On Valuing Equity-Linked Insurance and Reinsurance Contracts*

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Insurance companies are increasingly facing losses that have heavy exposure to capital market risks through the issuance of equity-linked insurance policies. In this paper, we determine the continuous premium rate that an insurer charges via the principle of equivalent utility. Using exponential utility, we obtain the resulting premium rate in terms of a risk-neutral expectation. We also consider the related problem of pricing double-trigger reinsurance contracts, paying a function of the risky asset and losses, once the insurer has fixed her premium rate. We solve the Hamilton-Jacobi-Bellman equation arising in the indifference pricing problem and show that the price satisfies a PDE with a non-linear shift term. Although a closed form solution is not, generally, attainable, we obtain analytical results in some special cases. Finally, we recast the pricing PDE as a linear stochastic control problem and provide an explicit finite-difference scheme for solving the PDE numerically.

1. Introduction

With the $S\&P$ 500 index yielding returns of 10% over the last year and 16% over the last two years, it is no wonder that individuals seeking insurance are more often opting for equity-linked insurance contracts rather than fixed payment contracts. Equity-linked insurance contracts are highly popular options for policyholders because they also provide downside protection. From the insurer's perspective, such contracts induce claim sizes that are correlated to the fluctuations in the value of the $S\&P$ 500 index and, as such, posses significant market risk in addition to the traditional mortality risk. Determining the premium rate for this class of contracts is a daunting task which, due to the non-hedgable nature of the contracts, requires a delicate balancing of the insurer's risk preference, mortality exposure, and market exposure. In this work, we adopt the principle of equivalent utility, also known as utility-based pricing to value such contracts (see e.g. Bowers, et.al. (1997)). This pricing principle prescribes a premium rate at which the insurer is indifferent between (i) taking on the risk and receiving no premium or (ii) taking on the risk while receiving a premium. We review the methodology in more detail at the end of §2.

Equity-linked life insurance policies have been considered in many previous works. Young (2003), for example, studied equity-linked life insurance policies with a fixed premium and with a death benefit that was linked to an index. She demonstrates that the insurance premium satisfies a non-linear, Black-Scholes-like PDE, where the nonlinearity arises due to the presence of mortality risk. Young and Zariphopoulou (2002, 2003) also use utility-based methods to price insurance products with uncorrelated insurance and financial risks. Insurance risks often result in economies that are incomplete, and in such incomplete markets, equivalent utility pricing methods are both useful and powerful. Even when the risky asset itself has non-hedgable jump risks, Jaimungal and Young (2005) studied, the indifference pricing methodology

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yields tractable and intuitively appealing results. Our work here extends these earlier studies in two main directions: firstly, by considering equity-linked losses that arrive at Poisson times; and secondly, by simultaneously considering the valuation of a reinsurance product within one consistent framework.

We assume that the equity-linked claims (losses) arrive at Poisson times and that the insurer may both invest continuously and update her holdings in the equity on which the claims are written. As with all utility-based approaches, this requires a specification of the real world (as opposed to risk-neutral) evolution of equity returns and claim arrivals. We present our specific modeling assumptions in §2. In §3, we then determine the premium rate for this portfolio of insurance claims through the principle of equivalent utility. We focus exclusively on exponential utility for several reasons: firstly, we find that the optimal investment strategies dictated by exponential utility is independent of the insurer's wealth. Secondly, the difference between the holdings, with and without the insurance risk, in the equity, reduces to the hedge in Black and Scholes (1973). Thirdly, in the limit at which the investor becomes risk-neutral, the premium reduces to the risk-neutral expected losses over the insurer's investment time horizon. From this, we derive the two Hamilton-Jacobi-Bellman (HJB) equations, corresponding to the premium problem, and solve them explicitly for any level of risk-aversion and any equity-linked loss function. We find that the resulting premium q is proportional to risk-neutral expectation of an exponentially weighted average of the equity-linked loss function. The premium for an insurer who is almost risk-neutral is also investigated via an asymptotic expansion of the exact result.

Any insurer who takes on equity-linked insurance risks is exposed to potentially large losses in the event of good market conditions and/or poor underwriting; consequently, in §4, we consider the related problem of pricing double-trigger reinsurance contracts once the insurer has fixed her premium rate. At maturity, the reinsurance contract pays a function of the total observed losses and the equity value to the insurer. The insurer pays an upfront single benefit premium for this contract. We prove that the price satisfies a Black-Scholes-like PDE with a non-linear shift term due to the presence of the non-hedgable mortality risk. If the reinsurance payoff does not depend on the loss level, we show that the indifference price reduces to the Black-Scholes price of the corresponding equity option. We also investigate the price that a near risk-neutral insurer would be willing to pay, and find that the price can be written in terms of an iterated risk-neutral expectation. In §4.2, we provide a probabilistic interpretation of the indifference price in terms of a dual optimization problem. Withinn this framework, the indifference price is the minimum of the risk-neutral expected value of the reinsurance contract with a penalty term, where the minimum is computed over the activity rate of a doubly stochastic Poisson process driving the claim arrivals. Subsequently, §4.3 provides numerical examples for the reinsurance contract price in two special cases: (i) a stop-loss payoff and (ii) a double-trigger stop-loss payoff.

2. The Model

To model the problem for insurers exposed to equity-linked losses, we assume that there is a risky asset whose price process follows a Geometric Brownian motion, and that losses follow a compound Poisson process with claim sizes depending on the price of the risky asset at the loss arrival time. More specifically, let $\{S(t)\}_{0\leq t\leq T}$ denote the price process for a risky asset; let $\{L(t)\}_{0\leq t\leq T}$ denote the loss process for the insurer; let $\mathcal{F}^S \equiv \{\mathcal{F}^S\}_{0\leq t\leq T}$ denote the natural filtration generated by $S(t)$; let $\mathcal{F}^L \equiv \{\mathcal{F}^L\}_{0\leq t\leq T}$ denote the natural filtration generated by $L(t)$; let $\mathcal{F} \equiv \mathcal{F}^S \vee \mathcal{F}^L$ denote the product filtration generated by the pair $\{S(t), L(t)\}$; and let $(\Omega\ ,\ \mathbb{P},\ \mathcal{F})$ represent the corresponding filtered probability space with statistical probability measure $\mathbb{P}$.

We assume that the insurer is able to invest continuously in the risky asset $S(t)$ and a risk-free money market account with constant yield of $r \geq 0$. Furthermore, the risky asset's price process satisfies the

SDE:

$$dS(t) = S(t)\{\mu\, dt + \sigma\, dX(t)\}\ , \tag{1}$$

where $\{X(t)\}_{0\le t\le T}$ is a standard $\mathbb{P}$-Brownian process, and $\mu > r$. Equivalently,

$$S(t) = S(0)e^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma\, X(t)}\ . \tag{2}$$

The loss process is assumed to following a compound Poisson process with deterministic hazard rate $\lambda(t)$, and loss sizes of size $g(S(t), t)$, where t is the arrival time of a loss. Notice that the loss size depends on the price of the risky asset prevailing at the time the loss arrives. This is a defining feature of equity-linked insurance products and introduces a new dimension to the optimal stochastic control problem associated with pricing the premium stream. The loss process may be written in terms of an underlying Poisson counting process: $\{N(t)\}_{0\le t\le T}$ as follows

$$L(t) = \sum_{n=1}^{N(t)} g(S(t_i), t_i)\,, \tag{3}$$

where t_i are the arrival times of the Poisson process. We implicitly assume that $g(S, t) \ge 0$ and is bounded for every finite pair $(S, t) \in [0,\infty) \times [0, T]$.

