Incorporating Risk Aversion and Model Misspecification into Structural Models of Default

Sebastian Jaimungal [a] and Georg Sigloch [b]

[a]Department of Statistics, University of Toronto,
100 St. George Street, Toronto, Ontario, Canada M5S 3G3

[b]Department of Mathematics, University of Toronto,
40 St. George St. Toronto, Ontario, Canada M5S 2E4

It is well known that purely structural models of default cannot explain short term credit spreads, while purely intensity based models of default lead to completely unpredictable default events. Neither of these features are realistic. Additionally, investor preference may play an important role in introducing correlation of defaults as well as setting spreads themselves. Leung, Sircar and Zariphopoulou (2008) recently introduced a structural model, in which default of the reference entity is triggered by a (non-traded) credit worthiness index, and utilized indifference pricing to value defaultable bonds. We build on this base structural model and add a new distressed regime which allows for unpredictable defaults, thus creating a hybrid model of default. Furthermore, in an unrelated paper, Uppal and Wang (2003) study portfolio optimization when model parameters are unknown. By combining the hybrid default model with the uncertain parameter portfolio optimization problem, we succeed in determining corporate bond spreads and CDS spreads using indifference valuation. Our framework therefore allows for risk aversion, parameter uncertainty and both structural and intensity default features.

1. Introduction

There are two main approaches to modeling default events – the reduced form approach and the structural approach.

In the reduced form approach, a credit event is modeled as the first arrival of an exogenous jump process such as a Poisson counting process or, more generally, a Cox process (see e.g. Lando (1994) and Schönbucher (2003)). So called *risk-neutral hazard rates* are then extracted from market prices of traded default triggered securities – defaultable bonds and credit default swap (CDS) are prime examples. Subsequently, reduced form approaches do not use any information about the internal structure of the company, like firm value or corporate debt. This can be an advantage, since the relevant data may be hard to observe or extract from available information.

In contrast, the structural approach models default as a consequence of a company becoming *unhealthy*. A popular structural model is the firm value model, which measures the company's health by its firm value, which itself is the viewed as the sum of the company's equity and debt. This interpretation makes the firm value a traded asset. Default occurs when the value of the firm is less than its outstanding debts or some percentage of the outstanding debt.

In the first paper using a structural approach, Merton (1974) models a company's equity as a European call option on its firm value with its debts used as a strike level. The company defaults if at maturity the value of the firm is below the company's debts. The advantage of this model is its simplicity even though it lacks realism. Black and Cox (1976) extend this idea to the more realistic case where the company defaults the instant its firm value drops below a critical level, turning the problem into a first passage time one.

There have been numerous extensions, modifications and increases in the sophistication of the firm value model over the last several decades. A limited list of important contributions to the field include Leland (1977) extension of the debt to a coupon paying bond; Kim, Ramaswamy, and Sundaresan (1993) and Longstaff and Schwartz (1995) inclusion of stochastic interest rates; Leland (1994) and Leland and Toft (1996) extension to endogenously specifying the default boundary as a result of equity holders maximizing the value of the firm; Duffie and Lando (2001) model of incomplete market information; and Fouque, Sircar, and Solna (2006) integration of stochastic volatility in firm value models.

In a recent paper, Leung, Sircar, and Zariphopoulou (2008) introduce a market model with a money market account and a defaultable risky asset, and use utility indifference pricing to price defaultable bonds on this risky asset. In that work, the firm's stock price and its asset value are modeled as correlated geometric Brownian motions; however, in contrast to previous models, although the asset value is assumed observable it is not tradable. Default of the firm is triggered by the asset value hitting a barrier D. The non-tradability of the firm's asset value makes the market incomplete. This contrasts with Sircar and Zariphopoulou (2007) where the authors analyze the effect of risk aversion within a reduced form approach.

Here, we are interested in addressing how risk aversion *and* model uncertainty combine to affect bond values and CDS rates. To achieve this, the paper is structured in two parts. The first part assumes complete knowledge of the model parameters while the second introduces model uncertainty. We now describe these parts in some detail.

In the first part of this article, we assume complete knowledge on the model parameters, and adopt a similar setting to Leung, Sircar, and Zariphopoulou (2008). Specifically, we assume the *perceived health* of a company is measured by a creditworthiness index (CWI; treated as the *firm's asset value* in Leung, Sircar, and Zariphopoulou (2008)). Since the health of a company is typically determined by more complex factors than the prices of its stocks and bonds, we assume that the CWI is not tradable. It is natural that the company's health will be correlated with its equity value, therefore we assume the CWI is positively correlated to the firm's stock price. However we extend the model in Leung, Sircar, and Zariphopoulou (2008) in several aspects. Firstly, there is no reason to assume that the defaultable stock is the only available tradable asset. A real world investor is always able to invest in many liquid stocks, and more importantly, investors will try to diversify their portfolios. As a consequence, we consider a market in which the investor also trades in a correlated non-defaultable index. This setting can easily be extended to several default-free risky assets, but we will not do this here. Secondly, experience shows that it is not reasonable to assume that default of a company can be completely anticipated. Consequently, we assume that after the CWI crosses a certain threshold D, the state of the company changes from *healthy* to *distressed*. At this point the company does not default, and instead enters a state of financial distress, in which default is now triggered by an exogenous Poisson process. In this context, D can be interpreted as a rough upper estimate of an otherwise unknown default barrier, after whose hitting investors become nervous and withdraw their investments from the firm. Another interpretation of D is that of the level at which rating agencies downgrade the credit rating of the company.

