities are given by

\[
\rho_{1m} \simeq 0.5830 + 0.1303 \frac{\epsilon^2(1 - \alpha)}{\alpha} + 0(\epsilon^2)
\]

\[
\rho_{2m} \simeq \rho_{1m}^2 + \frac{1}{4} \left[ 1 + \frac{\epsilon^2(1 - \alpha)}{\alpha} \right] + 0(\epsilon^2)
\]

(2.8)

It may be noted that the above mixture normal distribution has mean 0 and variance 1. The general formulae for computing these quantities for any symmetric distribution with mean 0 and variance 1 are given in the appendix. The expressions in (2.8) are approximation of these formulae for small \(\epsilon\).

**Case 2.** \(u < 1\),

When \(u < 1\), the above approximations cannot be used, since the moments of the overshoot cannot be well approximated. Clearly for small \(u\), we expect the random walk \(X_n\) to cross the boundary much sooner. For example, if \(u = 0\), then even the first observation will cross the boundary since the absolute value of any random variable is positive and hence \(AAI = 1\). Thus, for small \(u\), Srivastava and Wu (1991) used a different technique to approximate \(E(X_N^2)\). It is shown in the appendix that for any symmetric pdf with \(f''(0) = 0\), it can be approximated by

\[
AAI = E(N) = E(X_N^2)
\]

\[
= 1 + \xi_1 u + \xi_2 u^2 + \xi_3 u^3 + 0(u^3), \quad u < 1.
\]

(2.9)

where

\[
\xi_1 = 2f(0), \quad \xi_2 = \xi_1^2, \quad \xi_3 = \frac{1}{3}f''(0) + \xi_1^3,
\]

(2.10)
and \( f'(0) \) and \( f''(0) \) are respectively the first and second derivatives of \( f \) evaluated at zero. It can be easily be verified that \( f'(0) = 0 \) for normal, uniform and mixture normal distributions with mean zero and variance one. This assumption, however, is needed only to simplify the computation. Expressions for \( \xi_1, \xi_2, \xi_3 \) can easily be obtained in the general case except it is a little more involved.

From these results, the AAI for the normal case can be compared with the uniform and mixture normal. It may, however, be noted that from (2.9)

\[
AAI \simeq 1
\]

for \( u = 0 \), no matter what the distribution is, as it should be. Calculating the value of \( \xi_i, \) we get

(i) For normal : \( \xi_{1\phi} = 0.798, \xi_{2\phi} = 0.636, \xi_{3\phi} = 0.376, \)

(ii) For uniform : \( \xi_{1u} = 0.289, \xi_{2u} = 0.083, \xi_{3u} = 0.024, \)

(iii) For mixture normal:

\[
\xi_{1m} = 0.798 \left[ \frac{1 - \alpha}{\sigma_1} + \frac{\alpha}{\sigma_2} \right], \quad \xi_{2m} = \xi_{1m}^2,
\]

\[
\xi_{3m} = 0.508 \left[ \frac{1 - \alpha}{\sigma_1} + \frac{\alpha}{\sigma_2} \right]^3 - 0.133 \left[ \frac{1 - \alpha}{\sigma_1^3} + \frac{\alpha}{\sigma_2^3} \right],
\]

See Table 1.

2.2 Evaluation of MSD.

Next, we need to evaluate the mean squared deviation.

\[
MSD = \frac{E(\sum_{j=1}^{N} Z_j^2)}{AAI}
\]

It is shown in the appendix that

\[
MSD = \left[ \frac{\lambda^2 \sigma_z^2}{6} \right] \left[ \frac{E(X_{kN})}{E(X_{kN}^2)} \right] + C_1 \sigma_a^2, \quad (2.11)
\]

where

\[
C_1 = \left( 1 - \frac{\lambda^2 b}{6} \right), \quad b = E(w_j^4). \quad (2.12)
\]
Thus,
\[
\frac{MSD}{\lambda^2 \sigma_a^2} = \frac{E(X_N^4)}{6E(X_N^2)} - \frac{b}{\lambda^2} + \frac{1}{\lambda^2}. \tag{2.13}
\]

This has a common factor $\frac{1}{\lambda^2}$, which does not depend on the form of the distribution of the disturbance. Clearly, there is no point in including $\frac{1}{\lambda^2}$ in the comparison of this standardized quantity for different distributions. Consequently, we define
\[
g(.) = \frac{MSD}{\lambda^2 \sigma_a^2} - \frac{1}{\lambda^2} = \frac{E(X_N^4)}{6E(X_N^2)} - \frac{b}{\lambda^2} \tag{2.14}
\]
where $E(X_N^2)$ and $E(X_N^4)$ are the second and fourth moments of a randomly stopped standardized (mean 0 and variance 1) random walk. Thus $g(.)$ neither depends on $\lambda$ nor $\sigma_a^2$. Box and Luceño (1994) defines the same $g(.)$ on different grounds. According to them, when $\lambda = 1$, $g(.)$ give an idea as to how much the standardized mean squared deviation will increase by using the control procedures due to the adjustment cost as compared to the case when it is not needed (in the absence of adjustment cost) and adjustments are made to each inspected item. No matter which points of view is taken, clearly $g(.)$ is the most appropriate quantity that should be used for comparison as it does not depend on $\lambda$ and $\sigma_a^2$. However, Box and Luceño (1994) used $1 + g(.)$ with $\lambda = 1$ and $\sigma_a^2 = 1$ for comparison. From the exact results developed in this paper, clearly there is no need to take $\lambda = 1$ and $\sigma_a^2 = 1$ as $g(.)$ does not depend on either of them. To be able to compare with Box and Luceño’s results, the comparison in this paper has also been carried out with $1 + g(.)$.