Since our assumptions on the dynamics of the risky asset and the loss process have been addressed, we turn attention to the dynamics of the wealth process for the insurer. There are two separate situations of interest: (i) the insurer does not take on the insurance risk, however, the insurer does invest in the risky asset and the riskless money-market account; and (ii) the insurer takes on the insurance risk in exchange for receiving a continuous premium of q and simultaneously invests in the risky asset and the riskless money-market account. Let $\{W(t)\}_{0\le t\le T}$ and $\{W^L(t)\}_{0\le t\le T}$ denote, respectively, the wealth process of the insurer who does not take on the insurance risk (as in case (i)) and the wealth process of the insurer who does take on the insurance risk (as in case (ii)). The process $\pi \equiv \{(\pi(t), \pi_0(t))\}_{0\le t\le T}$ denotes an $\mathcal{F}_t$-adapted self-financing investment strategy, where $\pi(t)$ and $\pi_0(t)$ represent the amount invested in the risky asset and the amount in the money-market account, respectively. The wealth process dynamics then satisfy the following two SDEs:

$$\begin{cases} dW(u) &= [r\, W(u) + (\mu - r)\, \pi(u)]\, du + \sigma\, \pi(u)\, dX(u)\,, \\ W(t) &= w\,, \end{cases} \tag{4}$$

$$\begin{cases} dW^L &= \left[r\, W^L(u_-) + (\mu - r)\, \pi(u_-) + q\right] du + \sigma\, \pi(u_-)\, dX(u) - dL(u)\,, \\ W^L(t) &= w\,, \end{cases} \tag{5}$$

where w represents the wealth of the insurer at the initial time t; and for each process f, $f(u_-)$ represents the value of the process prior to any jump at u.

To complete the model setup, we suppose that the insurer has preferences according to an exponential utility of wealth $u(w) = -\frac{1}{\hat\alpha}e^{-\hat\alpha\, w}$ for some $\hat\alpha > 0$. The parameter $\hat\alpha$ is the absolute risk-aversion $r_{\hat\alpha}(w) \equiv -u''(w)/u'(w) = \hat\alpha$ as defined by Pratt (1964). We further assume the insurer seeks to maximize her expected utility of terminal wealth at the investment time horizon T. This results in two separate stochastic optimal control problems. We denote the value function of the insurer who does not accept the insurance risk by $V(w, t)$, and denote the value function of the insurer who does accept the insurance risk by $U(w, S, t; q)$. Explicitly, the value functions are defined as follows:

$$V(w, t) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[u(W(T))\,|\, W(t) = w\right]\,, \qquad \text{and} \tag{6}$$

$$U(w, S, t; q) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[u(W^L(T))|W^L(t) = w, S(t) = S\right]\ . \tag{7}$$

Here, $\mathcal{A}$ is the set of admissible, square integrable, and self-financing, $\mathcal{F}_t$-adapted trading strategies for which $\int_t^T \pi^2(s)\,ds < +\infty$. This restriction is necessary for the existence of a strong solution to the wealth process SDEs (4) and (5) (see Fleming and Soner, 1993).

A priori, it is not obvious that V should depend solely on the wealth process and time; similarly, it is not obvious that U should be independent of the loss L. However, through the explicit solutions in the next section, we determine that this is indeed the case, which is a familiar result when using exponential utility. Although the value functions are found to depend on the insurer's wealth, the optimal investment strategy is, in fact, independent of the wealth. This too is a consequence of exponential utility.

Next, the indifference premium is defined as the premium q such that the two value functions are equal:

$$V(w,t) = U(w,S,t;q)\,. \tag{8}$$

Intuitively, this implies that the insurer is equally willing either to accept the risk and receive a premium, or to decline the risk and receive no premium.

Once the indifference premium is obtained, the problem of pricing a reinsurance contract is considered in §4. The contract is assumed to pay an arbitrary function, $h(L(T), S(T))$, of the total observed losses and the risky asset's price at the time horizon T. The associated value function of an insurer who receives this payment will be denoted $U^R(w, L, S, t; q)$ and can be explicitly expressed as

$$U^R(w,L,S,t;q) = \sup_{\pi\in\mathcal{A}} \mathbb{E}[u(W^L(T) + h(L(T),S(T)))|W^L(t) = w, S(t) = S, L(t) = L]\,. \tag{9}$$

Notice that the reinsurance contract is relevant only at the terminal time, and its role is simply to increase the insurer's wealth by the contract value. Although the effect is explicitly felt at maturity, it will feed back into the optimal investment strategy which the insurer follows, and consequently, it will feed back into the value function itself.

The indifference price $P(L, S, t)$ of the contract is the amount of wealth the insurer who receives the reinsurance payment is willing to surrender so that the value function with the reinsurance payment is equal to the value function without the reinsurance payment. That is, the indifference price satisfies the equation:

$$U^R(w - P(L,S,t), L, S, t; q) = U(w,L,S,t;q)\,. \tag{10}$$

A posteriori, the price function is found to be independent of wealth for exponential utility. Furthermore, the indifference price is independent of the premium that the insurer charges. Rather, it equates the utility of the insurer who receives some premium rate q and is exposed to the equity-linked losses of an insurer who, in addition, receives a reinsurance contract payment and pays upfront for that reinsurance.

3. The HJB Equation For The Indifference Premium

Now that the stochastic model for the insurer has been described, and the pricing principle has been specified, we can focus on the details of the pricing problem itself. In the next subsection, the value function without the insurance risk is reviewed. The results of this section are essentially those of Merton (1969). These results are then used in §3.2 to solve the HJB equation for the insurer exposed to the insurance risk. In §3.3, we determine the indifference premium for a general loss function and provide specific examples. In §3.4, we address the issue of hedging the risk associated with this premium choice.

3.1. The Value Function Without The Insurance Risk

The value function of the insurer who does not take on the insurance risk is defined in (6), and we now use the dynamic programming principle to determine the optimal investment strategy and the value function itself. Given a particular investment strategy π, we determine that V satisfies the following SDE:

$$dV(W,s) \quad = \quad \left[V_t + (r\,W + (\mu - r)\,\pi)\,V_w + \tfrac{1}{2}\sigma^2\pi^2\,V_{ww}\right]ds + \pi\sigma\,V_w\,dX\,. \tag{11}$$

The subscripts denote the usual partial derivatives of V, and the time dependence of the various processes are suppressed for brevity. Through the usual dynamic programming principle, V solves the HJB equation:

$$\begin{cases} V_t + r\,w\,V_w + \max_\pi\left[(\mu - r)\,\pi\,V_w + \tfrac{1}{2}\sigma^2\pi^2\,V_{ww}\right] = 0, \\ V(w,T) = u(w). \end{cases} \tag{12}$$

We may assume that the optimal investment is provided by the first order condition, and the Verification Theorem confirms that the result holds. To this end, the optimal investment strategy is

$$\pi^*(t) = -\frac{\mu - r}{\sigma^2}\,\frac{V_w}{V_{ww}}\,. \tag{13}$$

On substituting π^* into (12), we find that V satisfies the PDE

$$V_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\frac{V_w^2}{V_{ww}} + r\,w\,V_w = 0\,. \tag{14}$$

Assuming that

$$V(w,t) = -\frac{1}{\hat\alpha}e^{-\alpha(t)\,w + \beta(t)}\,, \tag{15}$$

with $\beta(T) = 0$ and $\alpha(T) = \hat\alpha$, we determine that the HJB equation reduces to

$$-(\alpha_t + r\alpha)\,w + \beta_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 = 0\,. \tag{16}$$

The above must hold for all w and t; therefore,

$$\alpha(t) \quad = \quad \hat\alpha\,e^{r(T-t)} \quad \text{and} \tag{17}$$

$$\beta(t) \quad = \quad -\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2(T - t)\,, \tag{18}$$

resulting in the standard Merton optimal investment of (Merton, 1969)

$$\pi^*(t) = \frac{\mu - r}{\hat\alpha\,\sigma^2}e^{-r(T-t)}\,. \tag{19}$$

The above solution satisfies the requirements of the Verification Theorem and, therefore π^* corresponds to the optimal investment strategy for (6); and V, given in (15), is the solution of the original optimal stochastic problem.

