The Model of Lines for Option Pricing with Jumps

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Abstract

This article reviews a pricing model, suitable for variance-gamma jump processes, based on the method of lines. The method accuracy is studied using European style calls as a benchmark. Implementation details for continuously and discretely monitored barrier options, and American and Bermudan options are given.

1 Introduction

Lattice models introduced in [10] played an important role in extending the Black-Scholes model [5] to a wide class of exotic payoff structures. The emergence of jump models for asset prices proposed in the works of Merton [23], Bates [3], [4], Madan et al. [20] and Geman et al. [14], stimulated interest in adequate generalizations and replacements of lattice models to accommodate fat-tailed return distributions. In [1], we introduced a model based on the method of lines, henceforth referred to as the *model of lines*, which is simple to implement, flexible and admits an intuitive financial interpretation. In this article, a more complete description of the methodology is supplied, and the solution techniques are elaborated on.

In this paper, the underlying price process is postulated to be a variance-gamma pure jump process, which was introduced and studied extensively by Madan, Carr, Chang, Geman, Seneta and Yor in the series of articles, [8], [19], [20], [21]. The method of lines was first introduced in the financial literature by Carr [6] with the intent of pricing American style options in the standard Black-Scholes model. Carr recognized that this method prices random maturity contracts exactly, with maturities obeying the Erlang distribution for the n-th arrival of a Poisson process. The variance-gamma pure jump process can be obtained from the geometric Brownian motion by means of a stochastic time change driven by a gamma distribution, which is a generalization of the Erlang distribution where the arrival number n is taken to be continuous. To formulate the *model of lines*, Carr's equations for randomized maturity contracts, found in [6], will be recast in a form that yields exact pricing formulas for particular cases of the variance-gamma process. To accommodate the general variance-gamma model, a Richardson extrapolation scheme is proposed, and is demonstrated to work well with realistic model parameters.

While lattice models are based on recombining trees in which both calendar time and stock price are discretized, the *model of lines* postualtes only the discretization of calendar time. As such, each key date in the model corresponds to a continuous line for stock prices. In most situations, the solution of the equations for the pricing function along each line can be represented by simple polynomials with a finite number of terms. Following Bates [3], consider measuring time according to the ticks of a special financial clock, which runs at a speed proportional to the number of transactions per unit real time. According to the financial clock, transaction volume appears to be constant. Furthermore, log-returns are found to be nearly normally distributed when measured with respect to the financial clock. The *model of lines* developed here can be viewed as follows: each line corresponds to a fixed date in calendar time, while the corresponding duration in financial time - proportional to trading volume - is random and distributed as a Poisson exponential process. The time change over n-lines follows an Erlang distribution corresponding to the n^{th} arrival of a Poisson process. By formally taking the order n of the Erlang distribution to be a continuous variable, the arrival distribution becomes the gamma distribution. Gamma distributions provide the time change function for the variance-gamma pure jump process. Consequently, the *model of lines* provides a method for pricing derivative claims in the variance gamma model when the model parameters reduce the arrival distribution to an Erlang distribution. As will be demonstrate later on, the general case can often be recovered with high precision by interpolation or extrapolation methods.

The differential equations that arise in the *model of lines* are similar to the Black-Scholes equations, with the following important distinction: time derivatives are replaced by finite differences, while derivatives with respect to stock price remain intact. The pricing functions along each line are found to satisfy a system of inhomogeneous ordinary differential-difference equations, which admit simple analytic solutions for most options. Although the situation is similar to Carr's equations for American style options with randomized maturity [6], there are important differences. The key difference being that calendar time in Carr's solution must be reinterpreted as financial time in the *model of lines*. Because trading occurs in calendar time, not in financial time, the re-interpretation breaks risk neutrality. To restore risk neutrality, the stock price must be scaled and the option price discounted from one time-step to the next. Under this adjustment, the *model of lines* reproduces the *exact* - up to negligible roundoff errors - prices of European style options in which the underlying follows a variance-gamma process. The solution scheme for European style puts and calls can easily be modified to exactly price barrier and Bermudan options contingent on information on the lines only. Path-dependent options requiring continuous monitoring, such as American options and barrier options can also be priced efficiently. However, in these cases, the *model of lines* produces approximate prices, as the exercise boundaries are assumed to be piecewise constant between lines.

The *model of lines* enjoys the same calibration efficiencies and empirical explanatory power of the variance-gamma model. Nonetheless, it is still interesting to compare it with the better known stochastic volatility models. Diffusion models where volatility is stochastic have been considered by a number of authors, including Hull and White [17], Wiggins [29], Scott [27], Melino and Turnbull [22], Heston [15], [16]. Stochastic volatility models based on GARCH, such as in Duan [13], have the added advantage that the postulated process for the underlying asset is well justified by historical time series. As in jump models, the effect of stochastic volatility can formally be interpreted as inducing a random time change. Furthermore, these models are capable of explaining the skew of implied volatilities of medium and long dated options; however, intrinsic model limitations are encountered with short dated options. The observed steepness of implied volatility skews cannot be justified by a model where paths are continuous and volatility driven by a stationary process. Models with state and time dependent volatility, as in Rubinstein et al. [18],[25], Derman and Kani [11], [12] and Stutzer [28], are more effective with short dated options. Unfortunately, this approach requires the introduction of a highly non-stationary process that requires frequent readjustments. On the other hand, jump models reproduce the skew of short and medium dated options, while the predicted smiles for longer dated claims are flatter than observed. For a comparison among these models the reader is referred to the empirical studies [2], [26]. The present authors believe that a model which combines both jumps and stochastic volatility, possibly of the GARCH type, will perform considerably better than either models separately. In a forthcoming paper, we demonstrate how such a synthesis can be implemented by combining the *model of lines* with a two-level stochastic volatility process that gives rise to a recombining stochastic volatility tree. It suffices to say that the *model of lines* is not limited to pure jump models, but rather, is an essential element of a more elaborate pricing framework.

The remainder of this paper is organized as follows: Section 2 is composed of three parts: firstly, the variance gamma model is reiviewed; secondly, the financial interpretation of the method of lines in terms of randomized maturity options as developed by Carr [6] and Carr and Faguet [7] is discussed; thirdly, the *model of lines* is developed as a version of the method of lines that is appropriate for variance-gamma models. Section 3 contains the explicit solutions to the pricing problems for European options, continuously and discretely monitored barrier options, American and Bermudan options. Our implementation of the Richardson extrapolation algorithm is provided in Section 4 and Section 5 concludes the paper.