To complete the evaluation of MSD, we need to evaluate the fourth moment of the randomly stopped random walk. As in the case of the second moment, we shall consider the case of $u \geq 1$ separately from the case of $u < 1$.

**Case 1. $u \geq 1$.**

In this case we can write
\[
E(X_N^4) = u^4 + 4\rho_1 u^3 + 6\rho_2 u^2 + 4\rho_3 u + \rho_4 \tag{2.15}
\]
where $\rho_i$'s are the $i^{th}$ moment of the overshoot. Evaluation of $\rho_i$'s in the case of a symmetric distribution is given in the Appendix. For the three special cases, which we are considering in this
paper, they are given (\(\rho_1\) and \(\rho_2\) have already been given in section 2.1) as follows.

(i) Normal:

\[ \rho_{3\phi} = 0.796 \quad \rho_{4\phi} = 1.32 \]  \hspace{1cm} (2.16)

This was evaluated by Srivastava and Wu (1991). However, there was a numerical error in \(\rho_{4\phi}\) as given there; it was reported there to be 1.44.

(ii) Uniform:

\[ \rho_{3u} = 0.412 \quad \rho_{4u} = 0.46 \]  \hspace{1cm} (2.17)

(iii) Mixture normal: If \(\sigma_1^2 = 1 - \varepsilon\) and \(\sigma_2^2 = 1 + \varepsilon\frac{1-\alpha}{\alpha}\), then for small \(\varepsilon\),

\[ \rho_{3m} = 0.156 + 0.4364\frac{(1-\alpha)}{\alpha}\varepsilon^2 + 3\rho_{1m}\rho_{2m} - 2\rho_{1m}^3 \]

and

\[ \rho_{4m} = -\frac{h_2m}{24} + \frac{1}{2}\left[(1-\alpha)\sigma_1^6 + \alpha\sigma_2^6\right] + 3\rho_{2m}^2 + 4\rho_{1m}\rho_{3m} - 12\rho_{1m}\rho_{2m} + 6\rho_{1m}^4; \]

see Table 1 for numerical values for various values of \(\alpha\) and \(\varepsilon\).

Case 2. \(u < 1\).

Following Srivastava and Wu (1991) or as in Section 2.1, we get

\[ E(X_N^4) = b\left[1 + \xi_1 u + \xi_2 u^2 + \xi_4 u^3\right], \]  \hspace{1cm} (2.18)

for any symmetric (around zero) pdf f with mean 0, variance 1, fourth moment b, and \(f'(0) = 0\), where

\[ \xi_4 = \left(\frac{2}{b}\right)\xi_1 + \xi_3 \]  \hspace{1cm} (2.19)

while \(\xi_1, \xi_2,\) and \(\xi_3\) are as given in (2.10).

Thus, when \(u = 0\), \(E(X_N^4) = b\). The numerical values of \(\xi_4\) are as follows:

(i) Normal : \(\xi_{4\phi} = 0.908\)

(ii) Uniform : \(\xi_{4u} = 0.345\)

(iii) Mixture Normal : \(\xi_{4m} = 0.532\left[\frac{1-\alpha}{\sigma_1} + \frac{\alpha}{\sigma_2}\right] + \xi_{3m}\), see Table 1

2.3 Evaluation of C(L)
Having evaluated AAI and MSD, we can now write \(C(L)\) as

\[
C(L) = \frac{C_A}{E(X_N^2)} + C_0 \left[ \left( \frac{\lambda^2 \sigma_a^2}{\bar{\sigma}_a^2} \right) \left( \frac{E(X_N^4)}{E(X_N^2)} \right) + C_1 \sigma_a^2 \right], \tag{2.20}
\]

where \(C_0\) and \(C_1\) have been defined in (1.10) and (2.12) respectively. It has been shown above that \(E(X_N^2)\) and \(E(X_N^4)\) depends on \(L\), \(\lambda\) and \(\sigma_a^2\) through \(u = \left( \frac{L}{\bar{\sigma}_a^2} \right)\).

For the normally distributed disturbance, \(b = 3\) and \(C_1 = (1 - \frac{\lambda^2}{2})\). There is no such term in Box and Luceño (1994)'s equation (1.10) with \(m = 1\), which was obtained by Box and Kramer (1992) by simulation. Their cost function in the case of \(m = 1\) depends solely on \(\lambda \bar{\sigma}_a\) and \(L\) and not \(\lambda\). For small \(\lambda\), however, \(C_1 \simeq 1\), which they have in their equation (1.10) since in the case of \(m = 1\), \(\theta_m = \theta\). For large \(\lambda\), \(C_1 \simeq \frac{1}{2}\), not 1.

We shall now write \(C(L)\) in terms of \(\rho_i\)'s and \(\xi_i\)'s for the two cases separately.