3.2. The Value Function With Insurance Risk

While assuming the insurance company takes on the insurance risk and receives a premium rate of q, we must solve for the optimal investment and value function U, given in (7). Through straightforward methods, we establish the following HJB equation for the value function U:

$$\begin{cases} 0 = U_t + (rW + q)U_w + \mu\,S\,U_S + \tfrac{1}{2}\sigma^2\,S^2\,U_{SS} + \lambda(t)\left(U(w - g(S,t),S,t) - U(w,S,t)\right) \\ \qquad + \max_\pi\left\{\tfrac{1}{2}\sigma^2 U_{ww}\,\pi^2 + \pi\left[(\mu - r)U_w + \sigma^2 S(t) U_{ws}\right]\right\}\,, \\ U(w,S,T;q) = u(w)\,. \end{cases} \tag{20}$$

The shift term appears due to the presence of the claim arrivals, and can be explained by observing that a claim arrives in $(t, t + dt]$ with probability $\lambda(t)\,dt$, causing the wealth to drop by $g(S(t),t)$. At first sight, the presence of this non-linear shift term appears to render the problem intractable. However, on closer inspection, we find that, since utility is exponential, the HJB equation can be solved explicitly for arbitrary claims function g.

Theorem. 3.1 *The solution to the HJB system (20) is*

$$U(w,S,t;q) \quad = \quad V(w,t)\exp\left\{-q\,\frac{\hat{\alpha}}{r}\left(e^{r(T-t)}-1\right)+\gamma(S,t)\right\}, \tag{21}$$

where

$$\gamma(S(t),t)=\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \lambda(u)\,\left(e^{\alpha(u)\,g(S(u),u)}-1\right)du\,\middle|\,\mathcal{F}_t\right], \tag{22}$$

and the process $S(t)$ satisfies the following SDE in terms of the $\mathbb{Q}$-Wiener process $\{\overline{X}(t)\}_{0\le t\le T}$,

$$dS(t)=S(t)\,r\,dt+S(t)\,\sigma\,d\overline{X}(t)\,. \tag{23}$$

Furthermore, the optimal investment strategy is independent of wealth and equals

$$\pi^*(S,t)=\frac{e^{-r(T-t)}}{\hat{\alpha}}\left\{\frac{\mu-r}{\sigma^2}+S\,\gamma_S\right\}. \tag{24}$$

Proof. By assuming $U_{ww}<0$, the first order condition supplies the optimal investment strategy as

$$\pi^*(S,t)=-\frac{(\mu-r)U_w+\sigma^2 S(t)U_{ws}}{\sigma^2 U_{ww}}\,. \tag{25}$$

Substitute this into (20), and make the substitution

$$U(w,S,t;q)=V(w,t;q)\exp\{\gamma(S,t)\}\,, \tag{26}$$

where $V(w,t;q)$ denotes the value function of the insurer who receives a premium rate of q but does not accept the insurance risk. If we denote the wealth process for such an insurer as $W^q(t)$, then $W^q(t)=W(t)+q\,t$. Notice that $V(w,t;0)=V(w,t)$, and that $V(w,t;q)$ satisfies the HJB equation:

$$\begin{cases} 0= & V_t+(r\,w+q)\,V_w+\max_\pi\left[(\mu-r)\,\pi\,V_w+\frac{1}{2}\sigma^2\pi^2\,V_{ww}\right], \\ V(w,T;q)= & u(w). \end{cases} \tag{27}$$

Straightforward calculations yield the solution:

$$V(w,t;q)=V(w,t)\exp\left\{-q\,\frac{\hat{\alpha}}{r}\left(e^{r(T-t)}-1\right)\right\}. \tag{28}$$

Making the substitution (26) into (20), we discover, after some tedious calculations, $V(w,t;q)$ factors out of the problem, and the function $\gamma(S,t)$ satisfies the inhomogeneous linear partial differential equation:

$$\begin{cases} 0 & =\lambda(t)\,(e^{\alpha(t)g(S,t)}-1)+rS\gamma_s+\frac{1}{2}\sigma^2S^2\gamma_{SS}+\gamma_t\,, \\ \gamma(S,T) & =0\,. \end{cases} \tag{29}$$

The Feynman-Kac theorem directly leads to solution (34) from which we observe that $U_{ww}<0$ so that the maximization term is indeed convex. For smooth loss functions $g(S(t),t)$, the Verification Theorem implies, indeed, that (34) is the value function for the problem and strategy (25) is optimal. Accordingly, substituting the ansätz (26) into π^* leads to (24). □

Notice that if $\forall u\in(t,T]$, $g(S(u),u)>0\,\mathbb{Q}-a.s.$ then $\gamma(S,t)$ is strictly positive for every S. In the next section, under these conditions the indifference premium is show to also be positive.

3.3. The Indifference Premium

Now given both value functions, V and U, it is possible to obtain an explicit representation of the indifference premium.

Corollary 3.2 *The insurer's indifference premium rate q is independent of wealth and is given by*

$$q(S(t),t) = \frac{r}{e^{r(T-t)}-1}\,\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \lambda(u)\,\frac{e^{\alpha(u)\,g(S(u),u)}-1}{\hat{\alpha}}\,du \,\middle|\, \mathcal{F}_t\right]. \tag{30}$$

Proof. The indifference premium rate q is defined as the rate q such that $U(w,S,t;q) = V(w,t)$. Expression (30) then immediately follows from Theorem 3.1. □

Since q is essentially proportional to γ, based on the remarks at the end of the previous section, and if the claim sizes are positive $\mathbb{Q}$-a.s. over the time horizon $(t,T]$, then the indifference premium is positive.

It is important to discuss the dependence of q on the risky asset's price and time. In analyzing the value function U, we assumed that q was constant; however, on glancing at (30) it can be inferred that q is not a constant, and therefore, our assumptions are false, discrediting the analysis. This initial reaction is premature. The situation is best explained by appealing to the familiar case of a forward contract. On signing of a forward contract, the delivery price is set such that the contract has zero value. This delivery price is a function of the spot price of the asset and bond prices at the time of signing. Although the contract value on signing is zero, the forward price, at any future date, will not equal to the delivery price, and the contract's value is no longer zero. In the present context, the insurer is looking forward to a future time horizon, and is deciding on a rate to charge so that she is indifferent to taking the risk. Our analysis shows that the amount $q(S(t),t)$, which depends on the prevailing price of the risky asset, should be charged. This rate is fixed until the end of the time horizon, and does indeed render the insurer indifferent to the insurance risk *at the current time*. However, as time evolves, the prevailing indifference premium at that future point in time may be higher or lower than the rate the insurer initially set. Consequently, if the insurer took on the insurance risk at time t in exchange for $q(S(t),t)$ until the horizon end, then at some future time she may develop a preference either towards releasing the insurance risk or for holding onto it. With the forward contract analogy, it is no surprise then that the premium rate depended on the risky asset's price.

The indifference premium (30) also has a few very appealing properties which warrant discussing. Regardless of the risk-aversion level of the insurer, the expectation appearing in the premium calculation is always computed in a risk-neutral measure $\mathbb{Q}$. Furthermore, the risk-neutral distribution of claim sizes has not been distorted from the real world distribution. Indeed, the Radon-Nikodym derivative process which performs the measure change is

$$\eta(t) \equiv \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t = \exp\left\{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t + \frac{\mu-r}{\sigma}\,X(t)\right\}. \tag{31}$$

This is the same measure change that Merton (1976) uses in his jump-diffusion model and corresponds to risk adjusting only the diffusion component. Although the risk-aversion level does not feed into the probability measure used for computing expectations, it does manifest itself in the distortion of the claim sizes through the exponential term. The exponential term is inherited, indeed, from the utility function. Furthermore, notice that the factor in front of the expectation can be represented as $1/\int_t^T e^{r(T-u)}du$. The denominator of this expression is simply the continuous premium rate of \$1 per annum which is accumulated to end of the time horizon T. Consequently, the factor in front of the expectation can be viewed as a normalization constant. Finally, although the premium is a non-linear functional of the claim sizes $g(S(t),t)$, it is linear in the arrival rate of the claims $\lambda(t)$. This observation suggests that the generalization to multiple claims distributions is straightforward. In the theorem below, we provide the results for multiple claims distributions. The proof follows along the same lines as those in the previous two sections and we omit it for brevity.