2 The Variance Gamma Model

The variance gamma model introduced in [19], is an elegant extension of the standard geometric Brownian motion process for stock prices. In the variance-gamma model, the log-returns on stock prices are postulated to follow a Brownian motion

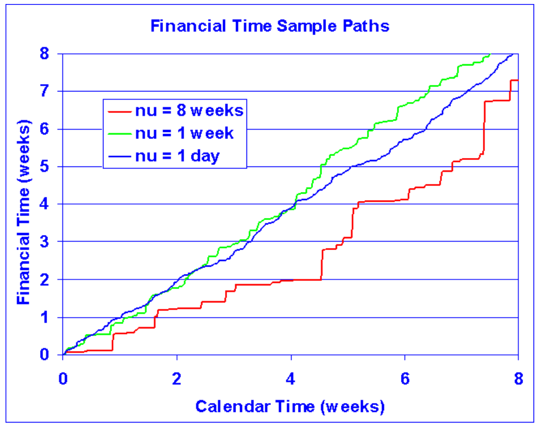

Figure 1: *Diagram showing several gamma process sample paths which describe financial time. Notice that these paths are quite different from typical diffusion process paths. Also, as ν tends towards zero the path becomes more deterministic*

not in calendar time, but rather in *financial time*, which flows faster or slower than real time depending on market activity (see figure 1). Financial time can in some sense be thought of as following the trading volume as opposed to clock tick-time. This has the advantage that when trading volumes rise, volatility increases and larger jumps are more likely to occur, as is found empirically. To further model financial time, a secondary process is introduced, which performs the time change from real-time to financial time. This secondary process is assumed to be a gamma process, which is essentially the continuous time counterpart to the Poisson process. To be specific, let S_t denote the stock price process at time t written as follows,

$$\ln\left(\frac{S_t}{S_0}\right) = \omega t + X_{\Gamma(t;\nu)}(\theta;\sigma) \tag{1}$$

where $X_\tau(\theta;\sigma)$ denotes a Brownian process with drift θ and volatility σ evaluated at time τ; $\Gamma(t;\nu)$ denotes a gamma process with a mean rate of one and variance rate of ν evaluated at time t; and ω is a factor necessary to maintain risk-neutrality. Assuming the risk free rate r is constant in time, ω is fixed as follows,

$$\mathbb{E}[S_t] = e^{rt}S_0 \qquad \Rightarrow \qquad \omega = r + \frac{1}{\nu}\ln\left(1 - \left(\theta + \frac{1}{2}\sigma^2\right)\nu\right) \tag{2}$$

The price of a European style option on that stock can be obtained by first conditioning on the financial time given by the random time change, and then integrating over all financial times with the appropriate density. Suppose the pay-off of the option at time T is $\phi(S_T)$, and denote the conditioned price of the option at current calendar time by $p(S_t, g)$, and the unconditioned price by $P(S_t)$. Then,

$$P(S_t) = \int_0^\infty dg \; \frac{g^{\frac{T-t}{\nu}-1} e^{-g/\nu}}{\Gamma(\frac{T-t}{\nu}) \nu^{\frac{T-t}{\nu}}} \; p(S_t, g) \tag{3}$$

and

$$\begin{aligned} p(S_t, g) &= e^{-r(T-t)} \mathbb{E}[\phi\left(S_T\right) \mid g] \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\sigma^2 g}} \; e^{-\frac{(x-\theta g)^2}{2\sigma^2 g}} \phi(e^{\omega(T-t)+x} S_t) \\ &= e^{-r(T-t)+(\theta+\frac{1}{2}\sigma^2)g} P_{BS}\left(e^{\omega(T-t)} S_t, g, \left(\theta + \frac{1}{2}\sigma^2\right), \sigma\right) \end{aligned} \tag{4}$$

where $P_{BS}(S, g, r, \sigma)$ denotes the Black-Scholes price of the European option maturing in time g, S denotes the spot, r the risk-free rate and σ the volatility. For particular pay-offs (puts and calls) it is possible to carry out the integral appearing in (3), the result can be expressed in terms of confluent hypergeometric functions [20]. However, in order to price path-dependent options, such as American or barrier options, it is necessary to solve a difficult integro-differential equation. It is conceivable that an eigenvalue decomposition for the pricing kernel could yield useful results; however, analytic tractability would suffer. In this paper, a solution to the pricing problem which leads to exact analytic expressions consisting solely of exponentials and polynomials is presented. Our solution uses the methods of lines framework, which is briefly discussed below.

The numerical method of lines can be thought of as a dimensional reduction of a partial differential equation to an ordinary differential equation. Consider the Black-Scholes differential equation,

$$\left\{-\partial_\tau + rS\partial_S + \frac{1}{2}\sigma^2 S^2 \partial_S^2\right\} P_{BS}(S, \tau) = rP_{BS}(S, \tau) \tag{5}$$

(τ denotes time to maturity.) If time is discretized, the time derivative can be approximated by a difference

$$\partial_\tau P_{BS}(S, \tau) \approx \frac{1}{\tau}(P_{BS}(S, \tau) - P_{BS}(S, 0)) \tag{6}$$

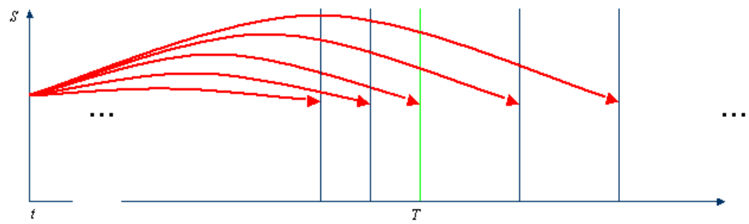

Figure 2: *The standard method of lines yields exact prices for an option that matures at a random time which is distributed according to an Erlang distribution)*

and the partial differential equation reduces to a sequence of one-dimensional ordinary differential equations. This technique was used by Carr [6] and by Meyer and Van Der Hoek [24] to price American options. Clearly, the results obtained using the method of lines yields only an approximate answer to the pricing problem. Carr proposed a very suggestive interpretation of the method of lines, which implies that this method prices random maturity contracts exactly (see figure 2). Starting from this intuition, we now argue that a slight modification can render the method of lines *exact* for the variance-gamma model.

To explore the connection between the method of lines and the variance-gamma model, first note that the representation given in equation (4) for the conditioned variance-gamma price in terms of the Black-Scholes price can be used to obtain a PDE for the conditioned price. If the time to maturity is taken to be equal to ν the unconditioned price, given by (3), is simply the Laplace-Carson transformation of the conditioned price. Consequently, as Carr demonstrated in [6], the PDE reduces to a sequence of ODE's. These step will now be carried out explicitly.

The conditioned price can be easily shown to satisfy the following PDE,

$$\begin{aligned} (-\partial_g + D_S)\, p(S_t, g) &= 0 && (7) \\ \lim_{g \to 0} p(S_t, g) &= \mathrm{e}^{-r\nu} \phi(\mathrm{e}^{\omega\nu} S_t) && (8) \end{aligned}$$

where the operator D_S is defined as,

$$D_S \equiv \frac{1}{2}\sigma^2 S^2 d_{SS} + \left(\theta + \frac{1}{2}\sigma^2\right) S d_S \qquad (9)$$

Applying the integral kernel appearing in the right hand side of (3) to the above PDE one finds,

$$-\frac{1}{\nu}\left[P(S_t) - \lim_{g\to 0} p(S_t, g)\right] + D_S P(S_t) = 0 \tag{10}$$

Consequently, the unconditioned price of a European option maturing at time $t + \nu$ is given by the solution to the ordinary differential equation,

$$-\frac{1}{\nu}\left[P(S_t) - e^{-r\nu}\phi(e^{\omega\nu} S_t)\right] + D_S P(S_t) = 0 \tag{11}$$

The first term of this equation is the analog of the discretized time in the usual method of lines framework. The main difference between the standard method of lines and the one constructed here, is that both the stock price level and the option price must be scaled from line to line. Furthermore, the variance-rate parameter ν now has a natural interpretation as the time-span between lines.