In the second part of this article, we address the very real fact that some model parameters may be uncertain. In particular, this concerns the CWI since usually the perceived health of a company can only be fully observed a few times a year, e.g. when the firm publishes its earnings. While it would be desirable to introduce the CWI as an unobservable quantity, in the indiffernce pricing setting this would lead to a highly non-tractable problem. Instead, we take a different approach, following ideas from Anderson, Hansen, and Sargent (2000), Maenhout (2004) and Uppal and Wang (2003), who introduce model uncertainty to portfolio optimization problems. In these approaches, the optimization problem is augmented to incorporate a minimax problem where one maximizes expected *penalized* utility of terminal wealth over all admissible trading strategies while minimizing over a set of measures equivalent to the historical one. Details on the methodology are provided in Section 6.

Due to the incomplete market framework, the market price of risk of the CWI must be incorporated into any risk-neutral valuation procedure. However, since we are explicitly investigating the affects of risk-aversion we do not take this risk-neutral approach. Another reason for avoiding the risk-neutral approach is that it assumes that the entity being valued is liquidly traded. Although there is a good market for CDS and defaultable bonds, it is far from being completely liquid – and in the current market conditions that liquidity has considerably dried up. There is an alternative approach to valuation in incomplete market settings which has been under intense study over the last several years – indifference valuation (also known as utility indifference and certainty equivalence). Indifference valuation circumvents the issue of tradability all together, and instead focuses on the optimal investment investment strategy in the presence and absence of the embedded risks. In particular, from the seller's viewpoint, utility indifference *pricing* compares (i) not taking on the default risk and receiving no premium or (ii) taking on the default risk while receiving a premium. For the buyer the setup is analogous. Some recent studies using indifference pricing include: Hodges and Neuberger (1989) demonstrate how transaction costs can be analyzed in this framework, Davis, Panas, and Zariphopoulou (1993) study the impact on derivative pricing in the presence of transaction costs, Young and Zariphopoulou (2002) apply these methods to insurance products, Henderson and Hobson (2002a) explore options on a non-tradable asset correlated to a tradable one, and Stoikov (2005) investigates the implications for volatility derivatives. It is important to note that the *indifference price* is a personal price and depends on the investor's level of risk aversion. Only if the market price is higher than this, will the seller engage in the transaction.

The remainder of this article is organized as follows: In section 2 we introduce the model. In section 3 we compute the value functions for the investment problem, i.e. the scenario in which the investor is invested in the tradable assets and the money market only. In sections 4 and 5 we compute the value functions corresponding to investments in defaultable bonds and credit default swaps. We combine the results from sections 3, 4 and 5 to determine bond yields and CDS spreads. Up to this point, the investor is assumed to have complete knowledge of the model parameters. In section 6 we introduce model uncertainty and examine how the results from the previous sections change in this generalized setting. The article closes with some concluding remarks in section 7.

2. The Model

We now outline our modeling framework and key assumptions.

Assumption 1. *The investor's utility function is exponential* $u(w) = -\frac{1}{\gamma}e^{-\gamma w}$.

Let a certain threshold $D > 0$ be given. We first consider the state before the CWI hits D for the first time, which we shall call the *healthy regime* henceforth. In this article, we work on the filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. Let $\{(B_t^{(1)},\ B_t^{(2)},\ B_t^{(3)}) : 0 \le t \le T\}$ denote correlated Wiener processes, let N_t denote a Poisson process, independent of the Brownian motions, with constant activity rate κ, and $\mathbb{F}$ denotes the natural filtration generated by the Wiener processes and the Poisson process: $\mathbb{F} \triangleq \{\mathcal{F}_t : 0 \le t \le T\}$ where $\mathcal{F}_t = \sigma\{(B_u^{(1)},\ B_u^{(2)},\ B_u^{(3)},\ N_u) : 0 \le u \le t\}$. The continuously compounded interest rate r is assumed constant throughout [1].

The non-defaultable index P_t, the defaultable stock S_t and the creditworthiness index C_t are modeled as correlated geometric Brownian motions

$$\begin{aligned} dP_t &= P_t\left(\mu_1\, dt + \sigma_1\, dB_t^{(1)}\right), \\ dS_t &= S_t\left(\mu_2\, dt + \sigma_2\, dB_t^{(2)}\right), \\ dC_t &= C_t\left(\nu\, dt + \eta\, dB_t^{(3)}\right). \end{aligned}$$

For our purposes it is convenient to write the variance-covariance matrix of P_t, S_t, C_t in the form

$$\begin{pmatrix} \boldsymbol{\Omega} & \boldsymbol{\omega} \\ \boldsymbol{\omega}^T & \eta^2 \end{pmatrix}.$$

[1]It is trivial to make interest rates deterministic. Extending to stochastic interest rates is both interesting and non-trivial as demonstrated by Young (2004) in the context of equity insurance linked products. In principal, stochastic hazard rates can also be handled as in Ludkovski and Young (2008)

Here $\mathbf{\Omega}$ is the variance-covariance matrix of P_t, S_t, and $\boldsymbol{\omega} = (\rho_{13}\sigma_1\eta,\ \rho_{23}\sigma_2\eta)$, and $d[B^{(i)}, B^{(j)}]_t = \rho_{ij}\ dt$.