Theorem. 3.3 *Suppose that the insurer is exposed to losses from m different sources of risk. Explicitly, the loss process is modeled as follows:*

$$L(t) = \sum_{j=1}^{m} \sum_{n=1}^{N_j(t)} g_j(S(t_j^i), t_j^i)\,, \tag{32}$$

where $\{N_j(t) : j = 1, \ldots, m\}$ are independent Poisson processes with arrival rates $\{\lambda_j(t) : j = 1, \ldots, m\}$ and $g_j(S, t)$ denote the loss functions for the i-th source of risk. Then, the value function of the insurer who takes on the insurance risk and receives a premium of $q(w, S, t)$ is

$$U(w, S, t; q) \quad = \quad V(w, t) \exp\left\{ -q\, \frac{\hat{\alpha}}{r} \left(e^{r(T-t)} - 1 \right) + \gamma(S, t) \right\}\,, \tag{33}$$

where

$$\gamma(S(t), t) = \sum_{j=1}^{m} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \lambda_j(u) \left(e^{\alpha(u)\, g_j(S(u),u)} - 1 \right) du \middle| \mathcal{F}_t \right]\,, \tag{34}$$

and the process $S(t)$ satisfies the following stochastic differential equation in terms of the $\mathbb{Q}$-Wiener process $\{\overline{X}(t)\}_{0 \le t \le T}$:

$$dS(t) = S(t)\, r\, dt + S(t)\, \sigma\, d\overline{X}(t)\,. \tag{35}$$

Furthermore, the insurer's indifference premium is independent of wealth and is explicitly

$$q(w, S, t) = \frac{r}{e^{r(T-t)} - 1} \sum_{j=1}^{m} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \lambda_j(u)\, \frac{e^{\alpha(u)\, g_j(S(u),u)} - 1}{\hat{\alpha}}\, du \middle| \mathcal{F}_t \right]\,. \tag{36}$$

3.3.1. Constant Losses And Risk Neutral Insurers

In this subsection, we consider claims which have constant losses $g(S, t) = l$ and a constant arrival rate $\lambda(t) = \lambda$. In this case, we determine the premium rate is

$$q = \frac{\lambda}{\hat{\alpha} \left(e^{r(T-t)} - 1 \right)} \left(Ei(\hat{\alpha}\, l\, e^{r(T-t)}) - Ei(\hat{\alpha}\, l) - (T-t) r \right)\,, \tag{37}$$

where $Ei(x)$ denotes the so called "exponential integral", defined as the following Cauchy principle value integral:

$$Ei(x) \equiv ⨍_{-\infty}^{x} \frac{e^t}{t}\, dt\,. \tag{38}$$

If the insurer is near risk-neutral, then a Taylor expansion in $\hat{\alpha}\, l$ can be carried out, and we find the indifference premium rate to linear order is

$$q = \lambda\, l \left(1 + \frac{1}{4} \left(e^{r(T-t)} + 1 \right) \hat{\alpha}\, l + o(\hat{\alpha}\, l) \right)\,. \tag{39}$$

As such, a risk-neutral insurer who is exposed to fixed losses, will charge a rate equal to the expected loss per unit time $\lambda\, l$ - an intuitively appealing result. As expected, the sign of the first order correction is positive. For losses that grow at most linearly, i.e. there exists $b(t) > 0$ and $S^*(t) > 0$ such that for each t and $S > S^*(t)$, $g(S, t) \le b(t)\, S$, the rate has the following perturbative expansion in terms of the risk-aversion parameter $\hat{\alpha}$:

$$q \quad = \quad \frac{\lambda\, r}{e^{r(T-t)} - 1} \sum_{n=1}^{\infty} \frac{\hat{\alpha}^{n-1}}{n!} \int_t^T e^{n\, r(T-u)}\, \mathbb{E}^{\mathbb{Q}} \left[\, g^n(S(u), u) \middle| \mathcal{F}_t \right] du\,. \tag{40}$$

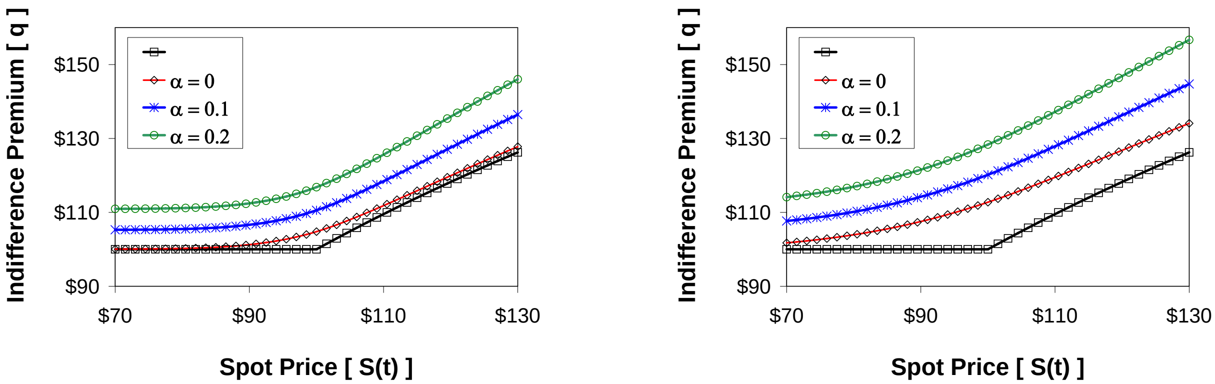

Figure 1. The dependence of the indifference premia on the underlying equity spot price for losses described in §3.3.2. The model parameters are $F = 1$, $K = 100$, $\beta = 1$, $r = 4\%$, $\sigma = 15\%$, and $\lambda = 100$. The terms in the left/right panels are one and five years respectively.

This series is shown to converge by appealing to the Lebesgue dominated convergence theorem and noting that $\mathbb{E}^{\mathbb{Q}}[S^n(u)|\mathcal{F}_t] = S(t)e^{n(r+\frac{1}{2}\sigma^2(n-1))(u-t)}$. In fact, the linearity condition can be weakened considerably; however, at this point we are concerned with aiding intuition and as a result, omit such details from the analysis. A risk-neutral insurer would then charge a premium rate of

$$q = \frac{r}{e^{r(T-t)} - 1} \int_t^T \lambda(u)\, e^{r(T-u)} \mathbb{E}^{\mathbb{Q}} \left[\, g(S(u), u) \middle| \, \mathcal{F}_t \right] du \,. \tag{41}$$

The above premium can be interpreted as the average risk-neutral expected loss per unit time, where the expected losses have been accumulated to maturity and then normalized (rather than discounted) back to time t.

In the next two subsections, we provide two explicit examples of the premium when the losses are functions of the logarithm of the stock index. While still maintaining the essential properties of linear claim sizes, we use the logarithm of the stock price because it allows for partially closed form solutions. To this end, we define $A(u)$ as the expectation appearing under the integral in the indifference premium (30), i.e.

$$A(u) \equiv \mathbb{E}^{\mathbb{Q}} \left[e^{\alpha(u)\, g(S(u),u)} \middle| \mathcal{F}_t \right] . \tag{42}$$

Then, the indifference premium q can be written in terms of $A(u)$ explicitly as

$$q = \frac{r}{\hat{\alpha}\left(e^{r(T-t)} - 1\right)} \int_t^T \lambda(u) \left\{ A(u) - 1 \right\} du \,. \tag{43}$$

3.3.2. Floor and Market Participation Claims

In our first explicit example, we consider insurance claims which pay a minimum of F and then grows proportionately to the logarithm of the excess spot price above the strike level K; that is,

$$g(S(t), t) = F + \beta \left(\log\left(S(t)\right) - \log(K) \right)^+ \,. \tag{44}$$

Straightforward, but tedious, calculations lead to the result:

$$A(u) = e^{F\alpha(u)} \left[\left(\frac{S(t)}{K} \right)^{\beta\alpha(u)} \Phi(-d_2(K))\, e^{\beta\alpha(u)\left(r - \frac{1}{2}\sigma^2(1-\beta\alpha(u))\right)(u-t)} + \Phi(d_1(K)) \right] , \tag{45}$$