This procedure can be applied recursively to obtain the price of any pay-off with cash flows occurring only at integer multiples of ν, and is given succinctly by the following differential-difference equations,

$$D_S P^{(n)}(S) \quad = \quad \frac{1}{\nu}\left(P^{(n)}(S) - e^{-r\nu} P^{(n-1)}\left(e^{\omega\nu} S\right)\right) \tag{12}$$

where $P^{(n)}(S)$ denotes the price function n time-steps of size ν prior to maturity. These equations must be supplemented by appropriate boundary conditions (see section 3) and a terminal time condition at maturity,

$$P^{(0)}(S) \quad = \quad \phi\left(S\right) \tag{13}$$

The system of equations (12) and (13) form what we term the *model of lines*.

There are several important features of the *model of lines* that should be elaborated on. Firstly, the drift which appears in (9) is not the risk-free rate. This is because in financial time the stock drifts according to θ rather than r. Secondly, in the operator D_S there is no constant term, i.e. the term $-rP$ in the usual Black-Scholes equation is missing. On reflection it is clear that such a term must be absent because the discounting occurs in real time and not financial time. Finally, once the price on one line is known, the price on the next line is determined from an option with a scaled spot and discounted price. The discounting of the price is natural, and can be thought of as the spot price of the previous line. The scaling of the spot itself can be understood from the fact that, although the drift of the stock in financial time is not equal to the risk free rate, risk neutrality must still be enforced, hence across each line additional drifting must be imposed.

Thus far, it was assumed that the stock paid no dividends; if, however, dividends are paid continuously at a constant rate d, the risk-neutrality condition is effectively modified to,

$$\mathbb{E}[S_t] = \mathrm{e}^{(r-d)t} S_0 \tag{14}$$

This implies that $\omega \to \omega - d$, while all other quantities remain intact. The price of a continuously dividend paying stock then reduces to that of its non-dividend paying cousin with the above adjustment to ω.

Within the *model of lines* framework, it is also possible to incorporate dividends that are paid on the lines. The methodology is straight-forward, on the lines the spot must be effectively reduced by the dividend pay-out, and amounts to the following alteration of the differential-difference equations (12) and (13),

$$D_S P^{(n)}(S) = \frac{1}{\nu}\left(P^{(n)}(S) - e^{-r\nu} P^{(n-1)}\left(e^{\omega\nu}(S - \mathbf{d_n})\right)\right) \tag{15}$$

$$P^{(0)}(S) = \phi(S) \tag{16}$$

where $\mathbf{d_n}$ denotes the dividend paid on the n^{th} line.

Extensions to situations with stochastic interest rates are also possible. In this case the *model of lines* would have to be written in the forward measure and the payoff parameterized in terms of the forward price, which follows a martingale process. Similarly, extensions of Black's formulas for interest rate derivatives such as caps and swaptions can be obtained. While this paper focuses on equity options, we are currently preparing articles on these and other extensions.

3 Option Pricing in the Model of Lines

The differential-difference equations (12), (13) are easier to solve if written in terms of the moneyness parameter

$$x_t = \ln(S_t/K) \tag{17}$$

rather than the stock price. It is also convenient to work with the scaled dimensionless prices $\tilde{P}^{(n)}(x)$ defined so that

$$P^{(n)}(x) = K\tilde{P}^{(n)}\left(x + n\omega\nu\right) \tag{18}$$

The differential equations for the scaled prices has the following simple form:

$$D_x\tilde{P}^{(n)}(x) \quad = \quad -\frac{e^{-r\nu}}{\nu}\tilde{P}^{(n-1)}\left(x\right) \tag{19}$$

where, the differential operator D_x is given by

$$D_x \equiv \frac{1}{2}\sigma^2 d_{xx} + \theta d_x - \frac{1}{\nu} \tag{20}$$

and the final time condition is now

$$\tilde{P}^{(0)}(x) \quad = \quad \frac{1}{K}\phi\left(x\right) \tag{21}$$

Additional boundary conditions depending on the particulars of the contract, such as early exercise clauses for Americans, must also be included.

In the following sections, the above equations are used to derive closed form solutions for the price functions of a number of standard option contracts.

3.1 European Options

Consider a European put option struck at K and maturing at time T. The terminal boundary condition is expressed through the payoff function

$$\tilde{P}^{(0)}(x, K) = \frac{1}{K}\phi(x) \equiv (1 - \mathrm{e}^x)_+ \tag{22}$$

Notice that the coefficients of the system of ODE's in (19) are constant in time. Furthermore, the boundary conditions do not depend on the time change. The general solution to our system of equations on the lines $n = 1, 2, ..$ therefore has the form

$$\tilde{P}^{(n)}(x, K) = \begin{cases} \mathrm{e}^{-d_+ x} \sum_{m=0}^{n-1} a_m^{(n)} x^m & , x > 0 \\ \mathrm{e}^{-rn\nu} - \mathrm{e}^{x-n\alpha} + \mathrm{e}^{-d_- x} \sum_{m=0}^{n-1} b_m^{(n)} x^m & , x < 0 \end{cases} \tag{23}$$

where, the constant

$$\alpha \equiv \omega\nu \tag{24}$$

has been introduced to lighten notations, and $d_\pm$ are the positive and negative solutions of the characteristic polynomial of the differential operator D_x, i.e.

$$d_\pm = \frac{\theta \pm \sqrt{\theta^2 + \frac{2}{\nu}\sigma^2}}{\sigma^2} \tag{25}$$

Most of the coefficients of the price function on the n^{th}-line can be computed using equation (19) and expressed in terms of the coefficients on the $(n-1)^{\mathrm{th}}$-line. The resulting recurrence relations to be solved backwards in time are

$$a_m^{(n)} = \frac{\gamma_+ a_{m-1}^{(n-1)} + \frac{m(m+1)}{2}\sigma^2 a_{m+1}^{(n)}}{m\,(\theta - \sigma^2 d_+)} \quad , \qquad 1 \leq m \leq n-1 \tag{26}$$

$$b_m^{(n)} = \frac{\gamma_- b_{m-1}^{(n-1)} + \frac{m(m+1)}{2}\sigma^2 b_{m+1}^{(n)}}{m(\theta - \sigma^2 d_-)} \quad , \qquad 1 \leq m \leq n-1 \tag{27}$$

Here the discount factors, $\gamma_\pm \equiv -\nu^{-1}\,\mathrm{e}^{-r\nu-\alpha d_\pm}$ have been introduced and $b_n^{(n)} = a_n^{(n)} \equiv 0$. The coefficients $a_0^{(n)}$ and $b_0^{(n)}$ are an exception, as the they are fixed by enforcing continuity of the pricing function and the delta ratio at the at-the-money point $x = 0$

$$\begin{pmatrix} 1 & -1 \\ -d_+ & d_- \end{pmatrix} \begin{pmatrix} a_0^{(n)} \\ b_0^{(n)} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-rn\nu} - \mathrm{e}^{-n\alpha} \\ -\mathrm{e}^{-n\alpha} - a_1^{(n)} + b_1^{(n)} \end{pmatrix} \tag{28}$$

The three equations (26), (27), (28)fully determine the price of the European put option in terms of a finite combination of elementary functions. It is quite surprising that although the underlying follows a variance-gamma process, the prices at integer multiples of the variance rate can be expressed as simply as (23).