**Case 1.** For \(u \geq 1\),

\[
C(L) \simeq \frac{C_A}{u^2 + 2 \rho_1 u + \rho_2} + C_0 \left[ \left( \frac{\lambda^2 \sigma_a^2}{\bar{\sigma}_a^2} \right) \left( \frac{u^4 + 4 \rho_1 u^3 + 6 \rho_2 u^2 + 4 \rho_3 u + \rho_4}{u^2 + 2 \rho_1 u + \rho_2} \right) + C_1 \sigma_a^2 \right] \tag{2.20a}
\]

\[
\simeq \frac{C_A}{u^2 + 2 \rho_1 u + \rho_2} + C_0 \left[ \left( \frac{\lambda^2 \sigma_a^2}{\bar{\sigma}_a^2} \right) \{u^2 + 2 \rho_1 u + \rho_2\} + 4(\rho_2 - \rho_1^2) \right] + C_1 \sigma_a^2 \tag{2.20b}
\]

**Case 2.** For \(u \leq 1\),

\[
C(L) \simeq \frac{C_A}{1 + \xi_1 u + \xi_2 u^2 + \xi_3 u^3} + C_0 \left[ \left( \frac{\lambda^2 \sigma_a^2}{\bar{\sigma}_a^2} \right) b \left( \frac{1 + \xi_1 u + \xi_2 u^2 + \xi_3 u^3}{1 + \xi_1 u + \xi_2 u^2 + \xi_3 u^3} \right) + C_1 \sigma_a^2 \right] \tag{2.21a}
\]

\[
\simeq C_A \left[1 - \xi_1 u - (\xi_3 - 2 \xi_1 \xi_2) u^3\right] + C_0 \left[ \left( \frac{\lambda^2 \sigma_a^2}{\bar{\sigma}_a^2} \right) b \left( 1 + \frac{2 \xi_1 u^3}{b} \right) + C_1 \sigma_a^2 \right] \tag{2.21b}
\]

2.4 **Optimum Value of the Action Limit.**

To obtain the optimum value of the action limit, we shall use equation (2.20b) in the case of \(u \geq 1\), and treat \(u^2 + 2 \rho_1 u + \rho_2\) as a new parameter. Differentiating (2.20) with respect to the new
parameter and equating it to zero, we find that the optimum value of \( u \), say \( u_0 \), must satisfy

\[
\frac{u_0^2}{2} + 2\rho_1 u_0 + \rho_2 = \left( \frac{6C_A}{\lambda^2 \sigma_2^4 C_0} \right)^{\frac{1}{2}}, \quad u \geq 1
\]

Hence, for large \( C_A \), we obtain

\[ u_0 = [(6R_A)^\frac{1}{4} - \rho_1] \]  \hspace{1cm} (2.22)

where

\[ R_A = \left( \frac{C_A}{\lambda^2 \sigma_2^4 C_0} \right) \]  \hspace{1cm} (2.23)

Clearly,

\[ L_0 = \lambda \sigma_4 [(6R_A)^\frac{1}{4} - \rho_1] . \]

To obtain the optimum value of \( u \) when \( u < 1 \), we shall consider the approximation (2.21b). Differentiating it and equating it to zero, we find that the optimum value \( u_0 \) is given by

\[ u_0^2 = \frac{C_A \xi_1}{\lambda^2 \sigma_2^4 C_0 \xi_1 - 3(\xi_3 - 2\xi_1 \xi_2)} \]

Thus

\[ u_0 = \left[ \frac{2C_A f(0)}{2\lambda^2 \sigma_2^4 C_0 f(0) - f''(0) + 24f^3(0)} \right]^{\frac{1}{2}} \]  \hspace{1cm} (2.24)

\[ \simeq (R_A)^{\frac{1}{2}} \]  \hspace{1cm} (2.25)

if \( C_0 \) is large. Thus, when \( C_0 \) is large and \( u \) is small, the optimum value of the action limit is not much affected from the departure of normality so long as the disturbance has a symmetric distribution with \( f'(0) = 0 \).

3. Effect of Non-normality.

Box and Luceño (1994) considered the effect of non-normality on AAI and MSD by simulation. They considered Barnard's (1959) model, uniform distribution and a mixture of normal distribution. By choosing the parameters of the mixture normal appropriately to match with Barnard's model, they found no significant difference in the performance of the two models. For our investigation we need the form of the pdf of disturbance which is not available for Barnard's model. Thus we shall consider only mixture normal and uniform distributions. But, before we begin the
comparison, let us recall that the average adjustment interval in terms of \( u = \frac{L}{\lambda \sigma_a^2} \), where \( L \) is the control limit, does not depend on \( \lambda \) and \( \sigma_a^2 \). Similarly, the function \( g(.) \), or, equivalently \( 1 + g(.) \) does not depend on \( \lambda \) and \( \sigma_a^2 \). For easy reference, we write their expressions here as follows:

\[
AAI = \begin{cases} 
  u^2 + 2\rho_1 u + \rho_2, & u \geq 1, \\
  1 + \xi_1 u + \xi_2 u^2 + \xi_3 u^3, & u < 1.
\end{cases}
\]

(3.1)

\[
1 + g(.) = \begin{cases} 
  (1 - \frac{b}{6}) + \frac{1}{6} \left[ \frac{u^4 + 4\rho_1 u^3 + 6\rho_2 u^2 + 4\rho_3 u + \rho_4}{u^2 + 2\rho_1 u + \rho_2} \right], & u \geq 1, \\
  (1 - \frac{b}{6}) + \frac{1}{6} \left[ \frac{1 + \xi_1 u + \xi_2 u^2 + \xi_3 u^3}{1 + \xi_1 u + \xi_2 u^2 + \xi_3 u^3} \right], & u < 1;
\end{cases}
\]

(3.2)

the quantities \( \rho_1, \ldots, \rho_4, \xi_1, \ldots, \xi_4 \) have been defined in the text. It is shown in Table 2, that there is not much significant difference between the two approximations of \( C(L) \) given in (2.20a) and (2.20b) for \( u \geq 1 \) and in (2.21a) and (2.21b) for \( u < 1 \) respectively. However, we have chosen the more accurate approximations, namely (2.20a) and (2.21a) in obtaining \( 1 + g(.) \) for comparison. The approximations in (2.20b) and (2.21b) have been used to obtain the optimum value of the control parameter \( L \) or equivalently \( u \).

