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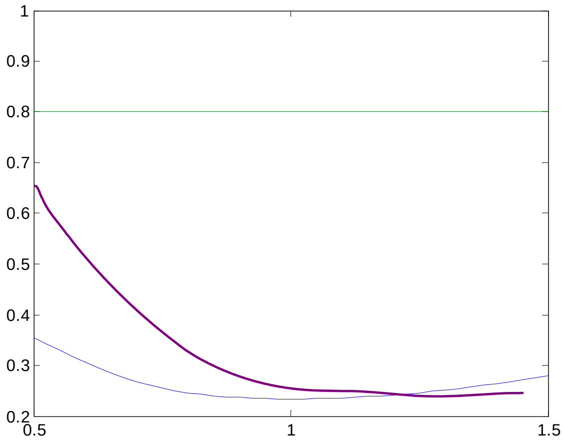
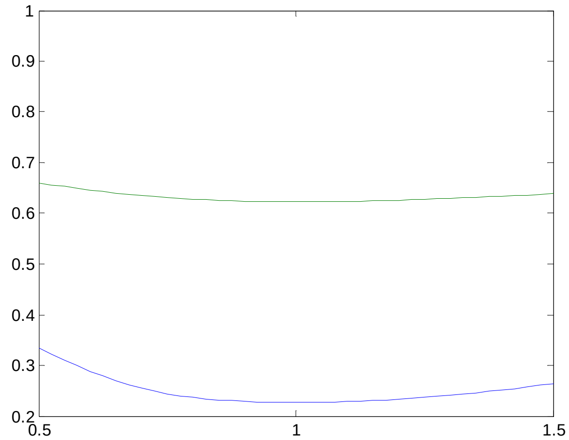
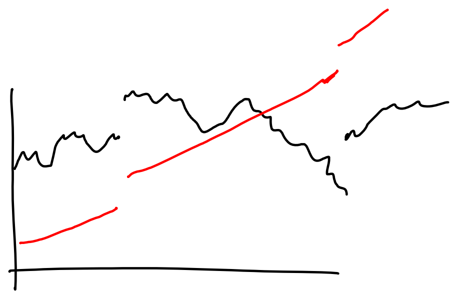
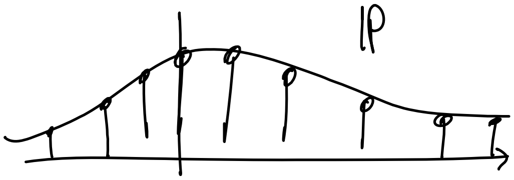
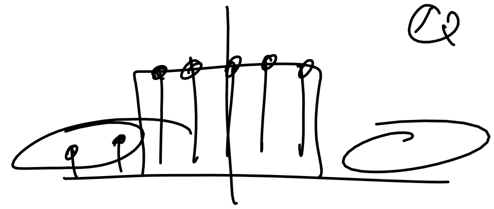






Compound Poisson Process





$$J_t = \sum_{n=1}^{N_t} j_n \quad ; \quad j_1, j_2, \ldots \quad \text{iid} \quad \overset{\text{c.d.f.}}{G(y)}$$

$$S_t = S_0 \, e^{(r - \frac{1}{2}\sigma^2) t + \sigma W_t + (J_t - A_t)}$$

recall when $G(y) = \delta(y - \gamma)$

$$A_t = (e^{\gamma} - 1)\lambda t$$

direct computation...

$$\mathbb{E}[S_t] = S_0 e^{rt} \, e^{-A_t} \, \mathbb{E}[e^{J_t}]$$

$$\mathbb{E}[e^{u J_t}] = \mathbb{E}[\, \mathbb{E}[e^{u J_t} \mid N_t]\,]$$

$$= \mathbb{E}[\, \mathbb{E}[e^{u(j_1 + \cdots + j_{N_t})} \mid N_t]\,]$$

$$= \mathbb{E}\left[\prod_{n=1}^{N_t} \mathbb{E}[e^{u j_n}]\right] \quad \text{independence}$$

$$= \mathbb{E}\left[\left(\mathbb{E}[e^{u j_1}]\right)^{N_t}\right] \quad \text{identical}$$

$$= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left(\mathbb{E}[e^{u j_1}]\right)^n$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t \, \mathbb{E}[e^{u j_1}])^n}{n!}$$

$$= \exp\left\{ -\lambda t + \lambda t \, \mathbb{E}[e^{u j_1}] \right\}$$

$$= \exp\left\{ (\underbrace{\mathbb{E}[e^{u j_1}]}_{\text{before mod } e^{ur}} - 1) \lambda t \right\}$$

so $A_t = (\mathbb{E}[e^{j_1}] - 1) \lambda t$

$g_t = \mathbb{E}_t[e^{u J_T}]$ is a mtg

$$= \mathbb{E}_t[e^{u(J_T - J_t)}] \, e^{u J_t}$$

$g_t = g(t, J_t) \Rightarrow g(t, J) = h(t) \cdot e^{uJ}$

$$dg_t = \partial_t g \, dt + \left(g(t, J_{t-} + j_{N_t}) - g(t, J_{t-}) \right) dN_t$$