Let

$$\tau_h \triangleq \inf\{t : \min_{0\leq s\leq t} C_t = D\}$$

be the first time that the CWI hits the threshold D. At this time the stock S_t does not default yet. However the investor realizes that from now on, the firm is in a state of financial distress. As a consequence, he completely liquidates his investment in S and from thereon only invests in the money market and the non-defaultable index. Since S has not defaulted yet, it is resonable to assume that the investor can sell S at the current market price S_{τ_h}.

After C_t has hit D for the first time, the firm enters a state of financial distress, which will be called the *distressed regime* henceforth. In the distressed regime default is triggered by the switching of the Poisson process N_t for which τ_d denotes the first arrival time of N_t after time τ_h, i.e.

$$\tau_d \triangleq \inf\{t > \tau_h \mid N_t > N_{t-}\}.$$

Since the investor has liquidated their position in the defaultable stock, the only sources of randomness in this state are $B_t^{(1)}$ and, if invested in a credit derivative written on the firm, also N_t.

An alternative realistic model for the distressed regime would be the following: in addition to the non-defaultable index the investor is still allowed to invest in the defaultable stock. Since default is triggered by the switching of a Poisson process and therefore cannot be anticipated, the investor loses at least a portion of the wealth invested in S. To our knowledge, this setup has not been investigated in previous papers. It makes the model more interesting, but also less mathematically tractable, and particularly so when the investor invests in credit derivatives. In this paper we will therefore concentrate on the simpler model introduced earlier.

3. The Investment Problem

Given our model above, we now proceed to describe the optimal portfolio problem for an investor who wishes to trade in the money-markets, the risky (but non-defaultable) index and the risky defaultable equity.

3.1. Trading Strategies and Wealth Process

In this subsection, we define the set of admissible trading strategies and their corresponding wealth processes.

Throughout this paper, we let $\boldsymbol{B}_t = (B_t^{(1)},\ B_t^{(2)})^T$, $\boldsymbol{\mu} = (\mu_1,\ \mu_2)^T$, and $\boldsymbol{r} =\ (r, r)^T$. Further, let $\boldsymbol{\pi} = (\boldsymbol{\pi}_t)_{t=0}^T$, where $\boldsymbol{\pi}_t = (\pi_t^{(1)}, \pi_t^{(2)}) \in \mathbb{R}^2$, be an $\mathcal{F}_t$-adapted process satisfying $\int_0^T \boldsymbol{\pi}_t^2\ dt < \infty$ almost surely. Then for all $t \in [0, T]$, $w \in \mathbb{R}$ the SDE

$$\begin{cases} dW_s &= \left[(\boldsymbol{\mu} - \boldsymbol{r})^T \boldsymbol{\pi}_s + r\ W_s\right]\ ds + \pi_s^{(1)}\sigma_1\ dB_s^{(1)} + \pi_s^{(2)}\sigma_2\ dB_s^{(2)}, \\ W_t &= w, \end{cases}$$

has a unique strong solution $W_s^{\boldsymbol{\pi}}$. We interpret $\pi_t^{(1)}$ and $\pi_t^{(2)}$ as the dollar amounts invested in P and S at time t, and $W_s^{\boldsymbol{\pi}}$ as the wealth process corresponding to the self-financing trading strategy $\boldsymbol{\pi}$ and the initial condition $W_t = w$. For convenience we will write W_s rather than $W_s^{\boldsymbol{\pi}}$ whenever w and $\boldsymbol{\pi}$ follow from the context.

In this paper we only consider Markovian strategies, which means that the investor makes their decisions according to the current state of W_t, P_t, S_t, and C_t. Furthermore, at any time the investor knows which of the three regimes the firm is currently in (healthy, distressed, post default). Consequently, the strategy will also depend on $\mathbb{I}\{t \geq \tau_h\}$ and $\mathbb{I}\{t \geq \tau_d\}$ and leads us to the following definition:

Defintion 1. *An admissible trading strategy is a function*

$$\boldsymbol{\pi}_t = (\pi_t^{(1)}, \pi_t^{(2)}) = \boldsymbol{\pi}(w, P, S, C, t, \mathbb{I}\{t > \tau_h\}, \mathbb{I}\{t > \tau_d\})$$

of the form

$$\boldsymbol{\pi}_t = \begin{cases} \boldsymbol{\pi}(W_t, P_t, S_t, C_t, t), & t < \tau_h, \\ \boldsymbol{\pi}(W_t, P_t, t), & t > \tau_h, \end{cases}$$

satisfying the following:

1. $\pi_t^{(2)} = 0$ *for* $t \geq \tau_h$,
2. $\int_0^T \boldsymbol{\pi}_t^2\ dt < \infty$ *almost surely,*
3. $\int_0^T \left(\exp\left\{-\gamma e^{r(T-t)W_T^{\boldsymbol{\pi}}}\right\}\right)^2\ dt < \infty.$

The last condition ensures that certain uniform integrability conditions are satisfied, which allow us to construct the necessary verification theorems. The set of admissible trading strategies will be denoted $\mathcal{A}$. In the interest of readability, we have opted to record and prove the relevant verification theorems in Appendix A.