3.1 Comparison of Normal with Uniform Distribution.

The pdf of a symmetric uniform distribution with mean 0 and variance 1 is given by

\[
f(w) = \begin{cases} 
  \frac{1}{2\sqrt{3}}, & -\sqrt{3} < w < \sqrt{3}; \\
  0, & \text{otherwise}.
\end{cases}
\]

The fourth moment for the above uniform distribution is given by

\[
b_u = \frac{9}{5}
\]

and for the standard normal, it is given by \( b_\phi = 3 \).

A comparison of AAI and \( 1 + g(.) \) for the normal with uniform is given in Table 2. It shows that the theoretical formulae given in this paper are a very good approximation of the exact result which were obtained by Box and Luceno (1994) by Monte Carlo simulation in their Table for \( \lambda = 1 \) and \( \sigma_a^2 = 1 \). As pointed out earlier, the assumptions of \( \lambda = 1 \) and \( \sigma_a^2 \) are unnecessary and the comparison holds for all values of \( \lambda \) and \( \sigma_a^2 \).
3.2 Comparison of Normal with Mixture Normal.

The pdf of a symmetric mixture normal is given by

\[ f(w) = (1 - \alpha)\phi(w; 0, \sigma_1^2) + \alpha\phi(w; 0, \sigma_2^2), \quad 0 < \alpha \leq 1 \]

where \( \phi(; 0, \sigma^2) \) denotes a normal pdf with mean 0 and variance \( \sigma^2 \).

To make the variance of the mixture normal distribution to be one, we must have

\[ (1 - \alpha)\sigma_1^2 + \alpha\sigma_2^2 = 1, \]

giving

\[ \alpha = \frac{1 - \sigma_1^2}{\sigma_2^2 - \sigma_1^2} \quad (3.1) \]

Without loss of generality, we shall assume that

\[ \sigma_1^2 \leq 1 \leq \sigma_2^2. \]

The fourth moment of the mixture normal, with \( \alpha \) given in (3.1), is given by

\[ b_m = 3(\sigma_1^2 + \sigma_2^2 - \sigma_1^2\sigma_2^2) \]

To study the effect of non-normality, we fix the value of \( \alpha \) and let \( \sigma_1^2 \) and \( \sigma_2^2 \) approach one at the same rate. Thus, if we let

\[ \sigma_1^2 = 1 - \varepsilon \quad (3.2) \]

then

\[ \sigma_2^2 = 1 + \frac{\varepsilon(1 - \alpha)}{\alpha} \quad (3.3) \]

and the fourth moment

\[ b_m = 3 \left[ 1 + \frac{\varepsilon^2(1 - \alpha)}{\alpha} \right] \quad (3.4) \]

A comparison of normal with mixture normal is given in Tables 3 and 4.
3.3 Comparison with Box and Lucoño’s Computation

Box and Lucoño (1994) consider the mixture of $N(0, \sigma_2^2)$ and $N(0, \sigma_2^2 + \sigma_6^2)$ with probability $(1 - \alpha)$ and $\alpha$ respectively. For their computation, they chose, with $\theta = 1 - \lambda$, $\sigma_2^2 = 1$, $\sigma_6^2 = \left(\frac{\lambda^2}{\delta^2}\right)$, $\alpha = \frac{\delta}{(1 - \theta^2)}$, $s = \frac{2}{(1 + \theta)}$. In order that the variance of the mixture normal be $\sigma_a^2$, we must have $\sigma_a^2 = \frac{1}{\theta}$. Thus, instead of five parameters ($\lambda$, $\sigma_2^2$, $\alpha$, $s$, $\sigma_a^2$), they need to select only two parameters ($\alpha$, $\sigma_6^2$) or equivalently ($\delta$, $\sigma_6^2$). For example, given $\delta$ and $\sigma_6^2$, we have

$$\lambda = \frac{\delta \sigma_6^2}{2} \left[ \left(1 + \frac{4}{\delta \sigma_6^2} \right)^{\frac{1}{2}} - 1 \right], \quad \sigma_a^2 = \frac{1}{\theta},$$

(3.7)

$$\alpha = \frac{\delta}{(1 - \theta^2)}, \quad s = \frac{2}{(1 + \theta)}, \quad \theta = 1 - \lambda.$$  

(3.8)

and thus they need to specify only $\delta$ and $\sigma_6^2$ in their computation. Since, $1 + g(.)$ and AAI do not depend on $\lambda$ and $\sigma_a^2$, we need to select only $\alpha$ and $\epsilon$. However, $\sigma_1^2 = (1 - \epsilon)$ in our notation is equal to $\left(\frac{\epsilon}{\sigma_a^2}\right)$ in their notation giving

$$\epsilon = \frac{\lambda}{(2 - \lambda)}.$$  

(3.9)

We compute $1 + g(.)$ for different values of $\delta$ and $\sigma_6^2$ in Table 5. Comparing it with Table 3 of Box and Lucoño (1994), it is clear that the theoretical approximations given in this paper are very accurate. There appears to be some difference for larger values of $u$. However, it is in this situation that the renewal theory provides a very accurate approximation. Since, they do not provide standard errors of their simulation, the accuracy of the theoretical result should be preferred. It may also be mentioned that their assumption of $\sigma_a^2 = 1$ and $\lambda = 1$ is redundant.