$\mathbb{E}_t[dg_t] = 0$

$$\Rightarrow \partial_t g + \mathbb{E}[g(t, J + j) - g(t, J)] \lambda = 0$$

$$\boxed{\partial_t g + \lambda \int_{-\infty}^{\infty} [g(t, J+y) - g(t, J)] \, dG(y) = 0}$$ PIDE.

$$e^{uJ} \partial_t h + \lambda \int_{-\infty}^{\infty} (h \, e^{u(J+y)} - h \, e^{uJ}) \, dG(y) = 0$$

$$e^{uJ} \left[\partial_t h + \left(\lambda \int_{-\infty}^{\infty} (e^{uy} - 1) \, dG(y) \right) h \right] = 0$$

$h(T) = 1$

$$h(t) = \exp\left\{ \left(\lambda \int_{-\infty}^{\infty} (e^{uy} - 1) \, dG(y) \right) (T - t) \right\}$$

so finally:

$$\boxed{g(t,J) = e^{\lambda \int_{-\infty}^{\infty} (e^{uy}-1)\, dG_T(y)\, (T-t) + uJ}}$$

$$\rightarrow \quad e^{\lambda \int_{-\infty}^{\infty} (e^{uy}-1)\, dG_T(y)\, t}$$

Sofar $\lambda\left(\mathbb{E}[e^{uj_1}]-1\right) t$

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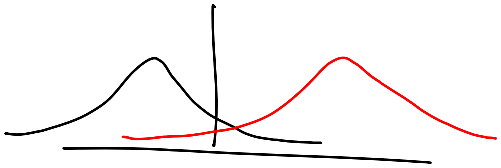
Merton jump-diffusion: $j_i \sim N(\hat{\mu}, \hat{\sigma}^2)$

asymmetry.



$$S_T = S_0 \, e^{(r - \frac{1}{2}\sigma^2) T + \sigma W_T + (J_T - A_T)}$$

$$\text{call} = e^{-rT} \, \mathbb{E}_0^{\mathbb{Q}}\left[(S_T - K)_+ \right]$$

$$= e^{-rT} \, \mathbb{E}_0^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)_+ \mid N_T \right]\right]$$

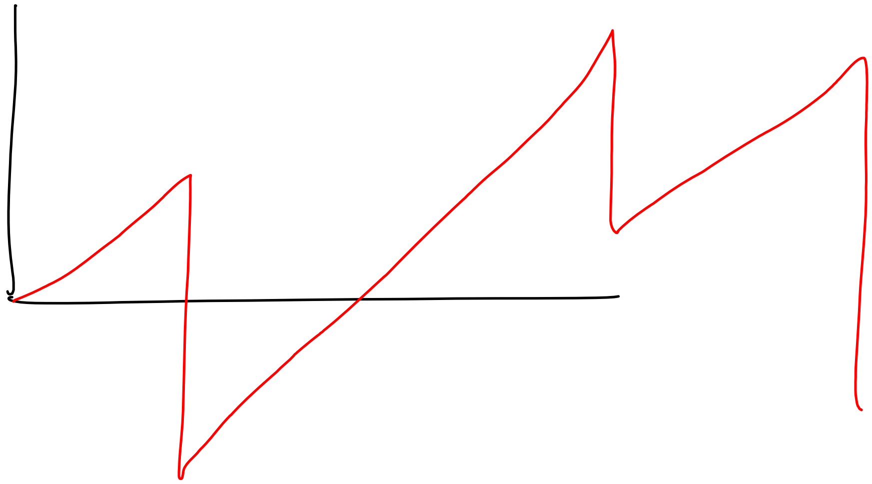
recall $J_T = \sum_{n=1}^{N_T} j_n$ so given N_T

$$J_T \big|_{N_T} \sim N(\hat{\mu} N_T ; \hat{\sigma}^2 N_T)$$

and $\sigma W_T + J_T \big|_{N_T} \sim N(\hat{\mu} N_T ; \hat{\sigma}^2 N_T + \sigma^2 T)$

can now rewrite inner $\mathbb{E}[\cdot]$ as Black-Scholes.