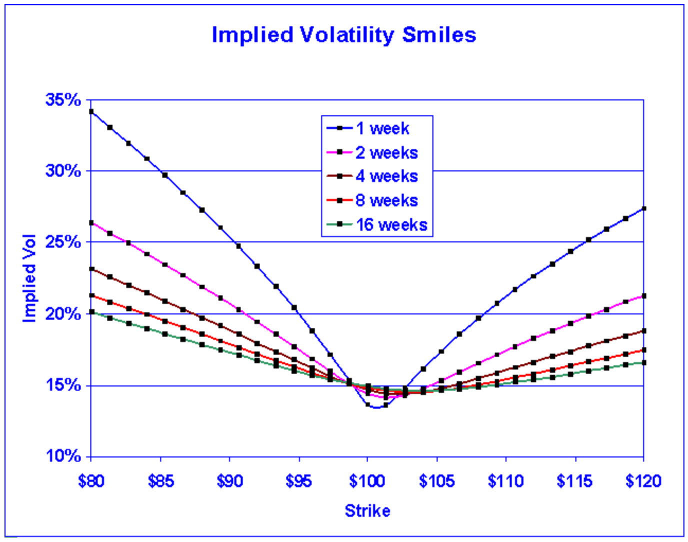

Figure 3: *The implied smiles of the variance-gamma model for maturities of 1, 2, 4, 8 and 16 weeks with parameters:* $\nu = 1$ *week,* $\sigma = 15\%$, $\theta = -20\%$ *and* $r = 5\%$

Volatility smiles for a variance-gamma process with $\sigma = 15\%$, $\theta = -20\%$, and $\nu = 1$ week are plotted in figure 3. The relative error between the implied volatilities obtained using the exact prices in [20] and those obtained using the *model of lines* were also calculated. The largest relative error for the smiles in figure 3 was found to be $\sim 10^{-3}\%$ while the average relative error over the smiles was found to be $\sim 10^{-5}\%$. These negligible discrepancies are due to computational round-off errors; there was also little difference in computation time between the two pricing schemes.

Put options, the stock and a bond provide a spanning set of assets for all European style claims. The price of European calls, $C^{(n)}(x, K)$, of the same maturity, T, is obtained from put-call parity, i.e.

$$C^{(n)}(x, K) = P^{(n)}(x, K) + K(e^x - e^{-rT}) \tag{29}$$

Furthermore, the price $F(S)$ of more general European style payoffs $\phi(S_T)$ can be reconstructed from put option prices as indicated in [9], where it is shown that a static replication argument leads to the pricing formula,

$$\begin{aligned} F^{(n)}(S) = &\quad \phi(S)e^{-rn\Delta t} + \phi'(S)[C^n(0,K) - P^{(n)}(0,K)] \\ &+ \int_0^S \phi''(K)P^{(n)}\left(\log\tfrac{S}{K}, K\right)dK + \int_S^\infty \phi''(K)C^{(n)}\left(\log\tfrac{S}{K}, K\right)dK \end{aligned} \tag{30}$$

3.2 Barrier Options

As an example of a barrier option, consider a pay-at-expiry, down-and-out put with barrier H such that

$$h = -\ln\frac{H}{K} > 0 \tag{31}$$

and rebate

$$R = \rho K \tag{32}$$

Since the boundary value is constant and the option price depends on the probability of breaching the barrier, but not on the time when it happens, the boundary condition does not depend on the financial time change, just as in the case of European options. Thus, the price of the barrier instrument must satisfy one of the following conditions,

$$\tilde{P}^{(n)}(x,K,H) = \begin{cases} \mathrm{e}^{-rn\nu}\rho & ; x \le n\alpha - h \quad \text{pay-at-expiry} \\ \\ \rho & ; x \le n\alpha - h \quad \text{pay-at-hit} \end{cases} \tag{33}$$

where α was defined in (24). Notice that in terms of the moneyness parameter x, the boundary moves upwards if α is positive and downwards otherwise. This is a consequence of equation (18). The sign of α largely depends on the size of the skewness parameter θ, and can be explained as follows: for a small kurtosis parameter (recall that ν was estimated in [20] to be ~ 0.2) equation (2) leads to the approximation $\alpha \sim (r - \theta - \frac{1}{2}\sigma^2)\nu$; consequently, if $\theta \lesssim r - \frac{1}{2}\sigma^2$ then $\alpha > 0$. For typical equity options, the implied volatility smile is negatively skewed implying a negative θ parameter, this in turn implies that α is typically positive. Nonetheless, it is entirely possible for α to be negative; as such, both signs of α will be discussed here.

If α is positive, then the boundary condition moves in towards $x = 0$ and eventually crosses it; while if α is negative the boundary condition continually moves away from $x = 0$. In the former case, the solution contains at most three regions, the

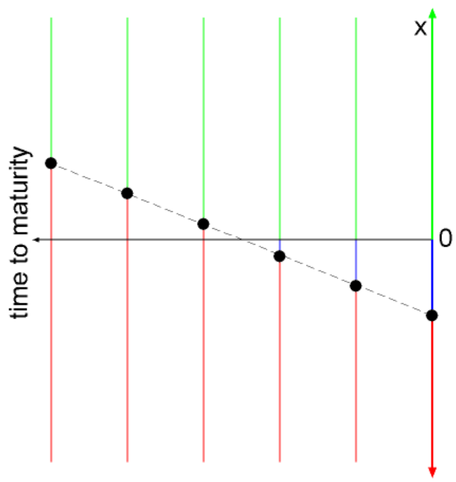

Figure 4: *The moving boundary and solution regions for a down and out barrier option with positive α. Notice that the number of regions is at most three and reduces to two when the boundary moves across the $x = 0$ line.*

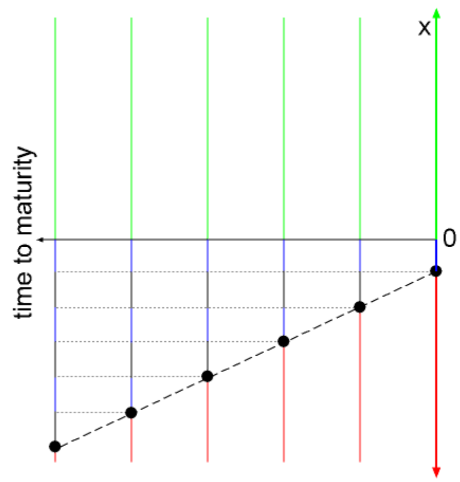

Figure 5: *The moving boundary and solution regions for a down and out barrier option with negative α. Notice that the number of regions constantly increases.*

region when the spot is above the strike, between the strike and the boundary, and below the boundary. When the boundary collides with $x = 0$, the solution then contains only two regions (see figure 4). However, with $\alpha < 0$, the number of regions constantly increases (see figure 5). In either case, the boundary is stationary in the original variables, even though it moves in $\tilde{P}$ parameterization.