3.2. Utility Maximization

We begin by maximizing the investor's terminal expected utility of wealth in the two regimes. The dynamics of the wealth process are given by

$$dW_t = \begin{cases} \left[(\boldsymbol{\mu} - \boldsymbol{r})^T \boldsymbol{\pi}_t + r\ W_t\right]\ dt + \pi_t^{(1)}\sigma_1\ dB_t^{(1)} + \pi_t^{(2)}\sigma_2\ dB_t^{(2)}, & t < \tau_h, \\ \left[(\mu_1 - r)\pi_t^{(1)} + r\ W_t\right]\ dt + \pi_t^{(1)}\sigma_1 dB_t^{(1)}, & t > \tau_h, \end{cases}$$

subject to $W_{\tau_h} = W_{\tau_h^-}$. In this setup, the investor is not exposed to any default risk since their position is liquidated once the firm enters the distressed regime. Consequently, in the distressed regime, the optimization problem reduces to that of the standard Merton (1969) investment problem with a money market account and risky asset P_t. The value function V in this case is well known, for exponential utility, to be

$$V(w,t) = \sup_{(\pi_1,0)\in\mathcal{A}} \mathbb{E}\left[u(W_T) \mid W_t = w,\ P_t = P\right] = -\tfrac{1}{\gamma}\exp\{a_t\, w - \tfrac{1}{2}\lambda^2(T-t)\}, \tag{1}$$

using the notation $a_t = -\gamma e^{r(T-t)}$ and $\lambda = (\mu_1 - r)/\sigma_1$.

Given the value function in the distressed regime, we now maximize expected terminal utility in the healthy regime through investment in the index P_t, the defaultable asset S_t and the money-market account. Here, the investor wishes to solve the following problem:

$$U(w,P,S,C) = \sup_{\boldsymbol{\pi}\in\mathcal{A}} \mathbb{E}\left[u(W_T) \mid W_0 = w,\ P_0 = P,\ S_0 = S,\ C_0 = C > D\right]\ ,$$

where $\mathcal{A}$ represent the set of admissible strategies. To invoke the dynamic programming principle, we introduce the time dependent value function

$$U(w,P,S,C,t) = \sup_{\boldsymbol{\pi}\in\mathcal{A}} \mathbb{E}\left[u(W_T) \mid W_t = w,\ P_t = P,\ S_t = S,\ C_t = C,\ t < \tau_h\right]\ .$$

Note that U is defined on the domain $\mathcal{D} \triangleq \mathbb{R} \times [0,\infty)^2 \times [D,\infty) \times [0,T]$.

Applying the standard Bellman principle we find that assuming U to be sufficiently regular, we expect it to satisfy the partial differential equation

$$\begin{cases} \partial_t U + \sup_{\boldsymbol{\pi}\in\mathbb{R}^2} \mathcal{L}^{\boldsymbol{\pi}} U = 0, \\ U(w,C,T) = u(w), \quad w \in \mathbb{R},\ C > D, \\ U(w,D,t) = V(w,t), \quad w \in \mathbb{R},\ t \in [0,T], \end{cases} \tag{2}$$

where $\mathcal{L}^{\boldsymbol{\pi}}$ is the infinitesimal generator of the processes (W_t, P_t, S_t, C_t), and in particular $\mathcal{L}^{\boldsymbol{\pi}} U =$

$\mathcal{K}U + \mathcal{K}^{\boldsymbol{\pi}}U$ with

$$\begin{aligned}
\mathcal{K}U &\triangleq rw\ \partial_w U + \mu_1 P\ \partial_P U + \mu_2 S\ \partial_S U + \nu C\ \partial_C U + \frac{1}{2}\sigma_1^2 P^2\ \partial_{PP}U + \frac{1}{2}\sigma_2^2 S^2\ \partial_{SS}U + \\
&\quad + \frac{1}{2}\eta^2 C^2\ \partial_{CC}U + \rho_{12}\sigma_1\sigma_2 PS\ \partial_{PS}U + \omega_1 PC\ \partial_{PC}U + \omega_2 SC\ \partial_{SC}U, \\
\mathcal{K}^{\boldsymbol{\pi}}U &\triangleq \frac{1}{2}\boldsymbol{\pi}^T\boldsymbol{\Omega}\boldsymbol{\pi}\ \partial_{ww}U + \boldsymbol{\pi}^T\left[(\boldsymbol{\mu} - \boldsymbol{r})\partial_w U + \boldsymbol{\Omega}(P\ \partial_{wP}U,\ S\ \partial_{wS}U)^T + \boldsymbol{\omega} C\ \partial_{wC}U\right]\ .
\end{aligned}$$

The first boundary condition in (2) is the obvious terminal condition, and the second boundary condition is due to the firm's switching to the distressed regime at time τ_h.

