MMF1952Y: Stochastic Calculus Main Results

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Probability Spaces

- A probability space is a triple **(Ω, $\mathbb{P}$, $\mathcal{F}$)** where
  - Ω is the set of all possible outcomes
  - $\mathbb{P}$ is a probability measure
  - $\mathcal{F}$ is a sigma-algebra telling us which

Stochastic Integration

- A **diffusion or Ito process** X_t can be "approximated" by its local dynamics through a driving Brownian motion W_t:

$$X_{t+\Delta t} - X_t = \underbrace{\mu(X_t, t)}_{\text{Adapted drift process}} \Delta t + \underbrace{\sigma(X_t, t)}_{\text{Adapted volatility process}} \underbrace{\left[W_{t+dt} - W_t\right]}_{\text{fluctuations}}$$

- $\mathcal{F}_t^X$ denotes the **information** generated by the process $\mathbf{X_s}$ on the interval **[0,t]**

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Stochastic Integration

- If a random variable **Z** is known given $\mathcal{F}_t^X$ then one says, $\mathbf{Z} \in \mathcal{F}_t^X$ and **Z** is said to be **measurable** w.r.t. $\mathcal{F}_t^X$
- If for every $\mathbf{t \geq 0}$ a stochastic process $\mathbf{Y_t}$ is known given $\mathcal{F}_t^X$ then, $\mathbf{Y_t}$ is said to be **adapted** to the filtration $\mathcal{F}^X := \{\mathcal{F}_t^X\}_{t \geq 0}$
- The "formula"

$$X_{t+\Delta t} - X_t = \mu(X_t, t)\, \Delta t + \sigma(X_t, t) \left[W_{t+dt} - W_t\right]$$

is an intuitive construction of a general diffusion process from a Brownian motion process

Stochastic Integration

- **Stochastic differential equations** are generated by "taking the limit" as **$\Delta t \to 0$**

$$dX_t = \mu(X_t, t)\,dt + \sigma(X_t, t)\,dW_t$$

- A natural integral representation of this expression is

$$X_t - X_0 = \int_0^t \mu(X_s, s)\,ds + \int_0^s \sigma(X_s, s)\,dW_s$$

- The first integral can be interpreted as an ordinary **Riemann-Stieltjes integral**
- The second term cannot be treated as such, since path-wise **W_t** is **nowhere differentiable**!

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Stochastic Integration

- Instead define the integral as the **limit of approximating sums**
- Given a **simple process** **g(s)** [piecewise-constant with jumps at **$a < t_0 < t_1 < \ldots < t_n < b$**] the stochastic integral is defined as

$$\int_a^b g(s)\,dW_s = \sum_{k=0}^{n-1} \underbrace{g(t_k)}_{\text{Left end valuation}} \left[W_{t_{k+1}} - W_{t_k}\right]$$

- Idea…
  - Create a sequence of approximating simple processes which converge to the given process in the **L^2** sense
  - Define the stochastic integral as the limit of the approximating processes

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Martingales

- Glimpses of **Martingales** appeared in the discrete-time setting
- The **expectation** of a random variable Y on a probability space **(Ω, $\mathbb{P}$, $\mathcal{F}$)**

$$\mathbb{E}^{\mathbb{P}}[Y] = \int_{\Omega} Y(\omega)\, d\mathbb{P}(\omega)$$

- Given a stochastic variable **Y** the symbol

$$\mathbb{E}^{\mathbb{P}}[Y|\mathcal{F}_t]$$

  represents the **conditional expectation** of **Y** given $\mathcal{F}_t^X$, i.e. the information available from time to **0** up to time **t**
- This expectation is itself, in general, a **random variable**

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Martingales

- Glimpses of **Martingales** appeared in the discrete-time setting
- First define conditional expectations with respect to a filtration $\{\mathcal{F}_t^X\}_{t\geq 0}$
- Given a stochastic variable **Y** the symbol

$$\mathbb{E}[Y|\mathcal{F}_t]$$

  represents the **conditional expectation** of **Y** given $\mathcal{F}_t^X$, i.e. the information available from time to **0** up to time **t**
- This expectation is itself, in general, a **random variable**

Martingales

- Two important properties of conditional expectations
- **Iterated expectations**: for s < t

$$\mathbb{E}[Y|\mathcal{F}_s] = \mathbb{E}[\,\mathbb{E}[\,Y|\mathcal{F}_t]|\mathcal{F}_s]$$

