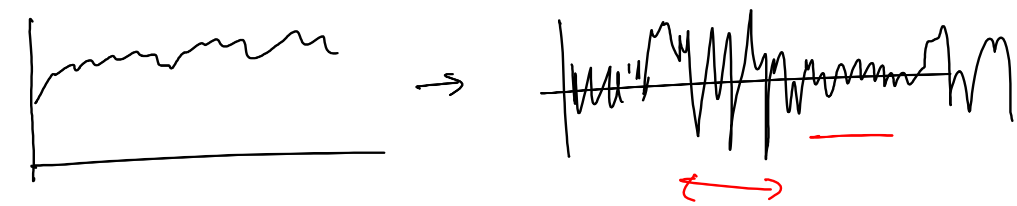
Tuesday, May 11, 2010
2:54 PM

Stochastic Volatility

- regime switching
- jumps
- diffusive model of vol.

$$\frac{dS_t}{S_t} = \mu\,dt + \underset{\xi}{\sigma}\,dW_t$$

$\hookrightarrow$ $F(t, S_t) \to$ PDE $\to$ Feynman-Kac $\mathbb{E}^{Q}[\,\cdot\,]$

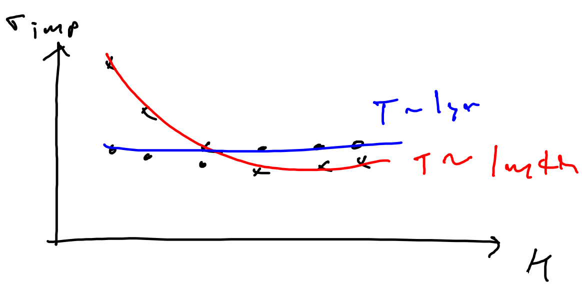


$R \uparrow \;\Rightarrow\;$ vol $\searrow$

$R \searrow \;\Rightarrow\;$ vol $\nearrow$

leverage effect.

$$P^*(K, T) = P_{BS}(K, T, \sigma_{imp}(K, T))$$



$$\frac{dS_t}{S_t} = \mu\, dt + \underline{\underline{\sigma(Z_t)}}\, dW_t$$

Z_t is a finite-state continuous homogeneous time Markov chain.

$$\mathbb{P}(Z_{t+\Delta t} = k \mid Z_t = j) = \left(\exp\{A \Delta t\}\right)_{jk}$$

generator matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix} \qquad \begin{array}{l} a_{ii} \le 0 \\ a_{ij} \ge 0 \quad i \neq j \\ \sum_j a_{ij} = 0 \end{array}$$

e.g. $A = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$

$$\mathbb{P}(Z_{\Delta t} = 2 \mid Z_0 = 1) = \left(e^{A \Delta t}\right)_{12}$$

$$= \mathbb{1} + \sum_{n=1}^{\infty} \frac{(A \Delta t)^n}{n!} = (\mathbb{1} + A \Delta t) + o(\Delta t)$$

$$= \begin{pmatrix} 1 - a \Delta t & a \Delta t \\ b \Delta t & 1 - b \Delta t \end{pmatrix} + o(\Delta t)$$

$\sim$

under $\mathbb{Q}$: $\quad A \to \hat{A}$

$\qquad W_t \to \hat{W}_t$

$\qquad \mu \to r$

$$\frac{dS_t}{S_t} = r\, dt + \sigma(Z_t)\, d\hat{W}_t$$

$$V_0 = e^{-rT} \mathbb{E}_0^{\mathbb{Q}}[(S_T - K)_+]$$

$S_t \to \ln S_t$

$$d(\ln S_t) = (r - \tfrac{1}{2}\sigma^2(Z_t))\,dt + \sigma(Z_t)\,d\hat{W}_t$$

$$\Rightarrow \quad S_T = S_0 \exp\left\{ \int_0^T (r - \tfrac{1}{2}\sigma^2(Z_s))\,ds + \int_0^T \sigma(Z_s)\,d\hat{W}_s \right\}$$