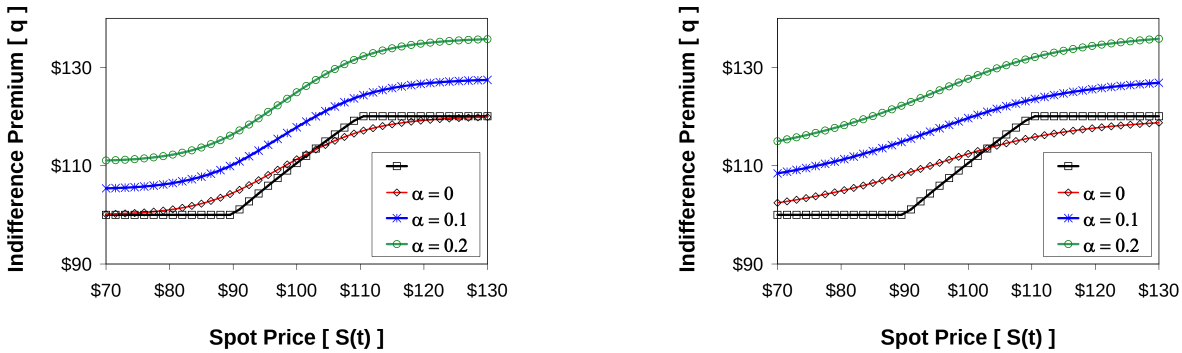

Figure 2. The dependence of the indifference premia on the underlying equity spot price for losses described in §3.3.3. The model parameters are $\theta = 1$, $\beta = 1$, $c_1 = 90$, $c_2 = 110$, $r = 4\%$, $\sigma = 15\%$, and $\lambda = 100$. The terms in the left/right panels are one and five years respectively.

where

$$d_1(K) = \frac{\log\left(\frac{K}{S(t)}\right) - (r - \frac{1}{2}\sigma^2)(u-t)}{\sigma\sqrt{u-t}}, \tag{46}$$

$$d_2(K) = d_1(K) - \beta\,\alpha(u)\,\sigma\,\sqrt{u-t}\,, \tag{47}$$

and $\Phi(\cdot)$ is the cumulative distribution function for the standard normal distribution.

In Figure 1, we illustrate how the premium depends on the underlying spot price for three choices of the risk-aversion parameter $\hat{\alpha}$, and for terms of one and five years respectively. The boxed line shows the pure loss function (44) scaled by the activity rate for comparison purposes. Naturally, as the risk-aversion parameter increases, the premia increases. Furthermore, since the loss is increasing as the spot grows, the premia increases as the maturity increases. This result is analogous to the pricing behavior of a call option in the Black-Scholes model.

3.3.3. Floor, Capped, and Market Participation Claims

In our second explicit example, we consider insurance claims which have a cap and a floor protection in addition to a participation in the risky asset's return. In this case, the claim sizes are

$$g(S(t),t) = \begin{cases} \theta\,, & S(t) < c_1\,, \\ \theta + \beta\left(\log\left(S(t)\right) - \log(c_1)\right)\,, & c_1 \le S(t) < c_2\,, \\ \theta + \beta\left(\log(c_2) - \log(c_1)\right)\,, & S(t) \ge c_2\,. \end{cases} \tag{48}$$

Once again, after some tedious calculations, we find that the integrand $A(u)$ reduces to

$$A(u) = e^{\alpha(u)\theta}\left\{\Phi(d_1(c_1)) + \left(\frac{S(t)}{c_1}\right)^{\beta\,\alpha(u)} e^{\beta\,\alpha(u)\left(r - \frac{1}{2}\sigma^2(1-\beta\,\alpha(u))\right)(u-t)}\left(\Phi(d_1(c_3)) - \Phi(d_2(c_1))\right) + \left(\frac{c_2}{c_1}\right)^{\beta\,\alpha(u)}\Phi(-d_1(c_2))\right\}. \tag{49}$$

In Figure 2, we illustrate how the premium depends on the underlying spot price for three choices of the risk-aversion parameter $\hat{\alpha}$, and for terms of one and five years respectively. The boxed line shows the pure loss function (48) scaled by the activity rate for comparison purposes. Once again, as the risk-aversion parameter increases, the premia increases. In this case, the loss is bounded from above and below; therefore, increasing maturity does not alter the premia as significantly as the uncapped case explored in the previous example. In fact, the premium actually decreases for larger spot prices when the term increases. This result is analogous with the pricing for a standard bull-spread option in the Black-Scholes model.

3.4. Hedging The Insurance Risk

Now that we have determined the indifference premium that the insurer charges, it is interesting to explore the hedging strategy that she would follow. In this incomplete market setting, it is impossible to exactly replicate the insurance claims; nonetheless, the insurer still holds different units of the risky asset when she is exposed to the insurance risk or is not exposed to the insurance risk. As a result, we can define an analog of the Black-Scholes Delta hedging parameter. To this end, we define the Delta as the excess units of the risky asset that the insurer holds when taking on the risk and receiving the premiums, and when there is an absence of insurance risk.

Corollary 3.4 *The Delta of the insurer's position is*

$$\Delta(S,t) \equiv \frac{1}{S}\left(\pi_U^* - \pi_V^*\right) \quad = \quad \frac{e^{-r(T-t)}}{\hat{\alpha}}\,\gamma_S(S,t)\,. \tag{50}$$

Proof. The optimal investment in the risky asset without the insurance risk appears in (13), and with the insurance risk appears in (24). □

Notice that this result is quite similar to the Black-Scholes Delta for an option. However, there is a subtle difference since the function $\gamma(S(t),t)$ appears in the result, rather than in the premium rate itself. Moreover, as $t \to T^-$, the Delta vanishes, this behavior contrast with the behavior of the Delta of an option. In the case of a European option, the Delta becomes equal to the derivative of the payoff function, and is zero only when the option has a constant payoff, namely, when the option is actually a bond. To understand why the Delta vanishes as maturity approaches in our case, suppose that the time horizon ends in $\Delta T \ll 1$ from now; then, the probability of a loss arriving is $\lambda\Delta T$ and therefore there is no need to hold additional shares of the risky-asset.

In Figure 3, we show how the Delta behaves as a function of the spot-level, risk-aversion parameter, and time to maturity for the examples in §3.3.2 and §3.3.3. The general shape of these curves is expected. In the first example, although the payoff grows logarithmically, it appears to grow linearly over the scale shown in diagram (see Figure 1), and therefore the Delta flattens out. While in the second example, the payoff is asymptotically flat outside of the participation region (see Figure 1), and therefore, the Delta decays in the tails.

4. The Indifference Price For Reinsurance

Now that we have determined the indifference premium that the insurers charges, we can address the dual problem of pricing a reinsurance contract which makes payments at the end of the time horizon. In section 2, we describe the value function associated with the insurer who takes on the insurance risk and receives the premium rate q and receives a reinsurance payment of $h(L(T),S(T))$. The value function of such an insurer was denoted U^R as defined in equation (9). The associated HJB equation for this value function is essentially the same as the one for U (see equation (20)); however, the boundary condition is now altered to account for the presence of the reinsurance, and we must also keep track of the loss process explicitly. Through the usual dynamic programming principle, we determine that U^R satisfies

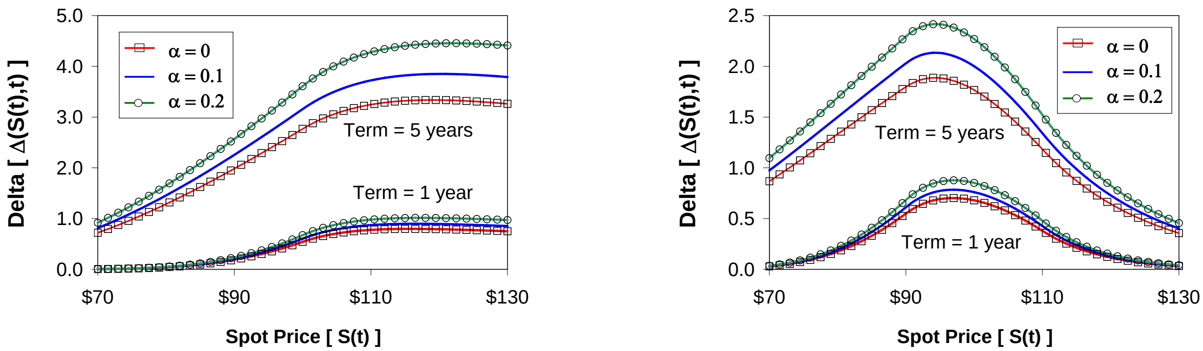

Figure 3. The dependence of the Delta on the underlying equity spot price for losses described in §3.3.2 and §3.3.3 (left/right panel respectively). The model parameters are the described in Figures 1 and 2.