The pricing scheme for $\alpha > 0$ will now be presented. Before the moving boundary crosses $x = 0$, the price function can be broken up into three regions and can be written as a series,

$$\tilde{P}^{(n)}(x,K,H) = \begin{cases} \mathrm{e}^{-d_+ x} \sum_{m=0}^{n-1} a_m^{(n,0)} x^m & , x > 0 \\ \mathrm{e}^{-rn\nu} - \mathrm{e}^{x-n\alpha} + \sum_{m=0}^{n-1} \left\{ \mathrm{e}^{-d_+ x} a_m^{(n,1)} + \mathrm{e}^{-d_- x} b_m^{(n,1)} \right\} x^m & , \mathbf{x^{(n+1)}} \leq x \leq 0 \\ \mathrm{e}^{-rn\nu} \rho & , x < \mathbf{x^{(n+1)}} \end{cases} \tag{34}$$

where the shifting barrier boundaries, $\mathbf{x^{(i)}}$, have been introduced,

$$\mathbf{x^{(i)}} \equiv (i-1)\alpha - h \tag{35}$$

The $a_m^{(n,i)}$ coefficients satisfy the recurrence relations given in (26) while the $b_m^{(n,i)}$ coefficients satisfy the recurrence relations given in (27). Once again, the coefficients with $m = 0$ cannot be obtained from the recurrence relations alone, and instead must be found by imposing the continuity of the scaled price and its derivative at $x = 0$, and continuity at the boundary $x = \mathbf{x^{(n+1)}}$,

$$\begin{pmatrix} 1 & -1 & -1 \\ -d_+ & d_+ & d_- \\ 0 & \mathrm{e}^{-\mathbf{x^{(n+1)}} d_+} & \mathrm{e}^{-\mathbf{x^{(n+1)}} d_-} \end{pmatrix} \begin{pmatrix} a_0^{(n,0)} \\ a_0^{(n,1)} \\ b_0^{(n,1)} \end{pmatrix} = \begin{pmatrix} c_0^{(n)} \\ c_0'^{(n)} \\ c_1^{(n)} \end{pmatrix} \tag{36}$$

where,

$$c_i^{(n)} = \begin{cases} \mathrm{e}^{-rn\nu} - \mathrm{e}^{-n\alpha}, & i = 0 \\ \mathrm{e}^{-h} + \mathrm{e}^{-rn\nu}(R-1) & \\ \quad - \sum_{m=1}^{n-1} \left\{ \mathrm{e}^{-d_+ \mathbf{x^{(n+1)}}} a_m^{(n,1)} + \mathrm{e}^{-d_- \mathbf{x^{(n+1)}}} b_m^{(n,1)} \right\} \left(\mathbf{x^{(n+1)}} \right)^m, & i = 1 \end{cases} \tag{37}$$

$$c_0'^{(n)} = -\mathrm{e}^{-n\alpha} - a_1^{(n,0)} + a_1^{(n,1)} + b_1^{(n,1)} \tag{38}$$

Equation (36) is the barrier analog of equation (28) for Europeans.

The moving boundary crosses the strike when $n \geq \frac{h}{\alpha}$, the solution then consists of only two regions,

$$\tilde{P}^{(n)}(x,K,H) = \begin{cases} \mathrm{e}^{-d_+ x} \sum_{m=0}^{n-1} a_m^{(n,0)} x^m & , x > \mathbf{x^{(n+1)}} \\ \\ \mathrm{e}^{-rn\nu} \rho & , x \leq \mathbf{x^{(n+1)}} \end{cases} \tag{39}$$

Of course, the $a_m^{(n,0)}$ coefficients still satisfy the recurrence relations given in (26), while the $m = 0$ coefficient is obtained by enforcing continuity at the boundary $x = \mathbf{x^{(n+1)}}$,

$$a_0^{(n,0)} = e^{d_+ \mathbf{x^{(n+1)}} - rn\nu} \rho - \sum_{m=1}^{n-1} \mathrm{e}^{-d_+ \mathbf{x^{(n+1)}}} a_m^{(n,0)} \left(\mathbf{x^{(n+1)}} \right)^m \tag{40}$$

This concludes the discussion of the $\alpha > 0$ scenario. If however, $\alpha < 0$, the number of regions on the n^{th}-line is equal to $n + 2$ as can be seen from figure 5. In this case the solution is given by,

$$\tilde{P}^{(n)}(x,K,H) = \begin{cases} \mathrm{e}^{-d_+ x} \sum_{m=0}^{n-1} a_m^{(n,0)} x^m, & x > 0 \\ \mathrm{e}^{-rn\nu} - \mathrm{e}^{x-n\alpha} + \sum_{m=0}^{n-1} \left\{ \mathrm{e}^{-d_+ x} a_m^{(n,1)} + \mathrm{e}^{-d_- x} b_m^{(n,1)} \right\} x^m, & \mathbf{x^{(1)}} < x \leq 0 \\ \mathrm{e}^{-rn\nu} \rho + \sum_{m=0}^{n-1} \left\{ \mathrm{e}^{-d_+ x} a_m^{(n,2)} + \mathrm{e}^{-d_- x} b_m^{(n,2)} \right\} x^m, & \mathbf{x^{(2)}} < x \leq \mathbf{x^{(1)}} \\ \vdots & \vdots \\ \mathrm{e}^{-rn\nu} \rho + \sum_{m=0}^{n-1} \left\{ \mathrm{e}^{-d_+ x} a_m^{(n,n+1)} + \mathrm{e}^{-d_- x} b_m^{(n,n+1)} \right\} x^m, & \mathbf{x^{(n+1)}} < x \leq \mathbf{x^{(n)}} \\ \mathrm{e}^{-rn\nu} \rho, & x \leq \mathbf{x^{(n+1)}} \end{cases} \tag{41}$$