$$\mathcal{G}_T \triangleq \sigma\left((Z_s)_{0 \le s \le T} \right)$$

$$\ln\left(\frac{S_T}{S_0}\right)\Big|_{\mathcal{G}_T} \sim \mathcal{N}\left(\int_0^T (r - \tfrac{1}{2}\sigma^2(Z_s))\,ds \,;\; \int_0^T \sigma^2(Z_s)\,ds \right)$$

$$V_0 = e^{-rT}\, \mathbb{E}_0^{\mathbb{Q}}\left[(S_T - K)_+ \right]$$

$$= \mathbb{E}_0^{\mathbb{Q}}\left[e^{-rT}\, \mathbb{E}_0^{\mathbb{Q}}\left[(S_T - K)_+ \mid \mathcal{G}_T \vee \mathcal{F}_0 \right] \right]$$

$$= \mathbb{E}_0^{\mathbb{Q}}\Big[S_0\, \Phi\left(d_+\left(\textstyle\int_0^T \sigma^2(Z_s)\,ds\right)\right)$$

$$- K e^{-rT}\, \Phi\left(d_-\left(\textstyle\int_0^T \sigma^2(Z_s)\,ds\right)\right) \Big]$$

$$d_\pm = \frac{\ln(S_0/K) + rT \pm \tfrac{1}{2}\int_0^T \sigma^2(Z_s)\,ds}{\left(\int_0^T \sigma^2(Z_s)\,ds\right)^{1/2}}$$

$$\int_0^T \sigma^2(Z_s)\,ds = \sigma^2(1) \cdot \underbrace{T_1}_{\text{occupation time of } Z_t = 1} + \sigma^2(2)\,(T - T_1)$$

$$V_0 = \int_0^T f_{T_1}(u)\, g(u)\, du$$

PDE approach ...

$$V_t = e^{-r(T-t)}\, \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)_+ \right]}_{\mathbb{Q}\text{-mtg.}}$$

$$e^{-rt} V_t = e^{-rT} \mathbb{E}_t^{\mathbb{Q}}[(S_T - K)_+] \quad \text{is a } \mathbb{Q}\text{-mtg!}$$

$\hookrightarrow V(t, S_t, Z_t)$

<u>side remarks...</u>

$$dF(Z_{t-}) = F(Z_{t+dt}) - F(Z_{t-})$$

$$= F(1)\, \mathbb{1}_{Z_{t+dt}=1} - F(Z_{t-})$$

$$+ F(2)\, \mathbb{1}_{Z_{t+dt}=2}$$

$$\mathbb{E}_t[dF(Z_{t-})] = F(1)(\mathbb{1} + A\,dt)_{Z_{t-}1} - F(Z_{t-})$$

$$+ F(2)(\mathbb{1} + A\,dt)_{Z_{t-}2}$$

$$Z_{t-}=1: \quad F(1)(\mathbb{1} + A\,dt)_{11} - F(1)$$

$$+ F(2)(\mathbb{1} + A\,dt)_{12}$$

$$= F(1)(1 + A_{11}\,dt) - F(1)$$

$$+ F(2)(0 + A_{12}\,dt)$$

$$= [F(1) A_{11} + F(2) A_{12}]\,dt$$

$$= (F(2) - F(1)) \underset{\substack{\shortparallel \\ a}}{A_{12}}\, dt$$

$$Z_{t-}=2: \quad (F(1) - F(2)) \underset{\substack{\shortparallel \\ b}}{A_{21}}\, dt$$

$$Z_{t-}=j: \quad \sum_k (F(k) - F(j))\, A_{jk}\, dt$$

$$dX_t = \mu(t, X_t, Z_{t-})\,dt + \sigma(t, X_t, Z_{t-})\,dW_t$$

$$dF(t, X_t, Z_{t-})$$

$$= \left(\partial_t F + \mu \partial_x F + \tfrac{1}{2}\sigma^2 \partial_{xx} F \right) dt$$