$$\text{call} = \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \cdot g(n)$$



$$g_t = \mathbb{E}_t^{\mathbb{Q}}[(S_T - K)_+] \quad \text{is a } \mathbb{Q}\text{-mtg.}$$

$$dg_t = (\partial_t + \overbrace{\mathcal{L}}^{(r-\gamma) S \partial_S + \frac{1}{2}\sigma^2 S^2 \partial_{SS}}) g \, dt + \sigma S \partial_S g \, dW_t$$

$$+ \left(g(t, S_{t-} e^{\tilde{y}_{N_t}}) - g(t, S_{t-}) \right) dN_t$$

$$\frac{dS_t}{S_{t-}} = r \, dt - \overbrace{\lambda(\mathbb{E}[e^{\tilde{y}_1}] - 1)}^{\gamma} dt$$

$$+ \sigma \, dW_t + (e^{\tilde{y}_{N_t}} - 1) \, dN_t$$

$$\boxed{(\partial_t + \mathcal{L}) g + \lambda \int_{-\infty}^{\infty} \left(g(t, S e^{y}) - g(t, S) \right) dG(y) = 0}$$

$$x = \ln S, \qquad g(t, e^x) = h(t, x)$$

$$\left(\partial_t + (r - \tfrac{1}{2}\sigma^2 - \gamma)\partial_x + \tfrac{1}{2}\sigma^2 \partial_{xx}\right) h$$

$$+ \lambda \int_{-\infty}^{\infty} \left(h(t, x+y) - h(t, x) \right) dG(y) = r h$$

Fourier transform in x ...

$$\partial_t \hat{h} + \left((r - \tfrac{1}{2}\sigma^2 - \gamma)(i\omega) + \tfrac{1}{2}\sigma^2 (i\omega)^2 \right) \hat{h}$$

$$+ \lambda \int_{-\infty}^{\infty} \left(\hat{h}(\omega) e^{i\omega y} - \hat{h}(\omega) \right) dG(y) = r \hat{h}(\omega)$$

$$\int_{-\infty}^{\infty} h(x+y) e^{-i\omega x} dx = \int_{-\infty}^{\infty} h(z) e^{-i\omega(z-y)} dz$$

$$= \int_{-\infty}^{\infty} h(z) e^{-i\omega z} dz \; e^{i\omega y}$$

$$= e^{i\omega y}\,\hat{H}(\omega)$$

$$Q(x) = (e^x - K)_+$$

$$\partial_t \hat{h}(\omega) + \psi(\omega)\,\hat{h}(\omega) = 0, \qquad \hat{h}(T,\omega) = \hat{Q}(\omega)$$

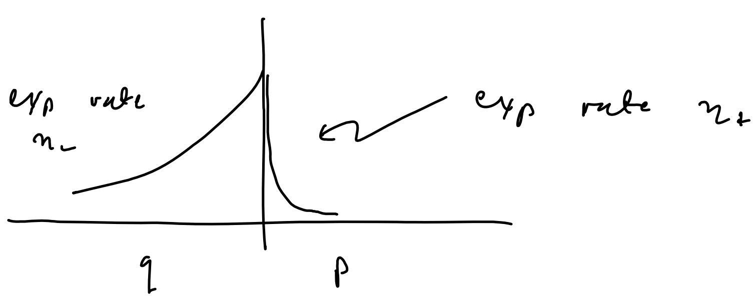
$$\psi(\omega) = (r - \tfrac{1}{2}\sigma^2 - \gamma)\,i\omega - \tfrac{1}{2}\sigma^2\omega^2$$
$$+ \lambda \int_{-\infty}^{\infty} (e^{i\omega y} - 1)\, dG_t(y) - r$$

sol:
$$\hat{h}(\omega) = e^{\psi(\omega)(T-t)}\,\hat{Q}(\omega)$$

$$h(t,x) = \int_{-\infty}^{\infty} e^{i\omega x}\, e^{\psi(\omega)(T-t)}\,\hat{Q}(\omega)\,\frac{d\omega}{2\pi}$$

For Merton $\displaystyle\int_{-\infty}^{\infty} e^{i\omega y}\, dG_t(y) = e^{i\omega\hat{u} - \frac{1}{2}\omega^2\hat{\sigma}^2}$

drift correction: $\gamma = \lambda\left(e^{\hat{u} + \frac{1}{2}\hat{\sigma}^2} - 1\right)$



$$dG_t(y) = q\,\eta_-\,e^{\eta_- x}\,\mathbb{1}_{x<0} + p\,\eta_+\,e^{-\eta_+ x}\,\mathbb{1}_{x\ge 0}$$

$\psi(\omega) = ?$

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Heston Model: $\qquad X_t = \ln S_t$

$$\frac{dS_t}{S_t} = r\,dt + \sqrt{v_t}\,dW_t$$

corr ρ.

$$dv_t = \kappa(\theta - v_t)\,dt + \eta\sqrt{v_t}\,dB_t$$

$$\mathbb{E}\left[e^{i\omega X_t}\right] = e^{A_t + B_t X_t + C_t v_t}$$