  - double expectation where the inner expectation is on a larger information set reduces to conditioning on the smaller information set

- **Factorization** of measurable random variables : if **Z** $\in \mathcal{F}_t^X$

$$\mathbb{E}[Z\,Y|\mathcal{F}_t] = Z\,\mathbb{E}[Y|\mathcal{F}_t]$$

  - If Z is known given the information set, it factors out of the expectation

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Martingales

- A stochastic process $\mathbf{X_t}$ is called an **$\mathcal{F}_t$-martingale** if

1. X_t is **adapted** to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$
2. For every t ≥ 0

$$\mathbb{E}[|X(t)|] < \infty$$

3. For every s and t such that 0 ≤ t < s

$$\mathbb{E}[X(s)|\mathcal{F}_t] = X(t)$$

This last condition states that the **expected future value is its value now**

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Martingales : Examples

- **Standard Brownian motion** is a Martingale

$$\begin{aligned}\mathbb{E}[W_T|\mathcal{F}_t] &= \mathbb{E}[(W_T - W_t) + W_t|\mathcal{F}_t] \\ &= \mathbb{E}[(W_T - W_t)|\mathcal{F}_t] + W_t = W_t\end{aligned}$$

- **Stochastic integrals** are Martingales

$$\begin{aligned}\mathbb{E}\left[\int_0^T g(s)\,dW_s\,|\mathcal{F}_t\right] &= \mathbb{E}\left[\int_0^t g(s)\,dW_s + \int_t^T g(s)\,dW_s\,|\mathcal{F}_t\right] \\ &= \int_0^t g(s)\,dW_s + \mathbb{E}\left[\int_t^T g(s)\,dW_s\,|\mathcal{F}_t\right] \\ &= \int_0^t g(s)\,dW_s\end{aligned}$$

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Martingales : Examples

- A stochastic process satisfying an SDE with **no drift term** is a Martingale

$$dX_t = g(t)\,dW_t$$

- A class of **Geometric Brownian motions** are Martingales:

$$\begin{aligned}\mathbb{E}\left[e^{-\frac{1}{2}c^2\,T + c\,W_T}\middle|\,\mathcal{F}_t\right] &= e^{-\frac{1}{2}c^2\,t + c\,W_t}\,\mathbb{E}\left[e^{-\frac{1}{2}c^2\,(T-t) + c\,(W_T - W_t)}\middle|\,\mathcal{F}_t\right] \\ &= e^{-\frac{1}{2}c^2\,t + c\,W_t}\end{aligned}$$

Ito's Lemma

- **Ito's Lemma**: If a stochastic variable X_t satisfies the SDE

$$dX_t = \mu(X_t,t)\,dt + \sigma(X_t,t)\,dW_t$$

then given any function $\mathbf{f(X_t, t)}$ of the stochastic variable $\mathbf{X_t}$ which is twice differentiable in its first argument and once in its second,

$$df(X_t,t) = \left[\left(\frac{\partial}{\partial t} + \mu(X_t,t)\,\frac{\partial}{\partial X_t} + \frac{1}{2}\sigma^2(X_t,t)\,\frac{\partial^2}{\partial X_t^2}\right) f(X_t,t)\right]\,dt + \left[\sigma(X_t,t)\frac{\partial}{\partial X_t}f\right]\,dW_t$$

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Ito's Lemma…

- Can be obtained heuristically by performing a Taylor expansion in $\mathbf{X_t}$ and $\mathbf{t}$, keeping terms of order **dt** and $\mathbf{(dW_t)^2}$ and replacing

$$(dW_t)^2 \to dt\ , \qquad (dt)^2 \to 0\ , \qquad dW_t\,dt \to 0$$

- Quadratic variation of the pure diffusion is O(dt)!
- Cross variation of dt and dW_t is $O(dt^{3/2})$
- Quadratic variation of dt terms is $O(dt^2)$