As usual, the $a_m^{(n,i)}$ coefficients satisfy the recurrence relation (26) while the $b_m^{(n,i)}$ coefficients satisfy the recurrence relation (27). Enforcing continuity in the pricing function $\tilde{P}$ and its derivative at each region switch except for the last, where only continuity is forced, allows the $m = 0$ coefficients to be calculated. The linear system of equations can be compactly represented as follows:

$$
\begin{pmatrix}
1 & -1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-d_+ & d_+ & d_- & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & y_+^{(1)} & y_-^{(1)} & -y_+^{(1)} & -y_-^{(1)} & 0 & 0 & \cdots & 0 & 0 \\
0 & -d_+y_+^{(1)} & -d_-y_-^{(1)} & d_+y_+^{(1)} & d_-y_-^{(1)} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & y_+^{(2)} & y_-^{(2)} & -y_+^{(2)} & -y_-^{(2)} & & 0 & 0 \\
0 & 0 & 0 & -d_+y_+^{(2)} & -d_-y_-^{(2)} & d_+y_+^{(2)} & d_-y_-^{(2)} & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & & & y_+^{(n-1)} & y_-^{(n-1)} & -y_+^{(n-1)} & -y_-^{(n-1)} \\
0 & 0 & 0 & \cdots & & & -d_+y_+^{(n-1)} & -d_-y_-^{(n-1)} & d_+y_+^{(n-1)} & d_-y_-^{(n-1)} \\
0 & 0 & 0 & \cdots & & & 0 & 0 & y_+^{(n)} & y_-^{(n)}
\end{pmatrix}
\begin{pmatrix}
a_0^{(n,0)} \\ a_0^{(n,1)} \\ b_0^{(n,1)} \\ \vdots \\ a_0^{(n,n+1)} \\ b_0^{(n,n+1)}
\end{pmatrix}
=
\begin{pmatrix}
c_0^{(n)} \\ c_0'^{(n)} \\ c_1^{(n)} \\ c_1'^{(n)} \\ \vdots \\ c_n^{(n)} \\ c_n'^{(n)} \\ c_{n+1}^{(n)}
\end{pmatrix}
\tag{42}
$$

where,

$$
y_\pm^{(i)} = \mathrm{e}^{-\mathbf{x}^{(\mathbf{i})} d_\pm} \tag{43}
$$

and the right hand side for the smooth pasting conditions of the price function are given by,

$$
c_i^{(n)} = \begin{cases}
\mathrm{e}^{-rn\nu} - \mathrm{e}^{-n\alpha}, & i = 0 \\
(\rho-1)\mathrm{e}^{-rn\nu} + \mathrm{e}^{-(h+n\alpha)} + \displaystyle\sum_{m=1}^{n-1} \Big\{ y_+^{(1)} \left(a_m^{(n,2)} - a_m^{(n,1)} \right) & \\
\qquad\qquad + y_-^{(1)} \left(b_m^{(n,2)} - b_m^{(n,1)} \right) \Big\} (-h)^m \,, & i = 1 \\
\displaystyle\sum_{m=1}^{n-1} \left\{ y_+^{(i)} \left(a_m^{(n,i+1)} - a_m^{(n,i)} \right) + y_-^{(i)} \left(b_m^{(n,i+1)} - b_m^{(n,i)} \right) \right\} \left(\mathbf{x}^{(\mathbf{i})} \right)^m, & 2 \le i \le n \\
-\displaystyle\sum_{m=1}^{n-1} \left\{ y_+^{(n+1)} a_m^{(n,n+1)} + y_-^{(n+1)} b_m^{(n,n+1)} \right\} \left(\mathbf{x}^{(\mathbf{n+1})} \right)^m, & i = n+1
\end{cases} \tag{44}
$$

while the right hand side for the smooth pasting conditions of the delta are as follows,

$$
c_i'^{(n)} = \begin{cases} -\mathrm{e}^{-n\alpha} - a_1^{(n,0)} + a_1^{(n,1)} + b_1^{(n,1)}, & i = 0 \\ \mathrm{e}^{-(h+n\alpha)} + \sum_{m=1}^{n-1} \left\{ y_+^{(1)} \left(m + d_+ h\right) \left(a_m^{(n,2)} - a_m^{(n,1)}\right) \right. & \\ \qquad \left. + y_-^{(1)} \left(m + d_- h\right) \left(b_m^{(n,2)} - b_m^{(n,1)}\right) \right\} (-h)^{m-1}, & i = 1 \\ \sum_{m=1}^{n-1} \left\{ y_+^{(i)} \left(m - d_+ \mathbf{x}^{(\mathbf{i})}\right) \left(a_m^{(n,i+1)} - a_m^{(n,i)}\right) \right. & \\ \qquad \left. + y_-^{(i)} \left(m - d_- \mathbf{x}^{(\mathbf{i})}\right) \left(b_m^{(n,i+1)} - b_m^{(n,i)}\right) \right\} \left(\mathbf{x}^{(\mathbf{i})}\right)^{m-1}, & 2 \leq i \leq n \end{cases} \tag{45}
$$

The evaluation algorithm for down-and-out puts admits an easy generalization to the case of barriers that are piecewise constant between the lines. In that case, the boundary value h for each time-step n is replaced with h_n.

The case when the barrier condition is discretely monitored on the lines can also be accommodated. In this situation, the price on each line is first evaluated assuming the absence of the barrier on that line. This means that the last region in the pricing formula (34), (39), and (41), are replaced by a series similar to that in the pricing formula for European options (23). However, before using this solution to evaluate the price on the next line, the pricing function needs to be truncated at the barrier value on the current line. As a result, the pricing function may not be continuous at the boundary of this last region, as should be expected for a discretely monitored barrier.

3.3 American Options

To price American options the same methods that apply in the barrier case can be utilized. As mentioned in the introduction, the *model of lines* applied to the American option pricing problem does not yield exact prices; rather, the prices obtained in this section are those of a piecewise constant barrier option, where the holder of the option is allowed to adjust the level of barrier over every time-interval. With this in mind, the main difference between the barrier option discussed in the previous section and the “American” option priced here, is that the moving boundary must now be located using the optionality clause, in addition to assuring that the smooth pasting requirements are satisfied. An additional smooth pasting requirement is obtained

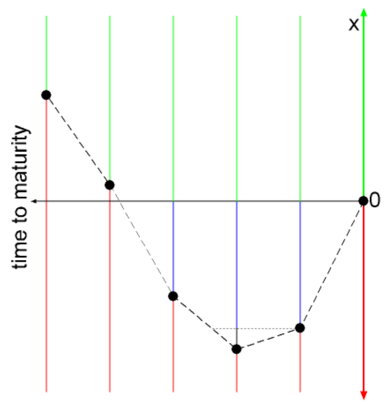

Figure 6: *The boundary for an American option with positive α. The number of solution regions at first increases, but it soon begins to decrease, and eventually reduces to two regions*

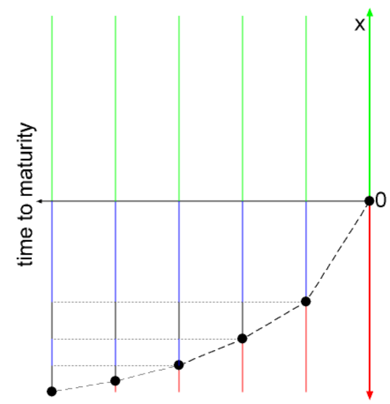

Figure 7: *The boundary for an American option with negative α. The number of regions in this scenario is constantly increasing*

by maximizing the options value; the standard arguments imply that maximizing the option's price is equivalent to forcing the hedge ratio at the exercise boundary to be equal to negative one. As in the case of European and barrier options, the boundary conditions are the same as those for the Black-Scholes partial differential equation.

