A Stochastic Justification of Some Simple Reliability Models

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Technical Report No. 0202 February 5, 2002

TECHNICAL REPORT SERIES

University of Toronto

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Abstract

In this paper we give conditions on the internal wear process under which the resulting time-to-failure model will be of the simple collapsible form (Oakes, 1995, Duchesne and Lawless, 2000) when the usage accumulation history is available.

We suppose that failure occurs when internal wear crosses a certain threshold and/or a traumatic event causes the item to fail (Cox, 1999 and Bagdonavičius and Nikulin, 2001). We model the infinitesimal increment in internal wear as a function of time, accumulated internal wear and usage history, and we derive conditions on this function to get a collapsible model for the conditional distribution of time-to-failure given the usage history.

1 Introduction

In the recent literature, several methods and models have been suggested to include the effect of the usage history on the lifetime distribution of various items. The accelerated failure time (AFT) model with time-varying covariates (Robins and Tsiatis, 1992) has been frequently used for this purpose. Recently, the collapsible model (Oakes, 1995, Duchesne and Lawless, 2000) was introduced. Attractive features of the collapsible model include easy interpretation of the effect of usage on lifetime and semiparametric or completely nonparametric modelling (Duchesne, 2000). The assumptions made by the model are, however, much stronger than that of the AFT model and a formal omnibus test of the "collapsibility assumption" remains an open problem. In this paper we show how collapsible models can arise from certain specific failure environements.

1.1 Definitions and notation

Let $\{X(t),\ t \geq 0\}$ be the internal wear process of the item. Set $X(0) \equiv 0$ and assume that $\{X(t)\}$ has right-continuous paths with finite left-hand limits w.p. 1. Let X^* be the threshold random variable, with $P[X^* > 0] = 1$. Define the time to failure of an item as $T = \inf\{t :\ X(t) \geq X^*\}$. Finally, let $\{\theta(t),\ t \geq 0\}$ be the usage rate of the item at time t and $y(t) = \int_0^t \theta(u)\, du$ be the cumulative usage at time t; we assume that $\theta(t) \geq 0$ for all $t \geq 0$. Let $\boldsymbol{\theta}_t \equiv \{\theta(s), 0 \leq s \leq t\}$ and $\boldsymbol{\theta} \equiv \{\theta(t), t \geq 0\}$ denote the usage accumulation histories, and let $\boldsymbol{\Theta}_t$ and $\boldsymbol{\Theta}$ represent the spaces of all possible usage histories $\boldsymbol{\theta}_t$ and $\boldsymbol{\theta}$, respectively, with $\theta(t)$ piecewise smooth, i.e. for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ there exists a countable set of time points $0 \leq t_1 < t_2 < \ldots$ with $t_i \to \infty$, such that $\theta(t) = a_i(t), t_i < t < t_{i+1}$ where a_i is continuous and smooth over $[t_i, t_{i+1}]$, $i = 1, 2, \ldots$. This family of usage histories is broad enough to include stepwise continuous usage histories (such as on-off or low intensity-high intensity usage patterns, for example).

By conditioning on the usage history, we make the assumption that as a process, usage evolves independently of the internal wear process, i.e.

$$P[\theta(t + \Delta t) \in A | \{\theta(s), X(s),\ 0 \leq s \leq t\}] = P[\theta(t + \Delta t) \in A | \boldsymbol{\theta}_t]$$

for any measurable set A. We also assume that X^* is independent of both processes $\{\theta(t)\}$ and $\{X(t)\}$.

This latter assumption allows us to suppose without loss of generality[1] in our proofs that the threshold is constant, i.e. $X^* = x^* > 0$ w.p. 1.

We now define the conditional models of interest. For the AFT model we use the definition of Robins and Tsiatis (1992):

Definition 1.1 *The accelerated failure time model is given by*

$$P[T > t|\theta_t] = G\left[\int_0^t \psi(\boldsymbol{\theta}_u;\beta)\,du\right], \tag{1}$$

where $\psi(\cdot,\cdot;\beta)$ is a positive $\boldsymbol{\Theta} \to [0,\infty)$ map, called time transformation function, that may depend on a vector of unknown parameters β, and $G[\cdot]$ is a survivor function.