Ito's Lemma…

$$
\begin{aligned}
df(X_t,t) &= f(X_{t+dt}, t+dt) - f(X_t,t) \\
&= f(X_t + dX_t, t+dt) - f(X_t,t) \\
&\approx \frac{\partial}{\partial t} f(x_t,t)\,dt + \frac{\partial}{\partial x_t} f(x_t,t)\,dx_t + \frac{\partial^2}{\partial x_t \partial t} f(x_t,t)\,dx_t dt \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial x_t^2} f(x_t,t)\,dx_t^2 + \frac{1}{2}\frac{\partial^2}{\partial t^2} f(x_t,t)\,dt^2 + \ldots
\end{aligned}
$$

$$
\begin{aligned}
(dX_t)^2 &= (\mu dt + \sigma dW_t)^2 \\
&= \mu^2 dt^2 + \sigma^2 (dW_t)^2 + 2\mu\sigma\,dt dW_t \approx \sigma^2 (dW_t)^2 \leftrightarrow \sigma^2\,dt
\end{aligned}
$$

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Ito's Lemma : Examples

- Suppose $\mathbf{S_t}$ satisfies the geometric Brownian motion SDE

$$
\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t
$$

then Ito's lemma gives

$$
\begin{aligned}
d \ln S_t &= (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma\,dW_t \\
\Rightarrow \quad \ln S_t - \ln S_0 &= (\mu - \tfrac{1}{2}\sigma^2)t + \sigma\,W_t
\end{aligned}
$$

Therefore $\mathbf{\ln S_t}$ satisfies a Brownian motion SDE and we have

$$
S_t = S_0 \exp\left\{(\mu - \tfrac{1}{2}\sigma^2)t + \sigma\,W_t\right\}
$$

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Ito's Lemma : Examples

- Suppose S_t satisfies the geometric Brownian motion SDE
- What SDE does S_t^β satisfy?

$$
\begin{aligned}
d(S_t^\beta) &= \left(0 + \mu\, S_t\, \beta S_t^{\beta-1} + \tfrac{1}{2}\sigma^2 S_t^2\, \beta(\beta-1) S_t^{\beta-2}\right) dt + \sigma S_t\, \beta S_t^{\beta-1} \\
\Rightarrow \frac{d(S_t^\beta)}{S_t^\beta} &= \beta\left(\mu + \tfrac{1}{2}(\beta-1)\sigma^2\right) dt + \beta\sigma\, dW_t
\end{aligned}
$$

- Therefore, $S^\beta(t)$ is also a GBM with new parameters:

$$
\frac{d(S_t^\beta)}{S_t^\beta} = \tilde{\mu}\, dt + \tilde{\sigma}\, dW_t
$$

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Dolean-Dade's exponential

- The **Dolean-Dade's exponential** $\mathcal{E}(\mathbf{Y}_t)$ of a stochastic process $\mathbf{Y}_t$ is the solution to the SDE:

$$
\frac{d\mathcal{E}(Y_t)}{\mathcal{E}(Y_t)} = dY_t
$$

- If we write

$$
dY_t = \mu_Y(t)\, dt + \sigma_Y(t)\, dW_t
$$

- Then,

$$
\mathcal{E}(Y_t) = \exp\left\{\int_0^t \left(\mu_Y(s) - \frac{1}{2}\sigma_Y(s)^2\right) ds + \int_0^t \sigma_Y(s)\, dW_s\right\}
$$

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Quadratic Variation

- In Finance we often encounter relative changes

$$\frac{dX_t}{X_t} = \mu_X(X_t, Y_t, t)\,dt + \sigma_X(X_t, Y_t, t)\,dW_t$$
$$\frac{dY_t}{Y_t} = \mu_Y(X_t, Y_t, t)\,dt + \sigma_Y(X_t, Y_t, t)\,dW_t$$

- The **quadratic variation** of the increments of X and Y can be computed by calculating the expected value of the product

$$d\,[X,Y]_t = \mathbb{E}\,[dX_t\,dY_t\,|\,\mathcal{F}_t] = \sigma_X\,\sigma_Y\,X_t\,Y_t\,dt$$

of course, this is just a fudge, and to compute it correctly you must show that

$$\lim_{||\mathcal{T}||\to 0} \sum_{j=0}^{n-1} (X(t_{j+1}) - X(t_j))(Y(t_{j+1}) - Y(t_j)) = \int_0^t \sigma_X(s)\,X(s)\,\sigma_Y(s)\,Y(s)ds\ a.s.$$

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Ito's Product and Quotient Rules