the following HJB equation:

$$\begin{cases} 0 = & U^R_t + (rW+q)U^R_w + \mu\, S\, U^R_S + \frac{1}{2}\sigma^2\, S^2\, U^R_{SS} \\ & +\lambda(t)\,(U^R(w-g(S,t),L+g(S,t),S,t) - U^R(w,L,S,t)) \\ & +\max_\pi \left\{\frac{1}{2}\sigma^2 U^R_{ww}\,\pi^2 + \pi\left[(\mu-r)U^R_w + \sigma^2 S(t) U^R_{ws}\right]\right\}, \\ U^R(w,L,S,t;q) = & u(w+h(L,S)). \end{cases} \tag{51}$$

The shift term now contains two types of shifting: the first, due to the decrease in the wealth of the insurer; and the second, due to the increase in the loss process. However, both shifts come from the same risk source. Once again, exponential utility allows us to obtain a solution of the HJB equation in a semi-explicit form.

Theorem. 4.1 *The solution to the HJB system (51) can be written as*

$$U^R(w,L,S,t) = U(w,S,t)\phi(L,S,t)\,, \tag{52}$$

where ϕ satisfies the non-linear PDE

$$\begin{cases} 0 = & \phi_t + r\,S\phi_S + \frac{1}{2}\sigma^2 S^2\left(\phi_{SS} - \frac{\phi_S^2}{\phi}\right) + \lambda(t)\,e^{\alpha(t)g(S,t)}\,(\phi(L+g(S,t),S,t) - \phi(L,S,t))\,, \\ \phi(L,S,T) = & e^{-\hat{\alpha}\,h(L,S)}\,. \end{cases} \tag{53}$$

Furthermore, the optimal investment in the risky-asset is

$$\pi^*(S,t) = \frac{e^{-r(T-t)}}{\hat{\alpha}}\left\{\frac{\mu-r}{\sigma^2} + S\left[\gamma_S + \frac{\phi_S}{\phi}\right]\right\}. \tag{54}$$

Proof. Assuming that $U^R_{ww} < 0$, we find the first order conditions allow the optimal investment strategy to be written,

$$\pi^*(t) = -\frac{(\mu-r)U^R_w + \sigma^2 S(t) U^R_{wS}}{\sigma^2 U^R_{ww}}\,. \tag{55}$$

On substituting the ansätz (52) and the optimal investment (55) into the HJB equation (51), we establish

$$\begin{cases} 0 = & \phi\left\{U_t + (rw+q)U_w + \mu S U_S + \frac{1}{2}\sigma^2 S^2 U_{SS} - \frac{1}{2}\frac{\left((\mu-r)U_w + \sigma^2 S U_{wS}\right)^2}{\sigma^2 U_{ww}}\right\} \\ & +U\left\{\phi_t + \left(\mu - (\mu-r)\frac{U_w^2}{U\,U_{ww}}\right) S\,\phi_S + \frac{1}{2}\sigma^2 S^2\left(\phi_{SS} - \frac{U_w^2}{U\,U_{ww}}\frac{\phi_S^2}{\phi} - 2\left(\frac{U_{wS}U_w}{U\,U_{ww}} - \frac{U_S}{U}\right)\phi_S\right)\right\} \\ & +\lambda(t)\,\left[U(w-g(S,t),S,t)\,\phi(L+g(S,t),S,t) - U(w,S,t)\,\phi(L,S,t)\right], \end{cases} \tag{56}$$

subject to the boundary condition $U(w,S,T)\phi(L,S,T) = u(w+h(L,S))$. From (20), the terms inside $\{\cdot\}$ in the first line of the above expression equals $-\lambda(t)\,[U(w-g(S,t),S,t) - U(w,S,t)]$; collecting this with the last line and making use of the identities

$$U(w-g(S,t),S,t) = U(w,S,t)\,e^{\alpha(t)g(S,t)}\,, \quad \frac{U_w^2}{U_{ww}U} = 1\,, \quad \text{and} \quad \frac{U_{wS}U_w}{U_{ww}U} = \frac{U_S}{U} = \gamma_S\,, \tag{57}$$

we find, then, equation (56) distills to (53). It can then be proven that $U^R_{ww} < 0$. Using the ansätz (52), the optimal investment π^* can be rewritten as (54). For smooth g, the Verification Theorem allows us to confirm that the constructed solution is the value function for the original problem and that the described strategy is clearly optimal. □

Corollary 4.2 *The insurer's indifference price $P(L(t),S(t),t)$ for the reinsurance contract satisfies the nonlinear shifted PDE:*

$$\begin{cases} r\,P = P_t + r\,S\,P_S + \frac{1}{2}\sigma^2\,S^2\,P_{SS} + \frac{\lambda(t)}{\alpha(t)}e^{\alpha(t)g(S,t)}\left(1 - e^{-\alpha(t)[P(L+g(S,t),S,t)-P(L,S,t)]}\right)\,, \\ P(L,S,T) = h(L,S)\,. \end{cases} \tag{58}$$

Proof. The indifference price is defined as the amount of initial wealth P the insurer is willing to surrender so that her value function with the reinsurance payment is equal to her value function without the reinsurance payment. Specifically, the price P satisfies

$$U^R(w-P,L,S,t) \quad = \quad U(w,S,t)\,. \tag{59}$$

Hence, $P(L,S,t) = -\frac{1}{\alpha(t)}\ln\phi(L,S,t)$, and on substituting ϕ in terms of P in (53), we obtain (58). □

Notice that if the payoff function $h(L,S)$ is independent of the loss level, i.e. $h(L,S) = h(S)$, then (58) reduces to

$$\begin{cases} r\,P = P_t + r\,S\,P_S + \frac{1}{2}\sigma^2\,S^2\,P_{SS}\,, \\ P(L,S,T) = h(S)\,. \end{cases} \tag{60}$$

The above price can be recognized as the price of a European option with payoff $h(S)$ in the Black-Scholes model (Black and Scholes, 1973). This result is expected since the reinsurance contract is now exposed only to the hedgable risk - the risky asset - and not to the non-hedgable claims risk. Therefore, our result should reduce to the no arbitrage Black-Scholes price for an insurer of any degree of risk-aversion.

4.1. Near Risk-Neutral Insurer

Let the price of a risk-neutral insurer, taken as the limit of a risk-averse insurer, be denoted $P^0(L,S,t) = \lim_{\hat{\alpha}\to 0^+} P(L,S,t)$. Then, the pricing PDE for P^0 follows from (58) and reduces to

$$\begin{cases} r\,P^0 = P^0_t + r\,S\,P^0_S + \frac{1}{2}\sigma^2\,S^2\,P^0_{SS} + \lambda(t)\Delta P^0 \\ P^0(L,S,T) = h(L,S)\,, \end{cases} \tag{61}$$

where ΔP^0 denotes the increase in the price due to a loss arrival, i.e. $\Delta P^0 \equiv P^0(L+g(S,t),S,t) - P^0(L,S,t)$. Consequently, through the Feynman-Kac Formula, a risk-neutral insurer would be willing to pay

$$P^0(L,S,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}h(L(T),S(T))\middle|\,\mathcal{F}_t\right] \tag{62}$$

for the reinsurance contract, where the $\mathbb{Q}$-dynamics of $S(t)$ appears in (35). Furthermore, the risk-neutral dynamics of $L(t)$ is unaltered from its real world dynamics, and in particular, the activity rate of the driving Poisson process remains at $\lambda(t)$ under the measure $\mathbb{Q}$. Although this market is incomplete, and therefore there exists many risk-neutral measures equivalent to the real world measure (Harrison and Pliska, 1981), the indifference pricing methodology selects a *unique* measure.