It is straightforward to see that U is independent of P and S, i.e. $U(w, P, S, C, t) = U(w, C, t)$. Therefore the two terms above simplify to

$$\begin{aligned}
\mathcal{K}U &= rw\ \partial_w U + \nu C\ \partial_C U + \frac{1}{2}\eta^2 C^2\ \partial_{CC}U, \qquad \text{and} \\
\mathcal{K}^{\boldsymbol{\pi}}U &= \frac{1}{2}\boldsymbol{\pi}^T\boldsymbol{\Omega}\boldsymbol{\pi}\ \partial_{ww}U + \boldsymbol{\pi}^T\left[(\boldsymbol{\mu} - \boldsymbol{r})\ \partial_w U + \boldsymbol{\omega} C\ \partial_{wC}U\right]\ .
\end{aligned}$$

Furthermore, a verification theorem (see appendix A.1) shows that if a function $\tilde{U}$ satisfies equation (2) on $\mathcal{D}$, then $U = \tilde{U}$.

The first order condition for the optimal investment in the risky assets $\boldsymbol{\pi}$ is

$$\boldsymbol{\Omega}\ \partial_{ww}U\ \boldsymbol{\pi} = -(\boldsymbol{\mu} - \boldsymbol{r})\ \partial_w U - \boldsymbol{\omega} C\ \partial_{wC}U\ ,$$

which yields

$$\boldsymbol{\pi}^* = -\frac{1}{\partial_{ww}U}\ \boldsymbol{\Omega}^{-1}\left[(\boldsymbol{\mu} - \boldsymbol{r})\ \partial_w U + \boldsymbol{\omega} C\ \partial_{wC}U\right]\ .$$

Due to the exponential utility assumption, wealth can be removed from (2) by writing $U(w, C, t) = u\left(w\, e^{r(T-t)}\right)\ g(C, t)$, and we get

$$\begin{cases}
\partial_t g + \nu C \partial_C g + \frac{1}{2}\eta^2 C^2 \partial_{CC} g \\
\quad - \frac{1}{2g}\ \left[(\boldsymbol{\mu} - \boldsymbol{r})g + \boldsymbol{\omega} C\ \partial_C g\right]\ \ \boldsymbol{\Omega}^{-1}\ \left[(\boldsymbol{\mu} - \boldsymbol{r})g + \boldsymbol{\omega} C\ \partial_C g\right] = 0 \\
\qquad\qquad\qquad\qquad g(D, t) = e^{-\frac{\lambda^2}{2}(T-t)}\ , \\
\qquad\qquad\qquad\qquad g(C, T) = 1\ .
\end{cases}$$

Following Zariphopoulou (2001) and Henderson and Hobson (2002b) we make a power transform substitution of the form

$$g(C, t) = e^{-\frac{1}{2}\Lambda^2\,(T-t)}\, G^{\beta}(\ln \tfrac{C}{D}, T - t),$$

where

$$\Lambda^2 = (\boldsymbol{\mu} - \boldsymbol{r})^T \boldsymbol{\Omega}^{-1} (\boldsymbol{\mu} - \boldsymbol{r})$$

and β is chosen such that the resulting PDE for G becomes linear. The PDE for G is

$$-\partial_\tau G+\left(\nu-\frac{1}{2}\eta^2-\boldsymbol{\omega}^T\boldsymbol{\Omega}^{-1}(\boldsymbol{\mu}-\boldsymbol{r})\right)\,\partial_x G+\frac{1}{2}\eta^2\,\partial_{xx}G+\frac{1}{2}\,\frac{(\partial_x G)^2}{G}\left[(\beta-1)\,\eta^2-\beta\,\boldsymbol{\omega}^T\boldsymbol{\Omega}^{-1}\boldsymbol{\omega}\right]=0.$$

The appropriate choice for β is then

$$\beta=\frac{1}{1-\frac{1}{\eta^2}(\boldsymbol{\omega}^T\boldsymbol{\Omega}^{-1}\boldsymbol{\omega})}\ ,$$

and the corresponding linear equation for $G(x,\tau)$ is

$$\begin{cases}-\partial_\tau G+\tilde{\nu}\ \partial_x G+\dfrac{1}{2}\eta^2\ \partial_{xx}G=0\\ \qquad\qquad G(0,\tau)=e^{\frac{1}{2\beta}(\Lambda^2-\lambda^2)\tau}\ ,\\ \qquad\qquad G(x,0)=1\ .\end{cases}\tag{3}$$

Here, $\widetilde{\nu}=\nu-\frac{1}{2}\eta^2-(\boldsymbol{\mu}-\boldsymbol{r})^T\,\boldsymbol{\Omega}^{-1}\,\boldsymbol{\omega}$ is the drift of the CWI under the *minimal entropy martingale measure*[2]. Due to the boundary condition along the barrier $C=D$, which is inherited from the subproblem of optimizing in the distressed regime, G is not simply the probability of remaining in the healthy regime under the MEMM. In fact, the PDE can be solved fairly easily through standard techniques (see appendix B) to find

$$\begin{aligned}G(x,\tau)&=1-\frac{x}{\sqrt{2\pi}\,\eta}\int_0^\tau\frac{e^{-(x+\widetilde{\nu}u)^2/(2\eta^2u)}}{u^{3/2}}\left[1-e^{+\frac{1}{2\beta}(\Lambda^2-\lambda^2)(\tau-u)}\right]du\\&=q_t(x,T;\widetilde{\nu})+\ e^{(\widehat{\nu}-\widetilde{\nu})x/\eta^2+\frac{1}{2\beta}(\Lambda^2-\lambda^2)\tau}\,(1-q_t(x,T;\widehat{\nu}))\ ,\end{aligned}\tag{4}$$

where

$$q_t(x,s;\theta)\triangleq\mathbb{Q}^\theta(\tau_h>s\mid\ln(C_t/D)=x),\quad\text{and}\quad\widehat{\nu}=\widetilde{\nu}+\sqrt{\widetilde{\nu}^2+\eta^2\cdot\frac{1}{\beta}(\Lambda^2-\lambda^2)}.$$