- **Ito's product rule** is the analog of the Leibniz product rule for standard calculus

$$\frac{d(X_tY_t)}{X_tY_t} = \frac{dX_t}{X_t} + \frac{dY_t}{Y_t} + \frac{d\,[X,Y]_t}{X_tY_t}$$
$$= (\mu_X + \mu_Y + \sigma_X\sigma_Y)dt + (\sigma_X + \sigma_Y)\,dW_t$$

- **Ito's quotient rule** is the analog of the Leibniz quotient rule for standard calculus

$$\frac{d(X_t/Y_t)}{X_t/Y_t} = \frac{dX_t}{X_t} - \frac{dY_t}{Y_t} + \frac{d\,[Y,Y]_t}{Y_t^2} - \frac{d\,[X,Y]_t}{X_t\,Y_t}$$
$$= \left(\mu_X - \mu_Y + \sigma_Y^2 - \sigma_X\sigma_Y\right)\,dt + (\sigma_X - \sigma_Y)\,dW_t$$

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Ito's Product and Quotient Rules

- We often encounter the inverse of a process
- Ito's quotient rule implies

$$\frac{d(1/Y_t)}{1/Y_t} = \left(-\mu_Y + \sigma_Y^2\right) dt - \sigma_Y \, dW_t$$

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Multidimensional Ito Processes

- Given a set of diffusion processes $\mathbf{X}_t^{(i)}$ (i = 1,…,n)

$$\begin{aligned} \frac{dX_t^{(1)}}{X_t^{(1)}} &= \mu^{(1)}(\vec{X}_t, t)\, dt + \vec{\sigma}^{(1)}(\vec{X}_t, t) \cdot d\vec{W}_t \\ \frac{dX_t^{(2)}}{X_t^{(2)}} &= \mu^{(2)}(\vec{X}_t, t)\, dt + \vec{\sigma}^{(2)}(\vec{X}_t, t) \cdot d\vec{W}_t \\ &\vdots \\ \frac{dX_t^{(n)}}{X_t^{(n)}} &= \mu^{(n)}(\vec{X}_t, t)\, dt + \vec{\sigma}^{(n)}(\vec{X}_t, t) \cdot d\vec{W}_t \end{aligned}$$

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Multidimensional Ito Processes

where $\mu^{(i)}$ and $\sigma^{(i)}$ are F_t – adapted processes and

$$\begin{aligned} \vec{\sigma}^{(i)}(\vec{X}_t, t) &\equiv \left(\sigma^{(i,1)}(\vec{X}_t, t), \dots, \sigma^{(i,m)}(\vec{X}_t, t)\right) \\ \vec{W}_t &\equiv \left(W_t^{(1)}, \dots, W_t^{(m)}\right) \\ d[W_t^{(i)}, W_t^{(j)}] &= \delta_{ij}\, dt \\ \text{and} \quad \delta_{ij} &\equiv \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \end{aligned}$$

In this representation the Wiener processes $\mathbf{W_t^{(i)}}$ are all independent

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Multidimensional Ito Processes

- Notice that the quadratic variation of between any pair of X's is:

$$\begin{aligned} d\left[X_t^{(i)}, X_t^{(j)}\right] &\equiv \mathbb{E}\left[dX_t^{(i)}\, dX_t^{(j)} \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[(X^{(i)}\, \vec{\sigma}^{(i)} \cdot d\vec{W}_t))(X^{(j)}\, \vec{\sigma}^{(j)} \cdot d\vec{W}_t) \middle| \mathcal{F}_t\right] \\ &= X^{(i)} X^{(j)} \vec{\sigma}^{(i)} \cdot \vec{\sigma}^{(j)}\, dt \end{aligned}$$

- Consequently, the correlation coefficient ρ_{ij} between the two processes and the volatilities of the processes are

$$\begin{aligned} \tilde{\sigma}_i &= \sqrt{\vec{\sigma}^{(i)} \cdot \vec{\sigma}^{(i)}} = ||\vec{\sigma}^{(i)}|| \\ \rho_{ij} &= \frac{\vec{\sigma}^{(i)} \cdot \vec{\sigma}^{(j)}}{||\vec{\sigma}^{(i)}||\, ||\vec{\sigma}^{(j)}||} = \frac{\vec{\sigma}^{(i)} \cdot \vec{\sigma}^{(j)}}{\tilde{\sigma}_i\, \tilde{\sigma}_j} \end{aligned}$$

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Multidimensional Ito Processes