$\mathcal{A}$

$$+ \sigma \partial_x F \, dW_t$$

$$+ \sum_{k=1}^{K} \left(F(t, X_t, k) - F(t, X_t, Z_{t-}) \right) \mathbb{1}_{Z_{t+dt} = k}$$

$$\mathbb{E}_t[dF]$$

$$= (\mathcal{A}) \, dt + \sum_{k=1}^{K} \left(F(t, X_t, k) - F(t, X_t, Z_{t-}) \right) A_{Z_{t-} k} \, dt$$

$$e^{-rt} V(t, S_t, Z_{t-}) = e^{-rT} \mathbb{E}^{\mathbb{Q}}[\varphi(S_T, Z_T)]$$

$$d(e^{-rt} V) = -r e^{-rt} V \, dt + e^{-rt} \, dV$$

$$= e^{-rt} \Big[-rV \, dt + (\partial_t + \mathcal{L}) V \, dt + \sigma S \partial_S V \, dW_t$$

$$+ \sum_{k=1}^{K} [V(t, S_t, k) - V(t, S_t, Z_{t-})] A_{Z_{t-} k} \, dt$$

$$+ (dM_t - dB_t) \Big]$$

$$\boxed{ -rV(t, S, j) + (\partial_t + \mathcal{L}^{(j)}) V(t, S, j) + \sum_{k=1}^{K} [V(t, S, k) - V(t, S, j)] A_{jk} = 0 }$$

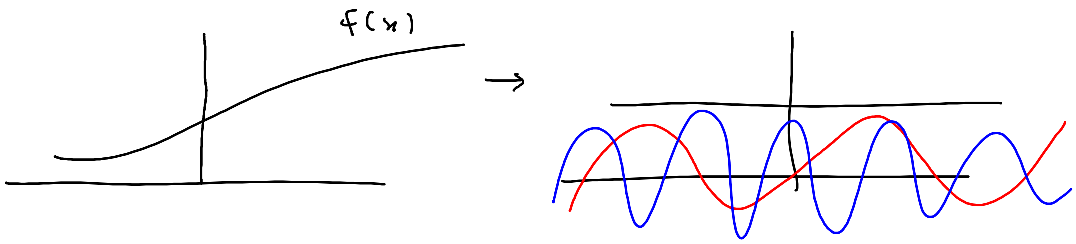
$j = 1, 2, \ldots, K$, s.c. $V(T, S, j) = \varphi(S, j)$

$$\mathcal{L}^{(j)} = r S \partial_S + \tfrac{1}{2} (\sigma(j))^2 S^2 \partial_{SS}$$

Tuesday, May 11, 2010
4:43 PM

$r=0$, $K=2$, $A = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$

$$\begin{cases} \left(\partial_t - \frac{1}{2}(\sigma^{(1)})^2 \partial_x + \frac{1}{2}(\sigma^{(1)})^2 \partial_{xx}\right) V^{(1)} + a(V^{(2)} - V^{(1)}) = 0 \\ \left(\partial_t - \frac{1}{2}(\sigma^{(2)})^2 \partial_x + \frac{1}{2}(\sigma^{(2)})^2 \partial_{xx}\right) V^{(2)} + b(V^{(1)} - V^{(2)}) = 0 \end{cases}$$



$$\hat{F}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} F(x)\,dx \qquad \left(\sum_{n=1}^{N} \hat{F}(\omega_n)\, e^{i\omega_n x}\right)$$

$$F(x) = \int_{-\infty}^{\infty} e^{+i\omega x}\, \hat{F}(\omega)\, \frac{d\omega}{2\pi}$$

$\hat{F}(\omega) = \mathcal{F}[F](\omega)$ Fourier transform

$F(x) = \mathcal{F}^{-1}[\hat{F}](x)$

$\mathcal{F}[\partial_x F](\omega) = i\omega\, \hat{F}(\omega)$

$\mathcal{F}[\partial_{xx} F](\omega) = -\omega^2\, \hat{F}(\omega)$

Fourier transform applied to PDE ..