It is interesting to investigate the first order correction in the risk-aversion parameter $\hat{\alpha}$ to gain some understanding of the perturbations around the risk-neutral price. If we assume that the payoff function is bounded from above, and hence the price is also bounded, then the price can be expanded in a power series in $\hat{\alpha}$. Specifically, write

$$P(L,S,t) = P^0(L,S,t) + \hat{\alpha} P^1(L,S,t) + o(\hat{\alpha})\,, \tag{63}$$

subject to $P^0(L,S,T) = h(L,S)$ and $P^1(L,S,T) = 0$. When inserting this ansätz into (58) and using (61), we determine $P^1(L,S,t)$ satisfies the following PDE:

$$\begin{cases} r\,P^1 = P^1_t + r\,S\,P^1_S + \frac{1}{2}\sigma^2\,S^2\,P^1_{SS} + \lambda(t)\Delta P^1 \\ \qquad\qquad +\lambda(t)\,e^{r(T-t)}\left\{g^2(S,t) - \left[\Delta P^0(L,S,t) - g(S,t)\right]^2\right\} + o(\hat{\alpha})\,, \\ P^1(L,S,T) = 0\,. \end{cases} \tag{64}$$

Through Feynman-Kac, the first order correction can be represented as a risk-neutral expectation as well, and we find the following result:

$$P^1(L,S,t) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \lambda(u)\,\left\{g^2(S(u),u) - \left[\Delta P^0(L(u),S(u),u) - g(S(u),u)\right]^2\right\}\,du\,\middle|\,\mathcal{F}_t\right]\,. \tag{65}$$

Interestingly, the payoff function $h(L,S)$ does not explicitly appear in P^1; rather, it feeds from the risk-neutral price function P^0 which does explicitly depend on the payoff. The sign of this correction term is difficult to discern on first observation. However, we may deduce that if (i) h is increasing in L, (ii) g is non-negative, and (iii) h is Lipschitz-continuous with Lipschitz constant 2, then the correction term is non-negative.

4.2. Probabilistic Interpretation of The Indifference Price

Although explicit solutions to the general pricing PDE (58) were not constructed, we follow Musiela and Zariphopoulou (2003) and show that the price function solves a particular stochastic optimal control problem. By using the convex dual of the non-linear term, the PDE is linearized and results in a pricing result similar to the American option problem. However, in the current context, the optimization is not over stopping times. Instead, we find that the optimization is over the hazard rate of the driving Poisson process.

Theorem. 4.3 *The solution of the system (58) is given by the value function*

$$P(S,L,t) \;=\; e^{-r(T-t)}\inf_{y\in\mathcal{Y}}\mathbb{E}^{\hat{\mathbb{Q}}}\left[h(L(T),S(T)) + \int_t^T e^{r(T-u)}\,\frac{\hat{\lambda}(u)}{y(u)\alpha(u)}\beta(y(u))\,du\,\middle|\,\mathcal{F}_t\right] \tag{66}$$

where $\mathcal{Y}$ is the set of non-negative $\mathcal{F}_t$-adapted processes, the loss process

$$L(t) = \sum_{n=1}^{\hat{N}(t)} g(S(t_i),t_i) \tag{67}$$

and t_i are the arrival times of the doubly-stochastic Poisson process $\hat{N}(t)$ where the $\mathcal{F}_t$-adapted hazard rate process is

$$\hat{\lambda}(t) = y(t)\,\lambda(t)\,e^{\alpha(t)g(S(t),t)} \tag{68}$$

in the measure $\hat{\mathbb{Q}}$. *Finally,* $S(t)$ *satisfies the SDE:*

$$dS(t) = r\,S(t)\,dt + \sigma\,S(t)\,d\hat{X}(t)\,, \tag{69}$$

where $\{\hat{X}(t)\}_{0\le t\le T}$ *is a* $\hat{\mathbb{Q}}$*-Wiener process.*

Proof. Let $\beta(x)$ denote the non-linear term in (58), i.e.

$$\beta(x) = 1 - e^{-x}\,, \tag{70}$$

and let $\hat{\beta}(y)$ denote its convex-dual so that

$$\hat{\beta}(y) = \max_x\,(\beta(x) - x\,y) = 1 - y + y\,\ln y\,. \tag{71}$$

Clearly, $\hat{\beta}(y)$ is defined on $(0,\infty)$ and is non-negative on its domain of definition. Furthermore,

$$\beta(x) = \min_{y\ge 0}\left(\hat{\beta}(y) + y\,x\right)\,. \tag{72}$$

Rewriting the exponential term in (58) in terms of $\hat{\beta}(y)$, we find that the PDE becomes linear in P:

$$\begin{cases} r\,P = & P_t + r\,S\,P_S + \frac{1}{2}\sigma^2\,S^2\,P_{SS} \\ & +\frac{\lambda(t)}{\alpha(t)}e^{\alpha(t)g(S,t)}\min_{y(t)\ge 0}\left(\hat{\beta}(y(t)) + y(t)\,\alpha(t)[P(L+g(S,t),S) - P(L,S)]\right)\,, \\ P(L,S,T) = & h(L,S)\,. \end{cases} \tag{73}$$

Through the usual dynamic programming principle, we find that the value function (66) satisfies the above HJB equation. The Verification Theorem, therefore, then implies that the solutions of (58) can be represented by (66). □

The pricing problem reduces to finding the activity rate that minimizes the Black-Scholes price of the reinsurance contract, subject to a penalty term, which is a function of the activity rate itself. It is useful to illustrate how the risk-neutral result of the previous subsection is recovered. In the limit in which $\hat{\alpha}\to 0^+$, the penalty term increases to infinity and the process y that minimizes (66) is clearly the one in which $\hat{\beta}(y(s)) = 0$ for all $s\in[t,T]$. This is achieved when $y(s) = 1$. The optimal hazard rate then becomes equal to its real world value $\hat{\lambda}(t) = \lambda(t)$, and the price reduces to (62).

4.3. Numerical Examples

In the absence of explicit solutions, we demonstrate how the pricing PDE can be used, nonetheless, to obtain the value of reinsurance contracts through a simple finite-difference scheme. Since we are not concerned with proving that the scheme converges in a wide class of scenarios, we take a practitioner's viewpoint and apply the scheme to situations in which the loss function and reinsurance contract itself are both bounded. To this end, it is convenient to rewrite the problem using the log of the forward-price process $z(t)\equiv \ln S(t) + r(T-t)$. Also, it is appropriate to scale the price function by the risk-aversion parameter and the discount factor by introducing the function

$$\overline{P}(L,z,t) \equiv \alpha(t)\,P(L,e^{z-r(T-t)},t)\,. \tag{74}$$

With these substitutions, the pricing PDE (58) becomes

$$\begin{cases} 0 & = \overline{P}_t - \frac{1}{2}\sigma^2\,\overline{P}_z + \frac{1}{2}\sigma^2\,\overline{P}_{zz} + \overline{\lambda}(z,t)\,\left(1 - e^{-(\overline{P}(L+\overline{g}(z,t),z,t) - \overline{P}(L,z,t))}\right)\,, \\ \overline{P}(L,z,T) & = \hat{\alpha}\,h(L,e^z)\,, \end{cases} \tag{75}$$

where $\overline{g}(z,t) = g(e^{z-r(T-t)},t)$ and $\overline{\lambda}(z,t) = \lambda(t)\,e^{\alpha(t)\overline{g}(z)}$. Here, we introduce a grid for the (L,z,t) plane with step sizes of $(\Delta L,\Delta z,\Delta t)$ so that

$$L_j = j\Delta L\,,\quad z_k = z_{min} + k\Delta L\,,\quad t_n = n\Delta t\,. \tag{76}$$