- The diffusions $X^{(i)}$ may also be written in terms of correlated diffusions as follows:

$$\begin{aligned} \frac{dX_t^{(1)}}{X_t^{(1)}} &= \mu^{(1)}(\vec{X}_t, t)\, dt + \tilde{\sigma}^{(1)}(\vec{X}_t, t)\, dB_t^{(1)} \\ \frac{dX_t^{(2)}}{X_t^{(2)}} &= \mu^{(2)}(\vec{X}_t, t)\, dt + \tilde{\sigma}^{(2)}(\vec{X}_t, t)\, dB_t^{(2)} \\ &\vdots \\ \frac{dX_t^{(n)}}{X_t^{(n)}} &= \mu^{(n)}(\vec{X}_t, t)\, dt + \tilde{\sigma}^{(n)}(\vec{X}_t, t)\, dB_t^{(n)} \end{aligned}$$

where $d[B_t^{(i)}, B_t^{(j)}] = \rho_{ij}\, dt$

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Multidimensional Ito's Lemma

- Given any function **f($X_t^{(1)}$, … , $X_t^{(n)}$, t)** that is twice differentiable in its first n-arguments and once in its last,

$$\begin{aligned} & df(X_t^{(1)}, \ldots, X_t^{(n)}, t) \\ &= \Bigg[\Bigg(\frac{\partial}{\partial t} + \sum_{i=1}^{n} \mu^{(i)}\, X_t^{(i)}\, \frac{\partial}{\partial X_t^{(i)}} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{n} X_t^{(i)}\, X_t^{(j)}\, \vec{\sigma}^{(i)} \cdot \vec{\sigma}^{(j)}\, \frac{\partial^2}{\partial X_t^{(i)} \partial X_t^{(j)}}\Bigg) f\Bigg]\, dt \\ &\quad + \sum_{i=1}^{n} \frac{\partial}{\partial X_t^{(i)}} f\, \vec{\sigma}^{(i)} \cdot d\vec{W}_t \end{aligned}$$

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Multidimensional Ito's Lemma

- Alternatively, one can write,

$$
\begin{aligned}
&df(X_t^{(1)},\ldots,X_t^{(n)},t) \\
&= \Bigg[\Bigg(\frac{\partial}{\partial t}+\sum_{i=1}^{n}\mu^{(i)}\,X_t^{(i)}\,\frac{\partial}{\partial X_t^{(i)}} \\
&\quad +\frac{1}{2}\sum_{i,j=1}^{n} X_t^{(i)}\,X_t^{(j)}\,\tilde{\sigma}^{(i)}\tilde{\sigma}^{(j)}\rho_{ij}\,\frac{\partial^2}{\partial X_t^{(i)}\partial X_t^{(j)}}\Bigg)\,f\Bigg]\,dt \\
&\quad +\sum_{i=1}^{n}\frac{\partial f}{\partial X_t^{(i)}}\,\tilde{\sigma}^{(i)}dB_t^{(i)}
\end{aligned}
$$

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Multidimensional Ito rules

- Product rule:

$$
\begin{aligned}
\frac{d(X_tY_t)}{X_tY_t} &= \frac{dX_t}{X_t}+\frac{dY_t}{Y_t}+\frac{d\,[X,Y]_t}{X_tY_t} \\
&= (\mu_X+\mu_Y+\vec{\sigma}_X\cdot\vec{\sigma}_Y)dt+(\vec{\sigma}_X+\vec{\sigma}_Y)\cdot d\vec{W}_t \\
&= (\mu_X+\mu_Y+\rho\sigma_X\sigma_Y)dt+\sigma_X dB_t^X+\sigma_Y\,dB_t^Y
\end{aligned}
$$

Multidimensional Ito rules

- Quotient rule:

$$
\begin{aligned}
\frac{d(X_t/Y_t)}{X_t/Y_t} &= \frac{dX_t}{X_t} - \frac{dY_t}{Y_t} + \frac{d\,[Y,Y]_t}{Y_tY_t} - \frac{d\,[X,Y]_t}{X_tY_t} \\
&= \left(\mu_X - \mu_Y + ||\sigma_Y||^2 - \vec{\sigma}_X \cdot \vec{\sigma}_Y\right) dt + (\vec{\sigma}_X - \vec{\sigma}_Y) \cdot d\vec{W}_t \\
&= \left(\mu_X - \mu_Y + \tilde{\sigma}_Y^2 - \tilde{\sigma}_X\tilde{\sigma}_Y\rho_{XY}\right) dt + \tilde{\sigma}_X dB_t^X - \tilde{\sigma}_Y dB_t^Y
\end{aligned}
$$