$$\partial_t \hat{V}^{(1)} + \frac{1}{2}(\sigma^{(1)})^2 \underbrace{(-i\omega \hat{V}^{(1)})}_{\mathcal{F}[\partial_x V^{(1)}]} + \frac{1}{2}(\sigma^{(1)})^2 \underbrace{(-\omega^2 \hat{V}^{(1)})}_{\mathcal{F}[\partial_{xx} V^{(1)}]}$$

$$+ a(\hat{V}^{(2)} - \hat{V}^{(1)}) = 0$$

$$\Rightarrow \begin{cases} \left(\partial_t - \frac{1}{2}(\sigma^{(1)})^2 (i\omega + \omega^2)\right) \hat{V}^{(1)} + a(\hat{V}^{(2)} - \hat{V}^{(1)}) = 0 \\ \left(\partial_t - \frac{1}{2}(\sigma^{(2)})^2 (i\omega + \omega^2)\right) \hat{V}^{(2)} + b(\hat{V}^{(1)} - \hat{V}^{(2)}) = 0 \end{cases}$$

$$\hat{V} = \begin{pmatrix} \hat{V}^{(1)} \\ \hat{V}^{(2)} \end{pmatrix}$$

$$B = \begin{pmatrix} -a - \frac{1}{2}(\sigma^{(1)})^2 (i\omega + \omega^2) & a \\ b & -b - \frac{1}{2}(\sigma^{(2)})^2 (i\omega + \omega^2) \end{pmatrix}$$

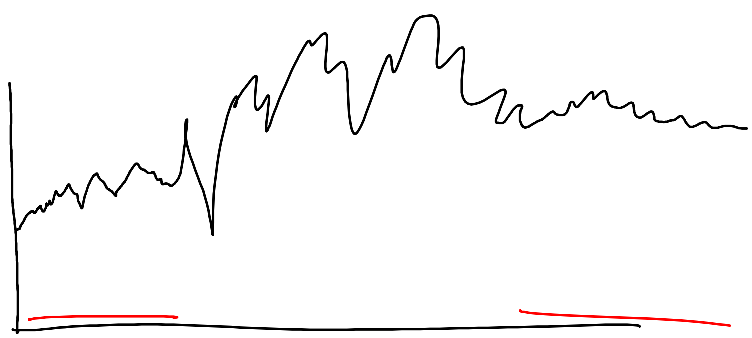
$$\Rightarrow \quad \partial_t \hat{V} + B(\omega)\hat{V} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\hat{V}(t,\omega) = e^{B(\omega)(T-t)} \hat{V}(T,\omega)$$

$$V(t,x) = \int_{-\infty}^{\infty} e^{i\omega x} \, e^{B(\omega)(T-t)} \, \hat{V}(T,\omega) \, \frac{d\omega}{2\pi}$$

$$\left(\hat{V}(T,\omega)\right)_j = \int_{-\infty}^{\infty} e^{-i\omega x} \, \Phi(x,j) \, dx$$

e.g. $(e^x - K)$



Tuesday, May 11, 2010
5:22 PM

jump models:

Poisson

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2) t + \sigma W_t + \gamma(N_t - \alpha t)}$$

$$\mathbb{E}_0[S_t] = S_0 e^{rt} \cdot e^{-\alpha\gamma t} \mathbb{E}_0[e^{\gamma N_t}]$$

$$\hookrightarrow e^{(e^\gamma - 1)\lambda t}$$

$$\mathbb{E}_0[e^{\gamma N_t}] = \sum_{n=0}^{\infty} e^{\gamma n} \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t e^\gamma)^n}{n!}$$

$$= e^{-\lambda t} e^{\lambda t e^\gamma} = e^{(e^\gamma - 1)\lambda t}$$

$$F(t, N_t) = \mathbb{E}_t[e^{\gamma N_T}]$$

$$dF = (F(t, N_{t-} + 1) - f(t, N_{t-}))\Delta N_t + \partial_t f \, dt$$

$$\mathbb{E}_t[dF] = (F(t, N_{t-}+1) - f(t, N_{t-}))\lambda \, dt + \partial_t f \, dt$$

$$\partial_t f(t, n) + (f(t, n+1) - f(t, n))\lambda = 0$$

(can use z-transform to solve for F

$$\sum_{n=0}^{\infty} F(t, n) z^n$$

solve.)