We then obtain the following explicit finite difference scheme for solving (75):

$$\begin{cases} \overline{P}(L_j, z_k, t_{n-1}) = & \frac{1}{2}\widetilde{\sigma}^2(1+\frac{1}{2}\Delta z)\overline{P}(L_j, z_{k-1}, t_n) + (1-\widetilde{\sigma})\overline{P}(L_j, z_k, t_n) \\ & +\frac{1}{2}\widetilde{\sigma}(1-\frac{1}{2}\Delta z)\overline{P}(L_j, z_{k+1}, t_n) + \overline{\lambda}(z_k, t_n)\left(1-e^{-\Delta\overline{P}(L_j, z_k, t_n)}\right), \\ \\ \overline{P}(L_j, z_k, T) = & h(L_j, e^{z_k}), \end{cases} \tag{77}$$

where

$$\widetilde{\sigma}^2 = \frac{\Delta t}{\Delta z^2}\sigma^2 \quad \text{and} \tag{78}$$

$$\Delta\overline{P}(L_j, z_k, t_n) = \overline{P}(L_j + \overline{g}(z_k, t_n), z_k, t_n) - \overline{P}(L_j, z_k, t_n). \tag{79}$$

However, to complete the description of the finite difference scheme, we need to impose appropriate boundary conditions. For the numerical examples in this section, we assume that claim sizes are generate losses as described in §3.3.3. Since such claims are asymptotically constant as S approaches zero and infinity, the price of the reinsurance must be asymptotically constant as well. Accordingly, we impose the boundary conditions $\overline{P}_S(L, z_{min}, t) = 0$ and $\overline{P}_S(L, z_{max}, t) = 0$ for all loss levels L and times t.

To illustrate the effects of the insurer's risk aversion level on the pricing results, we focus on two prototypical reinsurance payoff functions:

$$h_1(L, S) = \min(M, \max(0, L-m)) \quad \text{and} \tag{80}$$

$$h_2(L, S) = \mathbb{I}(S > S^*)\, \min(M, \max(0, L-m)). \tag{81}$$

The first reinsurance payoff function h_1 corresponds to a stop-loss reinsurance contract with payments starting at losses of m and attaining a maximum of M. This reinsurance contract makes payments that are independent of the risky asset's price at maturity; however, because the loss sizes are linked to the equity value, the value of the contract at initiation will depend on the spot price of the risky asset. The second reinsurance payoff function h_2 corresponds to a double-trigger reinsurance contract in which a stop-loss payment is made if the risky asset's price rises above a critical value S^*.

In Figure 4, we show how the price of the two reinsurance contracts depend on the prevailing spot price for several levels of risk aversion $\hat{\alpha}$. As expected, for both reinsurance contracts, the price increases as the insurer becomes more risk averse. Furthermore, for any given risk-aversion level, the price of the double-trigger stop-loss contract is lower than the pure stop-loss contract. This too is expected since the double-trigger contract pays nothing if the risky asset's price is below the trigger level at maturity. Finally, in the region of large risky asset prices, the two contracts asymptotically approach the same values.

5. Conclusions

In this paper, we obtained the premium an insurer requires if she takes on the risk of equity-linked losses. To do so, we employed the principle of equivalent utility with constant absolute risk-aversion, i.e. exponential utility, to value the contract, and although the insurer is risk-averse, we demonstrated that the premium is obtainable by computing a risk-neutral expectation of an exponentially weighted average of the claim sizes. In the limit in which the insurer becomes risk-neutral this expectation reduces to the expected loss per unit time. Furthermore, we examined the indifference price for and an insurer who took on the risk of the equity-linked insurance contract, but the general non-linear PDE that arises from the associated HJB equations was not solved. However, we were able to rewrite the non-linear PDE in terms of a dual linear optimization problem. This allowed us to provide a probabilistic interpretation of the pricing problem for the reinsurance contract: The price in the dual representation, is a minimum of

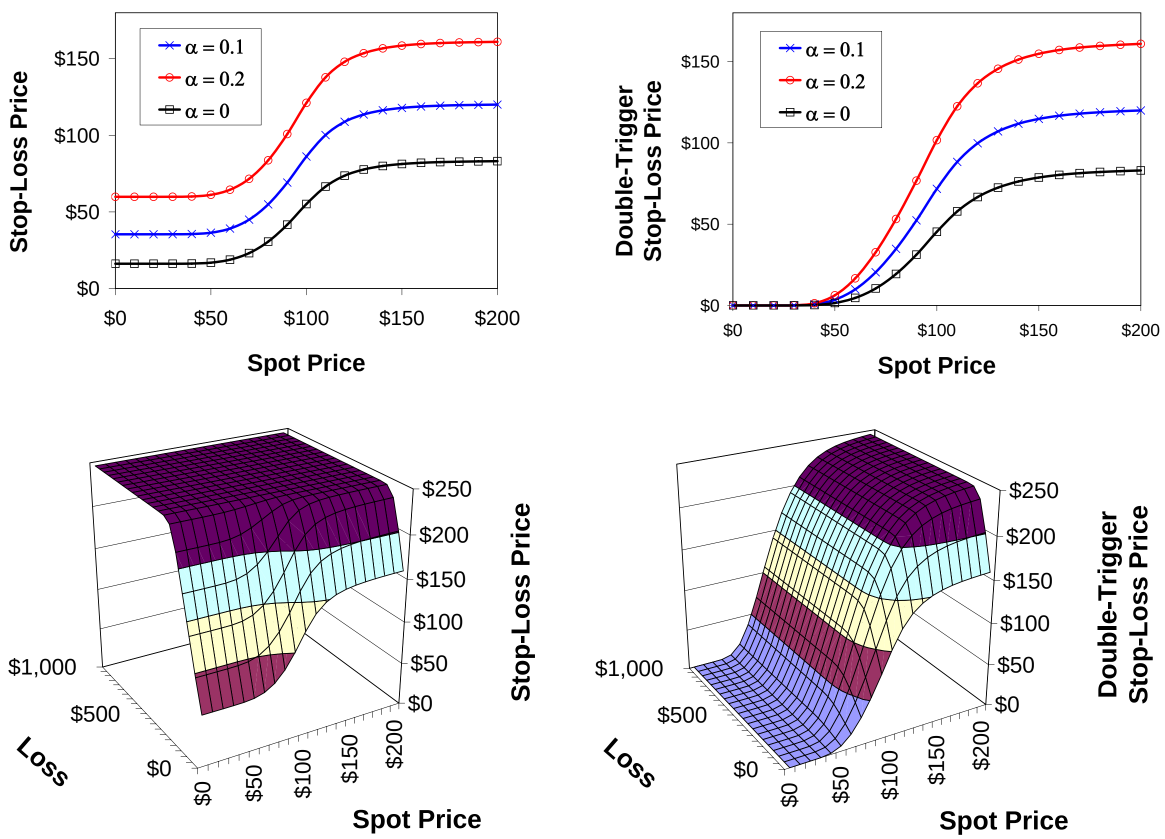

Figure 4. The indifference price for the reinsurance contracts (80) and (81) with losses described in §3.3.3. The model parameters are those used in Figure 2. In the bottom panels, the risk-aversion parameter is set to $\alpha = 0.2$. In all experiments, we used 1000 time steps and a grid of size 1000×1000 with $z_{min} = -10$, $z_{max} = 10$ and $L_{max} = 2000$.

the risk-neutral price plus a penalty term, where the optimization is over a stochastic activity rate for the Poisson processes driving the arrival times. In the limit of a risk-neutral insurer, we demonstrated that the price reduces to an expectation of the reinsurance payoff over a risk-neutral measure in which the distribution of losses are identical to the real-world distribution.

There are several avenues that are open for further exploration. For example, it would be interesting to obtain the distribution of the ruin times for insurers facing these equity-linked risks. A closed form result for general loss functions $g(S,t)$ is not likely. We suspect, however, that in cases when g is a piecewise linear function of the log-stock price, a semi-explicit form might be available. Another exploration could involve ruin-related problems: such as the optimal consumption problem for the insurer where the value function truncates at the time of ruin. This is similar to the questions Young and Zariphopoulou (2002) addressed in the context of fixed loss sizes. Extending their results to the case of equity-linked losses would be quite interesting. The Gerber and Shiu (1998) penalty function is another problem related to the time of ruin, and although it is not explicitly connected to question of indifference pricing, it would also be interesting to investigate its equity-linked extensions.

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