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Feynman – Kac Formula

- Suppose that a function f({X_1,…,X_n}, t) which is twice differentiable in all first n-arguments and once in t satisfies

$$
\begin{aligned}
r\,f &= f_t + \sum_{i=1}^{n} \mu^{(i)}\, X_t^{(i)}\, f_i + \frac{1}{2} \sum_{i,j=1}^{n} X_t^{(i)}\, X_t^{(j)}\, \vec{\sigma}^{(i)} \cdot \vec{\sigma}^{(j)} f_{i,j} \\
f(T) &= H(\vec{X}(T))
\end{aligned}
$$

then the **Feynman-Kac formula** provides a solution as :

$$
f(\vec{Y}(t),t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\vec{Y}(s),s)\,ds} H(\vec{Y}(T)) \middle| \mathcal{F}_t \right]
$$

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Feynman – Kac Formula

- In the previous equation the stochastic process $Y_1(t),\ldots, Y_2(t)$ satisfy the SDE's

$$\frac{dY_t^{(i)}}{Y_t^{(i)}} = \mu^{(i)}(\vec{Y}_t, t)\, dt + \vec{\sigma}^{(i)}(\vec{Y}_t, t) \cdot d\vec{W}'_t$$

and W'(t) are $\mathbb{Q}$ – Wiener processes.

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Self-Financing Strategies

- A **self-financing strategy** is one in which no money is added or removed from the portfolio at any point in time.
- All changes in the weights of the portfolio must net to zero value
- Given a portfolio with $\Delta^{(i)}$ units of asset $X^{(i)}$, the Value process is

$$V_t = \sum_i \Delta_t^{(i)}\, X_t^{(i)}$$

- The total change is:

$$dV_t = \sum_i \left\{ d\Delta_t^{(i)}\, X_t^{(i)} + \Delta_t^{(i)}\, dX_t^{(i)} + d[\Delta^{(i)}, X^{(i)}]_t \right\}$$

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Self-Financing Strategies

- If the strategy is **self-financing** then,

$$dV_t \;=\; \sum_i \Delta_t^{(i)}\, dX_t^{(i)}$$

- So that self-financing requires,

$$\sum_i \left\{ d\Delta_t^{(i)}\, X_t^{(i)} + d[\Delta^{(i)}, X^{(i)}]_t \right\} = 0$$

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Measure Changes

- The **Radon-Nikodym derivative** connects probabilities in one measure $\mathbb{P}$ to probabilities in an **equivalent** measure $\mathbb{Q}$

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}(\mathcal{A})\right] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{I}(\mathcal{A})\frac{d\mathbb{Q}}{d\mathbb{P}}\right]$$

- This random variable has $\mathbb{P}$-expected value of 1

$$\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1$$

- Its conditional expectation is a **martingale process**

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t \;:=\; \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_t\right]$$

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Measure Changes

- Given an event $\mathcal{A}$ which is $\mathcal{F}_T$-measurable, then,

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}(\mathcal{A})|\mathcal{F}_t\right] = \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{I}(\mathcal{A})\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_T|\mathcal{F}_t\right]}$$

For Ito processes there exists an F_t-adapted process s.t. the Radon-Nikodym derivative can be written

$$\begin{aligned}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t &= \mathcal{E}\left(\int_0^t \vec{\lambda}(\vec{W}_s, s)\cdot d\vec{W}_s\right)\\ &= \exp\left\{-\frac{1}{2}\int_0^u ||\vec{\lambda}(\vec{W}_s, s)||^2\, ds + \int_0^u \vec{\lambda}(\vec{W}_s, s)\cdot d\vec{W}_s\right\}\end{aligned}$$

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Girsanov's Theorem

- **Girsanov's Theorem** says that
  - if $W_t^{(i)}$ are **standard Brownian processes** under $\mathbb{P}$
  - then the $W_t^{(i)*}$ are **standard Brownian processes** under $\mathbb{Q}$ where

$$d\vec{W}_t^* = -\vec{\lambda}(\vec{W}_t, t)\, dt + d\vec{W}_t$$