4 Sampling Interval.

In this section, we consider the case when the cost of inspection is $C_1 \geq 0$ and instead of every item, we inspect every $m^{th}$ product. Thus, we observe $Z_{m}$, $Z_{2m}$, \ldots, $Z_{nm}$, \ldots An adjustment is made as soon as the forecasted value $\hat{Z}_{(n+1)m}$ of the $(n + 1)$ $m^{th}$ observation exceeds a control limit $L$, that is, at

$$N = \min\left\{ n \geq 1 : |\hat{Z}_{(n+1)m}| > L \right\},$$

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where
\[
\hat{Z}_{(n+1)m} = \lambda_m \left[ Z_{nm} + \theta_m Z_{(n-1)m} + \ldots + \theta_m^{n-1} Z_m \right] = \lambda_m Z_{nm} + \theta_m \hat{Z}_{nm},
\]
and
\[
\theta_m = 1 - \lambda_m,
\]

\[
\lambda_m = \frac{m \lambda^2}{2(1 - \lambda)} \left[ \left( 1 + \frac{4(1 - \lambda)}{m \lambda^2} \right)^{\frac{1}{2}} - 1 \right];
\]
it can be shown that \( \lambda_m \geq \lambda_1 \equiv \lambda \). Thus, the average adjustment interval (AAI) is given by

\[
AAI = mANI = mE(N),
\]

where ANI is the average number of inspections. If we assume that \( \hat{Z}_0 \) is now a random variable (instead of zero) with mean 0 and a small variance \((\lambda_m - \lambda)(1 - \lambda)\sigma_a^2\), then, it has been shown in Srivastava (1994) that the sequence of random variables \(Z_m, Z_{2m}, \ldots\) forms an IMA_1(\(\lambda_m, \sigma_m^2\)), where

\[
\sigma_m^2 = \frac{m \lambda^2 \sigma_a^2}{\lambda_m^2}.
\]

It should be noted that we are discussing the long-run average cost rate, and thus this technical device of assuming \( \hat{Z}_0 \) to be a random variable has no significant bearing. Indeed Box and Kramer (1992) assumes that \(Z_m, Z_{2m}, \ldots\) forms an IMA_1(\(\lambda_m, \sigma_m^2\)) without this assumption. Thus, \(\hat{Z}_{(n+1)m}\) is a random walk, a sum of \(mn\) iid random variables with mean 0 and variance \(\lambda_m^2 \sigma_m^2 = m \lambda^2 \sigma_a^2\).

As before, we can now write

\[
\hat{Z}_{(n+1)m} = m^\frac{1}{2} \lambda \sigma_a X_n \equiv m^\frac{1}{2} \lambda \sigma_a \sum_{i=1}^n w_i,
\]

where \(X_n\) is a standardized random walk, which is a sum of \(n\) iid random variables with mean 0 and variance 1. We shall assume that the pdf of \(w\), denoted by \(f(w)\) is symmetric around zero and the fourth moment is \(b\). Then

\[
ANI = E(N) = E(X_N^2),
\]

where \(N\) can be redefined as
\[ N = \min \left\{ n \geq 1 : |X_n| > u_m \right\}, \]

\[ u_m = \frac{L}{\lambda \sigma \sqrt{m}}. \]

Thus, if we compare ANI, the robustness study carried out in the previous section applies here as well and no further study is needed.

Next, we consider the MSD, which is given by

\[ MSD = \frac{E(\sum_{j=1}^{N} Z_j^2)}{\lambda AI} \]

When the disturbance is normally distributed, it is given by

\[ MSD = \lambda^2 \sigma^2 \left[ \frac{m E(X^4_N)}{6 E(X^2_N)} + \frac{(1 - \lambda)(1 + \lambda)}{\lambda^2} + \frac{1}{2} \right], \]

see Srivastava (1993, 1994), values of \( E(X^4_N) \) and \( E(X^2_N) \) can be obtained from the results in the previous section with \( u_m \) in place of \( u \). Extension of this result for any symmetric distribution for the disturbance term appears difficult at the present time. However, from the results of Srivastava and Wu (1992) for any symmetric random walk model, it is conjectured that

\[ MSD = \lambda^2 \sigma^2 \left[ \frac{m E(X^4_N)}{6 E(X^2_N)} + m \frac{(3 - b)}{6} + \frac{(1 - \lambda)(1 + \lambda)}{\lambda^2} + \frac{1}{2} \right], \]

where \( b = E(w_j^4) \); the exact result for random walk model is obtained from above with \( \lambda = 1 \). To study the effect of non-normality, we may consider the quantity

\[ \frac{E(X^4_N)}{E(X^2_N)} + (3 - b) \]

which is related to \( g \) and is the same as for the random walk model. Thus, it would appear to be robust for any \( m \) and \( 0 \leq \lambda \leq 1 \), not just for \( \lambda = 1 \).

To end, we may mention the optimum control limit and optimum inspection interval in the case
of normally distributed disturbance. We have

\[ L^* = \lambda \sigma_a \left[ (6R_A)^{\frac{1}{4}} - \rho_1 m^{\frac{1}{2}} \right], \]

and \( m^* \) is the solution of

\[ \left( \frac{6R_I}{m^2} \right)^{-1} - 6 \left[ \frac{\lambda m}{m} - \frac{1}{m} \left( 1 + \frac{4(1-\lambda)}{m\lambda^2} \right)^{-\frac{1}{2}} \right] = 0; \]

where \( \rho_1 = 0.583, \ R_A = \frac{C_A}{C_0 \lambda \sigma^2_a}, \) and \( R_I = \frac{C_I}{C_0 \lambda \sigma^2_a}. \) This is obtained by minimizing the overall cost

\[ C(L,m) = \frac{C_I}{m} + \frac{C_A}{AAI} + C_0 MSD, \]

see Srivastava (1993, 1994).