Intuitively, the optimal exercise boundary is expected to decrease as time to maturity lengthens. However, since the price of the option is obtained from the scaled prices by shifting its argument by α (see equation (18)), two situations corresponding to the sign of α can occur. If $\alpha > 0$ the relevant boundary for the scaled prices $\tilde{P}(x)$ decreases initially and eventually starts to increase and ultimately crosses the strike level. This situation is illustrated in figure 6. On the other hand, when $\alpha \leq 0$, the boundary will constantly decrease, forcing the number of regions to increase as time to maturity increases as depicted in figure 7. In either case the price on the n^{th}-line can be written in a general form, assuming that the number of regions is $\mathbf{b_n} + 2$, as follows,

$$
\tilde{P}^{(n)}(x) = \begin{cases}
\mathrm{e}^{-d_+ x} \displaystyle\sum_{m=0}^{n-1} a_m^{(n,0)} x^m, & x > 0 \\
\mathrm{e}^{-rn\nu} - \mathrm{e}^{x-n\alpha} + \displaystyle\sum_{m=0}^{n-1} \left\{ \mathrm{e}^{-d_+ x} a_m^{(n,1)} + \mathrm{e}^{-d_- x} b_m^{(n,1)} \right\} x^m, & \mathbf{x^{(n,1)}} < x < 0 \\
\mathrm{e}^{-r(n-1)\nu} - \mathrm{e}^{x-n\alpha} + \displaystyle\sum_{m=0}^{n-1} \left\{ \mathrm{e}^{-d_+ x} a_m^{(n,2)} + \mathrm{e}^{-d_- x} b_m^{(n,2)} \right\} x^m, & \mathbf{x^{(n,2)}} < x < \mathbf{x^{(n,1)}} \\
\vdots & \vdots \\
\mathrm{e}^{-r(n-\mathbf{b_n}+1)\nu} - \mathrm{e}^{x-n\alpha} + \displaystyle\sum_{m=0}^{n-1} \left\{ \mathrm{e}^{-d_+ x} a_m^{(n,b_n)} + \mathrm{e}^{-d_- x} b_m^{(n,b_n)} \right\} x^m, & \mathbf{x^{(n,b_n)}} < x < \mathbf{x^{(n,b_n-1)}} \\
1 - \mathrm{e}^{x-n\alpha}, & x < \mathbf{x^{(n,b_n)}}
\end{cases} \tag{46}
$$

With this ansatz, the position of the optimal exercise boundary on the n^{th}-line is given by $S_{ex}^{(n)} = S_0 \mathrm{e}^{\mathbf{x^{(n,b_n)}} - n\alpha}$. The coefficients $a_m^{(n,i)}$, satisfy the recurrence relation (26), while the $b_m^{(n,i)}$ coefficients satisfy the recurrence relation (27). Of course, the recurrence relations do not determine the $m = 0$ coefficients, rather they are obtained by enforcing continuity in the price and the delta at the end of every interval. Also the optimal exercise boundary must be solved for.

In the case of negative α, the number of regions is increasing, and as is clear from figure 7, $\mathbf{x^{(n,i)}} = \mathbf{x^{(n-1,i)}}$ for $i = 1, \ldots, \mathbf{b_{n-1}}$ and the number of regions between the strike level and the exercise boundary is $\mathbf{b_n} = n$. Consequently, all the regions for the new line (with the exception of the new optimal exercise point) are known from the previous line. The system of $2(n+1)$ equations which determines the new exercise point $\mathbf{x^{(n,b_n)}}$ in addition to the $m = 0$ coefficients, is very similar to the barrier case, only one more equation, which enforces continuity in the delta, must be added, and the system is now non-linear due to the new exercise point,

$$
\begin{pmatrix}
1 & -1 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
-d_+ & d_+ & d_- & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & y_+^{(1)} & y_-^{(1)} & -y_+^{(1)} & -y_-^{(1)} & 0 & 0 & \dots & 0 & 0 \\
0 & -d_+ y_+^{(1)} & -d_- y_-^{(1)} & d_+ y_+^{(1)} & d_- y_-^{(1)} & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & y_+^{(2)} & y_-^{(2)} & -y_+^{(2)} & -y_-^{(2)} & & 0 & 0 \\
0 & 0 & 0 & -d_+ y_+^{(2)} & -d_- y_-^{(2)} & d_+ y_+^{(2)} & d_- y_-^{(2)} & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & & & y_+^{(n-1)} & y_-^{(n-1)} & -y_+^{(n-1)} & -y_-^{(n-1)} \\
0 & 0 & 0 & \dots & & & -d_+ y_+^{(n-1)} & -d_- y_-^{(n-1)} & d_+ y_+^{(n-1)} & d_- y_-^{(n-1)} \\
0 & 0 & 0 & \dots & & & 0 & 0 & y_+^{(n)} & y_-^{(n)} \\
0 & 0 & 0 & \dots & & & 0 & 0 & -d_+ y_+^{(n)} & -d_- y_-^{(n)}
\end{pmatrix}
\begin{pmatrix}
a_0^{(n,0)} \\ a_0^{(n,1)} \\ b_0^{(n,1)} \\ \vdots \\ a_0^{(n,n)} \\ b_0^{(n,n)}
\end{pmatrix}
=
\begin{pmatrix}
c_0^{(n)} \\ {c'_0}^{(n)} \\ c_1^{(n)} \\ {c'_1}^{(n)} \\ \vdots \\ c_n^{(n)} \\ {c'_n}^{(n)}
\end{pmatrix}
\tag{47}
$$

where,

$$
y_\pm^{(i)} = \mathrm{e}^{-\mathbf{x}^{(\mathbf{n},\mathbf{i})} d_\pm} \tag{48}
$$

and the right hand side for the smooth pasting conditions of the price function are given by,

$$
c_i^{(n)} = \begin{cases}
\mathrm{e}^{-rn\nu} - \mathrm{e}^{-n\alpha}, & i = 0 \\
\mathrm{e}^{-r(n-i)\nu} - \mathrm{e}^{-r(n-i+1)\nu} + \sum_{m=1}^{n-1} \Big\{ y_+^{(i)} \left(a_m^{(n,i+1)} - a_m^{(n,i)} \right) & \\
\qquad\qquad + y_-^{(i)} \left(b_m^{(n,i+1)} - b_m^{(n,i)} \right) \Big\} \left(\mathbf{x}^{(\mathbf{i})} \right)^m, & 1 \le i \le n
\end{cases} \tag{49}
$$

while the right hand side for the smooth pasting conditions of the delta are as follows,

$$c_i'^{(n)} = \begin{cases} -\mathrm{e}^{-n\alpha} - a_1^{(n,0)} + a_1^{(n,1)} + b_1^{(n,1)}, & i = 0 \\ \displaystyle\sum_{m=1}^{n} \left\{ y_+^{(i)} \left(m - d_+ \mathbf{x}^{(\mathbf{i})} \right) \left(a_m^{(n,i+1)} - a_m^{(n,i)} \right) \right. & \\ \qquad \left. + y_-^{(i)} \left(m - d_- \mathbf{x}^{(\mathbf{i})} \right) \left(b_m^{(n,i+1)} - b_m^{(n,i)} \right) \right\} \left(\mathbf{x}^{(\mathbf{i})} \right)^{m-1}, & 1 \leq i \leq n \end{cases} \tag{50}$$

also, $a_m^{(n,n+1)} = b_m^{(n,n+1)} \equiv 0$. This is the complete solution, and although the system is non-linear, each guess of $\mathbf{x}^{(\mathbf{n},\mathbf{b_n})}$ renders it linear.