Here, as usual, $\Phi(y)$ denotes the standard normal cdf and $\mathbb{Q}^\theta$ is a measure induced the Radon-Nikodym derivative process

$$\left.\frac{d\mathbb{Q}^\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t}=\exp\left\{-\left(\tfrac{\nu-\theta}{\eta}\right)^2-\left(\tfrac{\nu-\theta}{\eta}\right)B_t^{(3)}\right\}\ .$$

4. The defaultable bond

We now include investment in a defaultable bond in addition to the money-market account, the risky (non-defaultable) index and the risky defaultable stock. In this case, the investor

[2]Since the process C_t is continuous the MEMM measure is equivalent to the minimal martingale measure (see Schweizer (1999)).

receives a notional of F at maturity if the reference entity does not default before the maturity date T, or receives a random recovery[3] R (with $0 \le R \le 1$) of the notional at default if default occurs prior to maturity. Consequently, if we let $\tau_1 \triangleq \tau_h \wedge T$, $\tau_2 \triangleq \tau_d \wedge T$, the dynamics of the wealth process is given by

$$d\overline{W}_t = \begin{cases} \left[(\boldsymbol{\mu} - \boldsymbol{r})^T \boldsymbol{\pi}_t + r\,\overline{W}_t\right]\; dt + \pi_t^{(1)}\sigma_1\; dB_t^{(1)} + \pi_t^{(2)}\sigma_2\; dB_t^{(2)}, & t < \tau_1, \\ \left[(\mu_1 - r)\pi_t^{(1)} + r\,\overline{W}_t\right]\; dt + \pi_t^{(1)}\sigma_1 dB_t^{(1)}, & \tau_1 < t < \tau_2, \\ \left[(\mu_1 - r)\pi_t^{(1)} + r\,\overline{W}_t\right]\; dt + \pi_t^{(1)}\sigma_1 dB_t^{(1)}, & t > \tau_2, \end{cases}$$

subject to

$$\begin{aligned} \overline{W}_{\tau_1} &= \overline{W}_{\tau_1^-} + F \cdot \mathbb{I}\{\tau_1 = T\}, \\ \overline{W}_{\tau_2} &= \overline{W}_{\tau_2^-} + RF \cdot \mathbb{I}\{\tau_2 < T\} + F \cdot \mathbb{I}\{\tau_2 = T\}. \end{aligned}$$

Mainly for notational purposes we assume that the Poisson process N_t, which drives default in the distressed regime, has a constant hazard rate κ. However the computations can easily be generalized to deterministic hazard rates.

The derivation of the corresponding value function is similar as for the investment problem, however the expression for the value function in the distressed regime is not as simple as in the pure investment problem.

4.1. Valuation in the Distressed Regime

The value function $\overline{V}$ corresponding to an investment in the defaultable bond in the distressed regime is

$$\overline{V}(w, P, t) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[u(\overline{W}_T)\;|\overline{W}_t = w,\; P_t = P,\; \tau_h < t < \tau_d\right],$$

As before, it is clear that $\overline{V}$ is independent of P, i.e. $\overline{V} = \overline{V}(w,t)$. The corresponding HJB equation is

$$\begin{cases} \partial_t \overline{V} + \sup_{\pi \in \mathbb{R}} \left\{ [rw + (\mu_1 - r)\pi]\;\partial_w \overline{V} + \frac{1}{2}\sigma_1^2 \pi^2 \partial_{ww} \overline{V} \right\} \\ \qquad\qquad + \kappa \left[V(w + \widetilde{R}F,\; t) - \overline{V} \right] = 0, \\ \qquad\qquad\qquad \overline{V}(w, T) = u(w + F), \quad w \in \mathbb{R}\,. \end{cases} \tag{5}$$

[3]The recovery rate is assumed to be random, but independent of the driving Brownian motions and N_t.

Here, V is the value function for the standard Merton investment problem (see Eq.(1)) and

$$\widetilde{R}_t \triangleq -\frac{1}{\gamma F e^{r(T-t)}} \log \mathbb{E}\, e^{-\gamma R F e^{r(T-t)}}, \quad \text{i.e.} \quad e^{-\gamma \widetilde{R}_t F e^{r(T-t)}} = \mathbb{E}\, e^{-\gamma R F e^{r(T-t)}} . \tag{6}$$

Note that if the recovery rate is assumed known, then $\widetilde{R}_t = R$.

The last term on the left hand side of Eq.(5) is due to a potential default and the corresponding switch from the distressed regime to the state of default. In addition, applying verification Theorem 6 from the appendix shows that any classical solution to Eq.(5) coincides with the value function $\overline{V}$.