5 Concluding Remarks.

In this paper, when the inspection cart is zero, we have obtained exact expressions for the average adjustment interval (AAI) and mean squared deviations (MSD) for an integrated moving average (IMA) process of order one in which the disturbance follow any symmetric distribution. These expression depend only on the second and fourth moment of a randomly stopped standardized (mean 0 and variance 1) and symmetric random walk, the fourth moment of the disturbance distribution, the variance \( \sigma^2_a \) of the disturbance and the nonstationarity parameter \( \lambda \) of the IMA process. It is shown that a quantity \( g(.) \) defined in Box and Luceño does not depend on \( \lambda \) and \( \sigma^2_a. \) Thus, all the comparison based on \( g(.) \) or \( 1 + g(.) \) holds for all \( \lambda \) and \( \sigma^2_a. \)

Using renewal theory, an approximation of the second and fourth moment of the randomly stopped random walk is given for large control limit. These approximation converges exponentially fast to the exact moments as shown by Stone (1965). When the standardized control limit is less than one, another approximation is given for these quantities, where, \( f'(0) = 0 \) is assumed for simplicity. From these results, an optimal control limit is given.

The calculations match with simulated results of Box and Luceño (1994) and confirms that the control scheme is robust for small to moderate departure for normality and for all \( \lambda \) not just for \( \lambda = 1 \) as calculated in the above mentioned paper.
The paper, however, provides expressions for AAI, MSD and optimal control limit for any symmetric distribution for the disturbance and thus can be used in case of severe non-normality.

When the sampling interval is also present, it is shown that AAI is robust. Although no expression for MSD is available, other than for the normal, it would appear from the results of symmetric random walk model that it is robust for all $\lambda$. 
APPENDIX

A. Evaluation of (1.12) and Proof of (2.4)

Recall that \( \hat{Z}_{n+1} = \lambda \sigma a_j = \lambda \sigma a \sum_{j=1}^{n} w_j s = \lambda \sigma a X_n \), where \( w_j \)'s are iid with mean 0 and variance 1 and \( X_n \) is a standard random walk. And

\[
N = \min\{n \geq 1 : |X_n| > u\}.
\]

Since,

\[
E[X_n^2 - n|X_{n-1}] = E[(X_{n-1} + w_n)^2 - n|X_{n-1}]
\]
\[= E[X_{n-1}^2 + 2X_{n-1} w_n + w_n^2 - n|X_{n-1}]\]
\[= [X_{n-1}^2 - (n-1)],\]

it follows that \( X_n^2 - n \) is a martingale. Hence, from the optional stopping theorem, see Chow, Robbins and Teicher (1965), we get

\[
E(X_N^2) = E(N)
\]

B. Evaluation of (1.9)

We shall assume that \( E(W_j^3) = 0 \) and \( E(W_j^4) = b \). Then

\[
E\left[ \sum_{j=1}^{N} Z_j^2 \right] = E\left[ \sum_{j=1}^{n} (\hat{Z}_j + a_j)^2 \right]
\]
\[= E\left[ \sum_{j=1}^{N} (\lambda \sigma a X_{j-1} + \sigma a w_j)^2 \right]
\]
\[= \sigma_a^2 \left[ \sum_{j=1}^{N} (\lambda^2 X_{j-1}^2 + 2\lambda X_{j-1} w_j + w_j^2) \right]
\]
\[= \sigma_a^2 \left[ \lambda^2 \sum_{j=2}^{N} X_{j-1}^2 + 2\lambda \sum_{j=2}^{N} X_{j-1} w_j + \sum_{j=1}^{N} w_j^2 \right]
\]

From Wald's identity

\[
E\left( \sum_{j=1}^{N} w_j^2 \right) = E(N)E(w_1^2) = E(N)
\]
We can verify that the following are martingales

(a) \[ \sum_{j=2}^{n} X_{j-1}^2 - nX_n^2 + \frac{n(n+1)}{2}; \]

(b) \[ X_n^4 - 6nX_n^2 + 3n^2 + n(3-b); \]

(c) \[ \sum_{j=2} X_{j-1}(X_j - X_{j-1}); \]

Hence, from the optional stopping theorem

\[ E \left[ \sum_{j=2}^{N} X_{j-1}(X_j - X_{j-1}) \right] = 0, \]

\[ E \left[ \sum_{j=2}^{N} X_{j-1}^2 \right] = E[N X_N^2] - \frac{1}{2} E[N(N + 1)] \]

\[ = \frac{1}{6} E[X_N^4 + 3N^2 + N(3-b)] - \frac{1}{2} E[N(N + 1)] \]

\[ = \frac{1}{6} E(X_N^4) + \frac{1}{2} E(N^2) + \frac{1}{6}(3-b)E(N) - \frac{1}{2} E(N^2) - \frac{1}{2} E(N) \]

\[ = \frac{1}{6} E(X_N^4) - \frac{1}{2} E(N) + \frac{1}{6}(3-b)E(N) \]

This gives

\[ E \left[ \sum_{j=1}^{N} Z_j^2 \right] = \sigma_a^2 \left[ \frac{\lambda^2}{6} E(X_N^4) - \frac{\lambda^2}{2} E(N_1) + \frac{\lambda^2}{6}(3-b)E(N) + E(N) \right] \]

\[ = \sigma_a^2 \left[ \frac{\lambda^2}{6} E(X_N^4) + (1 - \frac{\lambda^2}{2})E(N) + \frac{\lambda^2(3-b)}{6}E(N) \right] \]