One popular specification of (1) is the log-linear model with $\psi(\boldsymbol{\theta}_u;\beta) = \exp\{\beta g_\theta[\theta(u)]\}$, where $g_\theta[\cdot]$ is a completely specified 1-1 $[0,\infty) \to [0,\infty)$ map. The class of models we are mainly interested in is defined as follows:

Definition 1.2 *A collapsible model is a model described by*

$$P[T > t|\boldsymbol{\theta}_t] = G[\phi(t, y(t);\beta)], \tag{2}$$

where $y(t)$ is the cumulative usage, $G[\cdot]$ is a survivor function, and $\phi(\cdot,\cdot;\beta)$ is a positive $[0,\infty)^2 \to [0,\infty)$ map such that $\phi(t, y(t);\beta)$ is non-decreasing in t for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

The function ϕ can be viewed as a common scale in which the age of all the items can be compared, regardless of their usage history. The meaning of the collapsible model (2) is that survival past a certain time point t only depends on the usage history $\boldsymbol{\theta}_t$ through t and $y(t)$. Some forms of the ideal time scale ϕ in collapsible models lead to nice interpretations. For instance, if $\phi(t, y(t);\beta) = \beta_0 t + \beta_1 y(t)$, then living one time unit has the same effect on the item as β_0/β_1 units of usage (Oakes, 1995).

2 Failure caused by excessive wear only

We now suppose that internal wear is accumulated according to the stochastic integral equation

$$X(t) = \int_0^t \mu[s, \boldsymbol{\theta}_s, X(s)]\,ds + \int_0^t \sigma[s, \boldsymbol{\theta}_s, X(s)]\,d\gamma(s), \quad t \geq 0, \tag{3}$$

where μ and σ are non-negative functions, and $\{\gamma(s)\}$ is a stochastic process with $\gamma(0) \equiv 0$ and $P[\gamma(t) - \gamma(s) \geq 0] = 1$ for all $0 \leq s < t$, such as the gamma process for example.

2.1 Deterministic wear

By "deterministic wear", we mean that the second integral on the right-hand side of (3) is identically 0, i.e. wear is deterministic and can be written as $x(t) = \int_0^t \mu[s, \boldsymbol{\theta}_s, x(s)]\,ds$. When μ only depends on s and $\boldsymbol{\theta}_s$, we suppose that the mapping $s \mapsto \mu[s, \boldsymbol{\theta}_s]$ is continuous [resp. continuously differentiable] at $s = t$ whenever the mapping $s \to \boldsymbol{\theta}_s$ is continuous [resp. continuously differentiable] at $s = t$. When μ also depends on $x(s)$, we also require a mild boundedness condition on μ. Under these conditions, Duchesne and Rosenthal (2002) show that

Theorem 2.1 *If wear is deterministic and μ satisfies the conditions stated above, then we have a collapsible model if, and only if, there exist functions $\mu_1, \mu_2 : [0,\infty)^2 \to [0,\infty)$ which are continuously differentiable, with $\frac{\partial}{\partial y}\mu_1(x,y) = \frac{\partial}{\partial x}\mu_2(x,y)$ for all $x, y \geq 0$ such that for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$,*

$$\mu[s, \boldsymbol{\theta}_s, x(s)] = \mu_1[s, y(s)] + \mu_2[s, y(s)]\theta(s)$$

at all times $s \geq 0$ where $\theta(s)$ is smooth.

[1] Duchesne and Rosenthal (2002) show that if a model is collapsible for any fixed threshold $X^* = x^* > 0$, $X(t)$ is non-decreasing as a function of t, and X^* is independent of $\{X(t)\}$, then the model is also collapsible under a positive random threshold X^*.

Two important corollaries follow from this theorem.