Factoring out wealth (i.e. writing $\overline{V}(w,t) = u(w\, e^{r(T-t)})\, \overline{g}(t))$ provides the following equation for $\overline{g}$:

$$\begin{cases} \overline{g}' + \inf\limits_{\pi\in\mathbb{R}} \left\{ (\mu_1 - r)\pi a_t\, \overline{g} + \dfrac{1}{2}\pi^2\sigma_1^2 a_t^2\, \overline{g} \right\} - \kappa\, \overline{g} + \kappa\, e^{-\frac{1}{2}\lambda^2(T-t)+\widetilde{R}_t F a_t} = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{g}(T) = e^{-\gamma F}\,. \end{cases}$$

The infimum is attained at $\pi^{(1),*} = -\lambda^2/(2a_t)$, and on substitution leads to the linear ODE

$$\begin{cases} \overline{g}' - \left(\kappa + \dfrac{1}{2}\lambda^2\right)\, \overline{g} + \kappa\, e^{-\frac{1}{2}\lambda^2(T-t)+\widetilde{R}_t F a_t} = 0 \\ \qquad\qquad\qquad\qquad \overline{g}(T) = e^{-\gamma F}\,. \end{cases} \tag{7}$$

This ODE can be easily solved to find

$$\overline{g}(t) = e^{-\frac{1}{2}\lambda^2(T-t)} \cdot \left[e^{-\gamma F} \cdot e^{-\kappa(T-t)} + \int_t^T e^{-\gamma \widetilde{R}_s F e^{r(T-s)}}\ \kappa e^{-\kappa(s-t)} ds \right] . \tag{8}$$

Interestingly, it is possible to rewrite this result in terms of an expectation over the default time as follows:

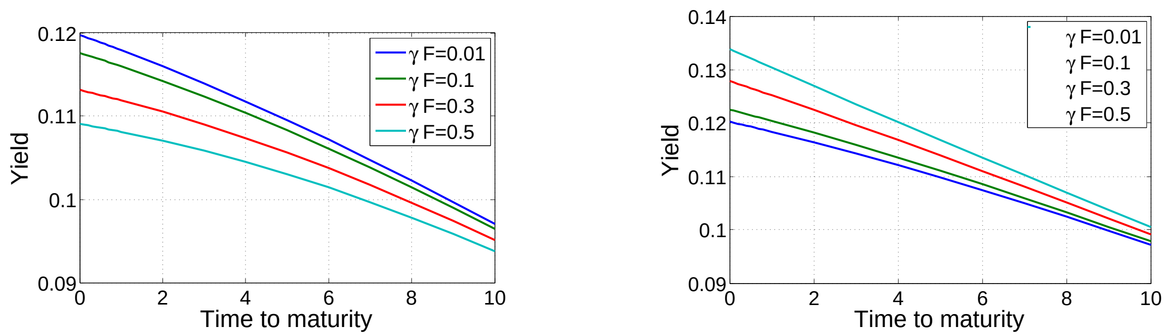
$$\overline{g}(t) = e^{-\frac{1}{2}\lambda^2(T-t)} \mathbb{E}\Big[\exp\Big\{ -\gamma\left(F\mathbb{I}_{\{\tau_d > T\}} + RFe^{r(T-\tau_d)}\ \mathbb{I}_{\{\tau_d \le T\}} \right) \Big\} \Big|\, \tau_h < t < \tau_d \Big] . \tag{9}$$

It is pleasing that an expectation over the risky bond's cash-flow accumulated to maturity arises in this context. This is of course a specific realization of the general duality result of Delbane, Grandits, Rheinlnder, Samperi, Schweizer, and Stricker (2002). However, this duality result is not so simple to apply in the healthy region.

Now it is a straightforward matter to determine the *indifference price* $\overline{p}$ of the risky bond in the distressed regime. The defining equation is $\overline{V}(w - \overline{p}, t) = V(w,t)$ resulting in an indifference price of

$$\overline{p} = -\frac{1}{\gamma} e^{-r(T-t)} \ln \left(e^{-\gamma F} \cdot e^{-\kappa(T-t)} + \int_t^T e^{-\gamma \widetilde{R}_s F e^{r(T-s)}}\ \kappa e^{-\kappa(s-t)}\ ds \right).$$

In Figure 1 we show the bond yield term structures with several levels of risk-aversion for both the seller and the buyer. Notice that as risk-aversion increases the buyer's yield increases as a more risk-averse investor demands a lower price and therefore a higher yield, while the opposite occurs for the seller. Interestingly, as term grows the spread decreases.



(a) Seller's Yields

(b) Buyer's Yields

Figure 1. The seller's and buyer's indifference yields for varying levels of risk-aversion in the distressed regime. The model parameters are: $C_0 = 1.1$, $r = 0.05$, $\mu_1 = 0.08$, $\mu_2 = 0.1$, $\nu = 0.01$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$, $\eta = 0.05$ $\kappa = 0.1$, $\rho_{12} = 0.5$, $\rho_{13} = 0.3$, $\rho_{23} = 0.8$, $D = 1$, $R = 0.3$.