Hence,
\[
MSD = \left( \frac{\lambda^2 \sigma_a^2}{6} \right) \left[ \frac{E(X_N^2)}{E(X_N^2)} \right] + \sigma_a^2 \left[ 1 - \frac{\lambda^2}{2} + \frac{\lambda^3 (3 - b)}{6} \right]
\]

\[
eq \left( \frac{\lambda^2 \sigma_a^2}{6} \right) \left[ \frac{E(X_N^2)}{E(X_N^2)} \right] + C_1 \sigma_a^2
\]

C. Evaluation of \( E(X_N^2) \) and \( E(X_N^4) \).

Case 1: \( a \geq 1 \)

For the case \( u \geq 1 \), we can write

\[
E(X_N^2) = E[|X_N| - u + u]^2
\]

\[
= u^2 + 2uE[|X_N| - u] + E[(|X_N| - u)^2]
\]

\[
= u^2 + 2u \rho_1 + \rho_2,
\]

and

\[
E(X_N^4) = u^4 + 4 \rho_1 u^3 + 6 \rho_2 u^2 + 4 \rho_3 u + \rho_4,
\]

where

\[
\rho_i = E[|X_N| - u]^i, \quad i = 1, 2, 3, 4.
\]

Using Renewal theory an approximation for \( \rho_i \) can be obtained for any distribution. It should be noted, however, that the renewal theory is set up for one sided boundry, but conditional argument can be used to obtain \( \rho_i \) for two-sided symmetric boundry.

Case 2: \( a < 1 \)

In this case, we use Taylor’s expansion repeatedly to expand the pdf round zero:

\[
f(w) = f(0) + wf'(0) + \frac{1}{2} w^2 f''(0) + \ldots
\]

For convenience of simplification, we have assumed that \( f'(0) = 0 \). Thus,
\[ E[X_N^2; \ N = 1] = \int_{|x| > u} x^2 f(x) \, dx \]
\[ = 1 - 2 \int_0^u x^2 f(x) \, dx \]
\[ = 1 - 2 \int_0^u x^2 [f(0) + \frac{x^2}{2} f''(0)] \, dx + 0(u^3) \]
\[ = 1 - \frac{2}{3} f(0) u^3 + 0(u^3). \]

\[ E[X_N^2; \ N = 2] = \int_{-u}^u f(x_1) \, dx_1 \int_{|x_2| > u} x_2^2 f(x_2 - x_1) \, dx_2 \]
\[ = \int_{-u}^u f(x_1) \, dx_1 \int_{-\infty}^\infty (x_1 + y)^2 f(y) \, dy + 0(u^3) \]
\[ = \int_{-u}^u f(x_1) \, dx_1 \int_{-\infty}^\infty (1 + x_1^2) f(y) \, dy + 0(u^3) \]
\[ = \int_{-u}^u f(x_1) \, dx_1 + \int_{-u}^u x_1^2 f(x_1) \, dx_1 + 0(u^3) \]
\[ = 2 \int_0^u f(x_1) \, dx_1 + 2 \int_0^u x_1^2 f(x_1) \, dx_1 + 0(u^3) \]
\[ = 2 uf(0) + \frac{u^3}{3} f''(0) + \frac{2}{3} u^3 f(0) + 0(u^3). \]

Similarly,
\[ E[X_N^2; \ N = 3] = 4 u^2 f^2(0) + 0(u^3) \]
\[ E[X_N^2; \ N = 4] = 8 u^3 f^3(0) + 0(u^3) \]
\[ E[X_N^2; \ N \geq 5] = 0(u^3). \]

Hence
\[ E(X_N^2) = 1 + 2uf(0) + 4u^2 f^2(0) + u^3 \left[ 2 f''(0) + 8 f^3(0) \right] + 0(u^3) \]
\[ \equiv 1 + \xi_1 u + \xi_2 u^2 + \xi_3 u^3 + 0(u^3), \]
where \( \xi_1 = 2f(0), \xi_2 = \xi_1^2, \xi_3 = \frac{1}{3} f''(0) + \xi_1^3 \). Similarly, it can be shown that
\[ E(X_N^4) = b[1 + \xi_1 u + \xi_2 u^2 + \xi_4 u^3] + 0(u^3) \]
where \( \xi_4 = \left[ \frac{1}{3} f''(0) + \frac{4}{3} f(0) + 8 f^3(0) \right] = \xi_3 + \frac{2}{3} \xi_1. \)
To write $C(L)$ in terms of $\rho_i$ and $\xi_i$, we note that

$$[1 + \xi_1 u + \xi_2 u^2 + \xi_3 u^3]^{-1} = 1 - \xi_1 u - (\xi_3 - 2\xi_1\xi_2)u^3 + 0(u^3)$$

and

$$[1 + \xi_1 u + \xi_2 u^2 + \xi_4 u^3] = [1 + \xi_1 u + \xi_2 u^2 + \xi_3 u^3][1 + (\xi_4 - \xi_3)u^3] + 0(u^3)$$

since $\xi_2 = \xi_1^2$. Similarly,

$$1 + 4\rho_1 u^{-1} + 6\rho_2 u^{-2} + 4\rho_3 u^{-3} + \rho_4 u^{-4} \simeq [1 + 2\rho_1 u^{-1} + \rho_2 u^{-2}]^2 + 4[1 + 2\rho_1 u^{-1} + \rho_2 u^{-2}][\rho_2 - \rho_1^2]u^{-2} + 0(u^{-3})$$

These expressions have been used to obtain the approximations (2.21b) and (2.20b) respectively.