Corollary 2.2 *Under the conditions of Theorem 2.1, collapsible models are the subset of AFT models with time-varying covariates with* $G[x]$ *and* $\psi(\boldsymbol{\theta}_u;\beta)$ *in (1) respectively given by* $G[x] = P[X^* > x]$ *and* $\psi(\boldsymbol{\theta}_u;\beta) = \mu_1[u, y(u)] + \mu_2[u, y(u)]\theta(u)$ *for continuously differentiable functions* μ_1, μ_2 *with* $\frac{\partial}{\partial y}\mu_1(x, y) = \frac{\partial}{\partial x}\mu_2(x, y)$ *for all* $x, y \geq 0$.

Corollary 2.3 *If* $\mu[s, \boldsymbol{\theta}_s, x(s)] \equiv \mu[s, \theta(s)]$, *then the only possible collapsible model is the linear collapsible model, i.e.* $P[T > t|\boldsymbol{\theta}_t] = G[t + \beta y(t)]$ *for some survivor function* $G[\cdot]$ *and* $\beta > 0$.

2.2 Stochastic wear

When the second integral on the right-hand side of (3) is not zero, it is more complicated to derive necessary conditions for the model to be collapsible. However sufficient conditions are available.

Theorem 2.4 *A set of sufficient conditions to obtain a collapsible model is that (i)* $\sigma[s, \boldsymbol{\theta}_s, X(s)] \equiv \sigma[s]$, *(ii)*$\mu[s, \boldsymbol{\theta}_s, X(s)] \equiv \mu[s, \boldsymbol{\theta}_s]$ *and (iii) there exist continuously differentiable functions* $\mu_1, \mu_2 : [0, \infty)^2 \to [0, \infty)$ *which are continuously differentiable, with* $\frac{\partial}{\partial y}\mu_1(x, y) = \frac{\partial}{\partial x}\mu_2(x, y)$ *for all* $x, y \geq 0$ *such that for all* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, $\mu[s, \boldsymbol{\theta}_s] = \mu_1[s, y(s)] + \mu_2[s, y(s)]\theta(s)$ *for all* $s \geq 0$ *such that* $\theta(s)$ *is smooth.*

It seems that Theorem 2.4 essentially gives the only way to obtain a collapsible model in this case. However this appears difficult to formulate in a precise theorem.

3 Failure caused by excessive wear and traumatic events

We now suppose that there are two possible causes of failure: internal wear crossing a threshold and the occurrence of a traumatic event that kills the item. Following Cox (1999) and Bagdonavičius and Nikulin (2001), we suppose that the hazard of a traumatic event is a function of the level of internal wear. That is, if we let K represent the time at which a traumatic event occurs, the hazard is given by

$$\lambda(t|\{X(s), \theta(s), 0 \leq s \leq t\}) = \lim_{h \downarrow 0} \frac{P[K \in [t, t+h)|K \geq t, \{X(s), \theta(s), 0 \leq s \leq t\}]}{h}. \tag{4}$$

The time to failure is now $U = \min(T, K)$, where T is as defined in Section 1. The conditional survivor function of interest becomes

$$G_U(t) = P[U > t|\boldsymbol{\theta}_t] = E\left[\exp\left\{-\int_0^t \lambda(s|\{X(s), \theta(s), 0 \leq s \leq t\})ds\right\} \times I[X(t) < x^*] \middle| \boldsymbol{\theta}_t\right]. \tag{5}$$

In this multiple failure cause setup under a stochastic wear accumulation with $\sigma[s, \boldsymbol{\theta}_s, X(s)] \equiv \sigma[s]$ and the additive hazards model $\lambda(t|\{X(s), 0 \leq s \leq t\}) = \lambda_0(t) + \beta X(t)$, Duchesne and Rosenthal (2002) show that we only get a collapsible model in the trivial case where $\mu[s, \boldsymbol{\theta}_s, X(s)] \equiv \mu[s]$, i.e. when the conditional survivor function (5) does not depend on $\boldsymbol{\theta}$. However we can get collapsible models in the multiple failure cause setup when the internal wear is deterministic:

Theorem 3.1 *Let internal wear be deterministic and let the hazard of a traumatic event be given by (4). Without loss of generality, let* $\lambda(t|\{x(s), \theta(s), 0 \leq s \leq t\}) \equiv \lambda^*(t|\boldsymbol{\theta}_t)$. *Then we obtain a collapsible model if, and only if, there exist continuously differentiable functions* μ_1, μ_2 *and* λ_1, λ_2 *with* $\frac{\partial}{\partial y}\mu_1(x, y) = \frac{\partial}{\partial x}\mu_2(x, y)$ *and* $\frac{\partial}{\partial y}\lambda_1(x, y) = \frac{\partial}{\partial x}\lambda_2(x, y)$ *for all* $x, y \geq 0$ *such that* $\mu[s, \boldsymbol{\theta}_s, x(s)] = \mu_1[s, y(s)] + \mu_2[s, y(s)]\theta(s)$ *and* $\lambda^*(s|\boldsymbol{\theta}_s) = \lambda_1[s, y(s)] + \mu_2[s, y(s)]\theta(s)$ *for all* $s \geq 0$ *such that* $\theta(s)$ *is smooth.*

4 Examples

4.1 Automobile systems' reliability

Duchesne and Lawless (2000) consider two collapsible models for the distribution of the time to failure, T, of automobile systems as a function of mileage accumulation history, $\boldsymbol{\theta}$:

$$\begin{aligned} P[T > t|\boldsymbol{\theta}_t] &= G_1[t^{1-\eta_1} y(t)^{\eta_1}] & (6)\\ \text{and} \qquad P[T > t|\boldsymbol{\theta}_t] &= G_2[(1-\eta_2)t + \eta_2 y(t)]. & (7)\end{aligned}$$

Both of these models can arise with failure caused by excessive wear only, if $\sigma[s, \boldsymbol{\theta}_s, X(s)] = \sigma[s]$, with $\mu[s, \boldsymbol{\theta}_s, X(s)] = (1-\eta_1)s^{-\eta_1} y(s)^{\eta_1} + \eta_1 s^{1-\eta_1} y(s)^{\eta_1 - 1}\theta(s)$ for model (6) and $\mu[s, \boldsymbol{\theta}_s, X(s)] = (1-\eta_2) + \eta_2\theta(s)$ for model (7). (Note that both specifications of μ statisfy Theorems 2.1 and 2.4.) A possible interpretation of μ for model (6) is that the infinitesimal increment in internal wear due to an increase of ds in age is proportional to $(y(s)/s)^{\eta_1} ds$ and the infinitesimal increment in internal wear due to an increase of $\theta(s)ds$ in cumulative mileage is proportional to $(s/y(s))^{1-\eta_1}\theta(s)ds$. Similarly, a possible interpretation of μ for model (7) is that the infinitesimal increment in internal wear due to an increase ds in age is proportional to ds and the infinitesimal increment in internal wear due to an increase $\theta(s)ds$ in cumulative mileage is proportional to $\theta(s)ds$.

4.2 Additive hazards model

Several authors (including Cox, 1999 and Bagdonavičius and Nikulin, 2001) suggest using the additive hazards model $\lambda(t|\{X(s), 0 \le s \le t\}) = \lambda_0(t) + \beta X(t)$ as a possible specification of (4). If wear is deterministic, then under this additive hazards model we get a collapsible model only if μ depends explicitly on the derivative of the usage rate $\theta(t)$. However if we let $\lambda(t|\{x(s), 0 \le s \le t\}) = \lambda^*(t|\theta(t)) = \lambda_0(t) + \beta\theta(t)$, i.e. traumatic events are more likely when the usage is more intense, then we automatically get a collapsible model when μ is as in Theorem 2.1, as then

$$P[T > t|\boldsymbol{\theta}_t] = \int_0^\infty \exp\left\{-\int_0^t \lambda_0(s)ds - \beta y(t)\right\} I\left[\int_0^t \mu[s, \boldsymbol{\theta}_s, x(s)ds < x\right] dF_{X^*}(x) = f(t, y(t))$$

for some function f.

Acknowledgements

This research was supported by individual operating grants to each author from the Natural Sciences and Engineering Research Council of Canada.

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