In the case of positive α, the boundary should first be assumed to lie below the old boundary ($\mathbf{x}^{(\mathbf{n},\mathbf{b_n})} < \mathbf{x}^{(\mathbf{n-1},\mathbf{b_{n-1}})}$). If no solution of system (47) exists, then the assumption is false. The bottom most region must then be deleted from the solution assumption and the system must be solved once again. If no solution still exists, delete yet another region and so on until a solution is found. Once the number of regions reduces to two, there will always be an optimal solution.

4 Extrapolation Techniques

Although the time-step dictates the ν parameter in the VG model, it is possible to use the *model of lines* to obtain good approximations to the variance-gamma prices when ν is different from Δt. Just as Carr [6] demonstrated that Richardson extrapolation to $\nu = 0$ reproduces the Black-Scholes value in the usual method of lines, we propose to use an extrapolation scheme to obtain the prices of options for $\nu \neq \Delta t$ and in particular for $\nu > \Delta t$. Figure 8 shows the exact implied Black-Scholes volatilities $\sigma_{BS}(\nu)$ for one-month European options with various strike levels as a function of the parameter ν. Quadratic polynomials in $\ln \nu$ were used to fit the first three points at $\nu = \Delta t = 1, 2$ and 4 weeks, and extrapolated to the fourth point at $\nu = 8$ weeks, i.e.

$$\sigma_{BS}(\nu) = A_0 + A_1 \log \nu + A_2 (\log \nu)^2 \tag{51}$$

The fitted curves in figure 8 visually demonstrate the quality of the approximation. The absolute error measured in terms of the implied Black-Scholes volatility for at-the-money instruments was found to be negligible. The largest errors appeared

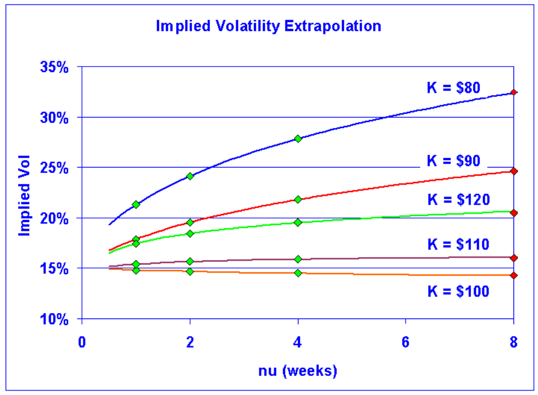

Figure 8: *Extrapolation of the implied volatility of one month European options for a VG model with $\nu = 8$ weeks using a fit to volatilities obtained with $\nu = \Delta t = 1, 2$ and 4 weeks. The model parameters were $\sigma = 15\%$, $\theta = -20\%$ and $r = 5\%$ and the spot was taken to be \$100*

in the out-of-the money option struck at 80% of the spot, for which the absolute error in implied volatility was 0.09%. Our conclusion is that extrapolation allows for the pricing of variance-gamma models with realistic parameter choices using the *model of lines*, albeit the prices thus obtained are approximate.

Just as in the case of European options, it is possible to use extrapolation to obtain the prices when $\nu \neq \Delta t$ for exotic options. In figure 9 the boundary of a Bermudan option which can be exercised every 8 weeks is plotted as a function of $\nu = \Delta t = 1, 2, 4$ and 8 weeks. The black dots in figure 9 form the predicted 8 week boundary obtained by extrapolation from the first three points. The extrapolation is based on a fit to a quadratic polynomial of $\ln \nu$ to the first three boundaries. The errors obtained by this extrapolation method are minimal with a maximum absolute error of \$0.18 for the longest maturity option. The at-the-money prices fitted to a linear function of ν are displayed in figure 10. The errors are once again negligible with the longest maturity option being underpriced by \$0.02.

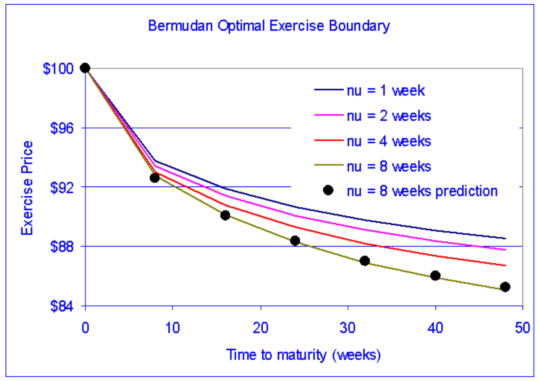

Figure 9: *The boundary of a Bermudan option which can be exercised every 8 weeks plotted for several values of* $\nu = \Delta t$*. The black dots show the boundary with* $\nu = 8$ *weeks obtained by extrapolation using the first three boundaries. The model parameters were* $\sigma = 15\%$, $\theta = -20\%$ *and* $r = 5\%$ *and the spot was taken to be $100*

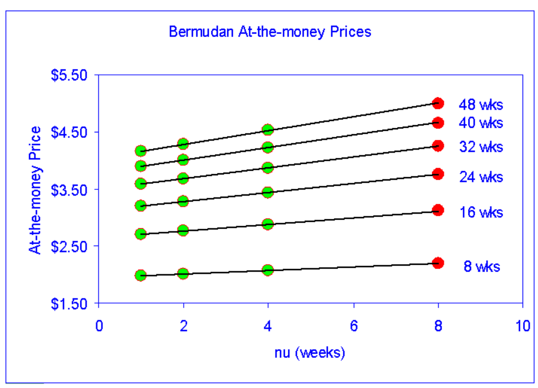

Figure 10: *The prices of the at-the-money options whose boundaries are shown in figure 9. The lines indicate a fit to the first three prices extrapolated to the fourth*

5 Conclusions

We proposed a new pricing model, coined the *model of lines*, which is appropriate for both path-dependent and path-independent options when the underlying asset follows a variance gamma process. The *model of lines* for jump processes is similar to the method of lines for American options in the Black-Scholes model. However, contrary to the numerical approximation scheme, the *model of lines* produced *exact* prices for a large class of options, the would be "errors" were reinterpreted as the effect of the jump component in the process for the underlying. The new methodology was applied to several pricing problems for European, American and barrier options, and the method was found to be both numerically efficient and simple to implement.

6 Acknowledgements

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