D. Expressions For $\rho_1$, $\rho_2$, $\rho_3$ and $\rho_4$.

Let $g(\lambda)$ be the characteristic function of the random variable $w$. It may be noted that $g(\lambda)$ will be real and symmetric since the pdf of $w$ is symmetric around zero. Thus, following Siegmund (1985) and Srivastava and Wu (1991), we find that

$$\rho_1 = -\frac{1}{\pi} \int_0^\infty \lambda^{-2} \log \left[ \frac{1 - g(\lambda)}{\lambda^2} \right] d\lambda \quad (0.1)$$

$$\rho_2 = \rho_1^2 + \frac{b}{12} \quad (0.2)$$

The values of $\rho_3$ and $\rho_4$ are obtained from the following relationships:

$$\rho_3 = 6\nu_3 + 3\rho_1\rho_2 - 2\rho_1^3 \quad (0.3)$$

$$\rho_4 = 24\nu_4 + 3\rho_2^2 + 4\rho_1\rho_3 - 12\rho_1^2\rho_2 + 6\rho_1^4, \quad (0.4)$$

where

$$\nu_3 = \frac{1}{\pi} \int_0^\infty \frac{\log[(1-g(\lambda))]}{\lambda^4} + \frac{1}{12} d\lambda \quad (0.5)$$
and

\[ \nu_4 = \frac{1}{2} \left[ -\frac{g^{(6)}(0)}{360} - \frac{b^2}{288} \right], \quad (0.6) \]

\[ g^{(6)}(0) = \left[ \frac{\partial^6 g(\lambda)}{\partial \lambda^6} \right]_{\lambda=0}, \quad b = E(w^4). \quad (0.7) \]

From the results of Stone (1965), it follows that \( \rho_i \)'s are very good approximations of \( E[|X_N| - u_i]^4 \); the convergence is exponentially fast.

We may note that the characteristic function \( g(\lambda) \) of the three distributions considered in this paper are as follows:

(i) \[ Normal: \quad g(\lambda) = e^{-\frac{\lambda^2}{2}}, \]

(ii) \[ Uniform: \quad g(\lambda) = \frac{\sin(\lambda\sqrt{3})}{\lambda\sqrt{3}}, \]

(iii) \[ MixtureNormal: \quad g(\lambda) = (1 - \alpha)e^{-\frac{\lambda^2\sigma_1^2}{2}} + \alpha e^{-\frac{\lambda^2\sigma_2^2}{2}}. \]

Thus, using the Mathematica Language, it is straightforward to compute the values of \( \rho_i \)'s. In the case of mixture normal, however, if we assume that \( \sigma_2^2 = 1 - \varepsilon \) and \( \sigma_2 = 1 + \frac{\varepsilon(1-\alpha)}{\alpha} \) and \( \varepsilon \) small which will usually be the case, we may expand \( g(\lambda) \) and obtain

\[ \rho_1 \simeq 0.583 + \frac{1}{4\pi} \varepsilon^2 (1-\alpha) \int_0^\infty \frac{\lambda^2}{e^{\frac{\lambda^2}{2}} - 1} d\lambda + O(\varepsilon^2) \]

\[ \simeq 0.583 + 0.1303 \frac{\varepsilon^2 (1-\alpha)}{\alpha} \]

Similarly, we can expand \( g(\lambda) \) to obtain \( \nu_3 \) for the mixture normal. In this case, since \( b - 3 = \frac{3\varepsilon^2 (1-\alpha)}{\alpha} \), \( \nu_3 \simeq 0.026 - \frac{(1-\alpha)}{\alpha} \varepsilon^2 \frac{1}{4\pi} \int_0^\infty \left( [2(e^{\frac{\lambda^2}{2}} - 1)]^{-1} - \frac{1}{e^{\lambda^2/2}} \right) d\lambda = 0.026 + 0.0728 \frac{(1-\alpha)}{\alpha} \varepsilon^2 \). To obtain \( \nu_4 \), we note that \( g^{(6)}(0) = -15 \) for normal, \( = \frac{27}{2} \) for uniform and for the mixture normal, it is equal to \( = -15[(1-\alpha)\sigma_1^6 + \alpha\sigma_2^6] \).
References


Department of Statistics
University of Toronto

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Table 1: Values of $\rho_i^{(1)}$'s and $\xi_i$'s for Mixture Normal

The $\rho_i^{(1)}$'s are based on the exact evaluation of the integrals given in the Appendix and the $\rho_i^{(2)}$'s are their approximations for small $\epsilon$. 
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Table 2: Comparison of AAI and $1 + g = (\text{MSD/} \lambda^2 \sigma^2) + (1 - 1/\lambda^2)$ Normal vs Uniform

$1+g$ in columns four and five are obtained from formulas (2.20a, 2.21a) and (2.20b, 2.21b) respectively. As noted, $1 + g$ does not depend on $\lambda$ or $\sigma^2$. The entries BL are from Box and Luceno's Table 2. There were no entries available from the BL table where dashes appear.
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Table 3: Comparison of Average Adjustment Interval
Note that AAI does not depend on $\lambda$ or $\sigma^2$.

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Table 4: Comparison of $1 + g = (\text{MSD}/\lambda^2\sigma_x^2) + (1 - 1/\lambda^2)$. Note that $1 + g$ does not depend on $\lambda$ or $\sigma_x^2$. Normal vs Mixture Normal
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Table 5: Comparison with Box and Luceno’s Table 3

BL = Box and Luceno, S = From this paper

$1 + g = (\text{MSD}/\lambda^2 \sigma_a^2) + (1 - 1/\lambda^2)$

Note that $1 + g$ does not depend on $\lambda$ or $\sigma_a^2$. The approximation uses (2.20a) and (2.21a).