OR NB: $f(t,N_t) = \mathbb{E}_t[\, e^{\gamma(N_T - N_t)}\,]\, e^{\gamma N_t}$

so $f(t,n) = h(t)\, e^{\gamma n}$

$\Rightarrow \quad \partial_t h(t)\, e^{\gamma n} + (\, e^{\gamma(n+1)} h(t) - e^{\gamma n} h(t))\, \lambda = 0$

$\Rightarrow \quad \partial_t h(t) + (e^{\gamma} - 1)\, \lambda\, h(t) = 0$

$\Rightarrow \quad h(t) = \exp\{-\lambda(e^{\gamma} - 1)\, t\}\, c$ and $h(T) = 1$

$\Rightarrow \quad h(t) = \exp\{\lambda(e^{\gamma} - 1)(T - t)\}$

$$\alpha = \frac{e^{\gamma} - 1}{\gamma}\, \lambda \;\Rightarrow\; \mathbb{E}_0[S_T] = S_0\, e^{rT}$$

$$e^{-rT}\, \mathbb{E}[\, (S_T - K)_+]$$

$$= e^{-rT}\, \mathbb{E}[\, (S_0\, e^{(r - \frac{1}{2}\sigma^2)T + \sigma W_T + \gamma(N_T - \alpha T)} - K)_+]$$

$$= e^{-rT}\, \mathbb{E}[\, \mathbb{E}[\, (\quad)_+ \mid N_T]\,]$$

$$= \mathbb{E}[\, S_0\, e^{\gamma(N_T - \alpha T)}\, \Phi(d_+(N_T)) - K\, e^{-rT}\, \Phi(d_-(N_T))\,]$$

$$d_\pm = \frac{\ln(S_0/K) + (r \pm \frac{1}{2}\sigma^2 - \gamma\alpha)T + \gamma N_T}{\sigma\sqrt{T}}$$

$$= \sum_{n=0}^{\infty} g(n)\, \frac{(\lambda T)^n}{n!}\, e^{-\lambda T}$$

"
$$\partial_t F + \mathcal{L} F + \lambda(\, F(t, S, n+1) - F(t, S, n)) = 0$$
"

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t + \gamma(N_t - \alpha t)}$$

$$dS_t = S_{t-}((r - \alpha\gamma)\,dt + \sigma\, dW_t) + S_{t-}(e^{\gamma} - 1)\,\Delta N_t$$

$$(S_{t-} \rightsquigarrow S_{t-} e^{\gamma})$$

NB: $$\mathbb{E}_t[dS_t] = S_{t-}((r - \alpha\gamma)\,dt + 0) + S_{t-}(e^{\gamma} - 1)\,\lambda\, dt$$

$$= S_{t-}\, r\, dt$$

r=0

$$V(t, S_t) = \mathbb{E}^{\mathbb{Q}}_t[(S_T - K)_+]$$

$$dV = (\partial_t V + \mathcal{L} V)\,dt + \sigma S \partial_S V\, dW_t + [V(t, S_{t-} e^{\gamma}) - V(t, S_{t-})]\,\Delta N_t$$

$$(\partial_t V + \mathcal{L} V) + \underbrace{(V(t, S e^{\gamma}) - V(t, S))\,\lambda}_{\int_{-\infty}^{\infty} (V(t, S e^{z}) - V(t, S))\,\lambda\, dF(z)} = 0$$

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