4.2. Valuation in the Healthy Regime

Given the value function in the distressed regime, we are now in a position to solve for the value function in the healthy regime. In this case, the value function is defined as

$$\overline{U}(w,P,S,C,t) \triangleq \sup_{\boldsymbol{\pi}\in\mathcal{A}} \mathbb{E}\left[u(\overline{W}_T) \mid \overline{W}_t = w,\ P_t = P,\ S_t = S,\ C_t = C,\ t < \tau_h\right] .$$

Then we expect $\overline{U}$ to satisfy the HJB equation

$$\begin{cases} \partial_t \overline{U} + \sup_{\boldsymbol{\pi}\in\mathbb{R}^2} \mathcal{L}^{\boldsymbol{\pi}} \overline{U} = 0, \\ \qquad \overline{U}(w,C,T) = u(w+F), \quad w \in \mathbb{R},\ C > D, \\ \qquad \overline{U}(w,D,t) = \overline{V}(w,t), \quad w \in \mathbb{R}, \end{cases} \tag{10}$$

i.e. the only difference between this HJB equation and Eq. (2) is the modified boundary conditions along the distress barrier $C = D$. As in the investment problem, the verification Theorem 6 from the appendix guarantees that any solution of (10) coincides with the value function $\overline{U}$.

Once again, it is clear that $\overline{U}$ is independent of P and S. Writing

$$\overline{U}(w,C,t) = u(w\,e^{r(T-t)})\ e^{-\frac{\Lambda^2}{2\beta}(T-t)}\ \overline{G}^{\beta}(\ln\tfrac{C}{D},\ T-t)$$

as before, implies $\overline{G}$ satisfies the linear PDE

$$\begin{cases} -\partial_\tau \overline{G} + \widetilde{\nu}\ \partial_x \overline{G} + \dfrac{1}{2}\eta^2\ \partial_{xx}\overline{G} = 0\ , \\ \qquad\qquad \overline{G}(0,\tau) = e^{\frac{\Lambda^2}{2\beta}\tau}\ \overline{g}(T-\tau)^{1/\beta}\ , \\ \qquad\qquad \overline{G}(x,0) = e^{-\frac{\gamma F}{\beta}}\ , \end{cases} \tag{11}$$

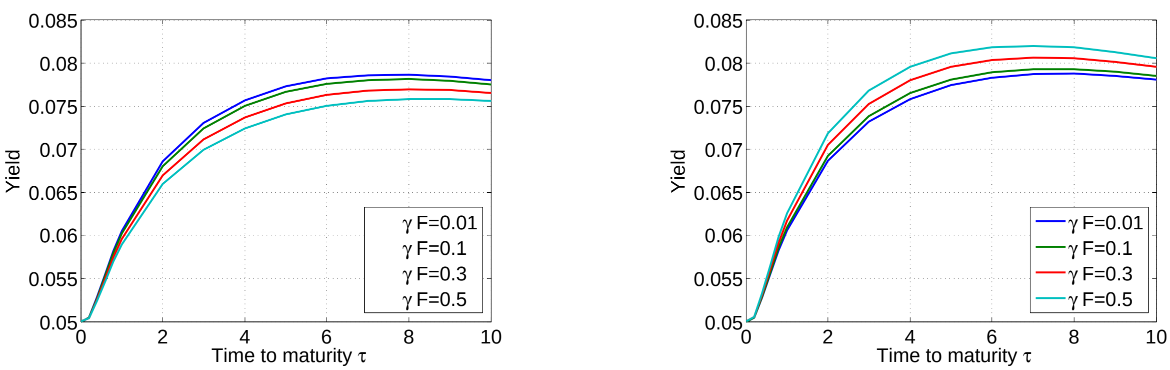
whose solution can be written

$$\overline{G}(x,\tau) = e^{-\frac{\gamma F}{\beta}} q_t(T;\widetilde{\nu}) + \frac{x}{\eta\sqrt{2\pi}} \int_0^\tau \frac{e^{-(x+\widetilde{\nu}u)^2/(2\eta^2 u)}}{u^{3/2}} e^{\frac{\Lambda^2}{2\beta}(\tau-u)}\ (\overline{g}(T-\tau+u))^{1/\beta}\ du. \tag{12}$$

Given (4) and (12), the indifference value $\overline{p}$ of the defaultable bond can be found by setting $U(w,S,C,t) = \overline{U}(w-\overline{p},S,C,t)$ from which we find

$$\overline{p}_t(x,T) = e^{-r\tau}\ \frac{\beta}{\gamma}\ \ln\frac{G(x,\tau)}{\overline{G}(x,\tau)}$$

with $x = \ln\frac{C}{D}$ and $\tau = T-t$. Unfortunately, we cannot simplify this expression any further;



(a) Seller's Yields

(b) Buyer's Yields

Figure 2. The seller's and buyer's indifference yields for varying levels of risk-aversion in the distressed regime. The model parameters are as in Figure 1

however, it is easy to numerically integrate using any standard quadrature routine. In Figure 2, the yield curves for different levels of risk-aversion are shown for the same set of parameters as in Figure 1. Notice that in the healthy regime there is a definite hump shape in the risky yield despite the flat risk-free term structure. The hump is due to the non-zero recovery of 30% assumed in the example. Once again we observe the increasing/decreasing of the buyer's/seller's yields as risk-aversion increases.

5. The Credit Default Swap

In this section we will address how to determine the CDS rate consistent with indifference valuation. To this end, suppose that the investor sells (or purchases) a CDS and receives (or pays) a continuous premium rate of A paid on a notional of F up until default time or maturity which ever occurs first. If default occurs first, the investor provides (or receives) a random[4] payment of $(1 - R)\,F$ (with $0 \leq R \leq 1$) and all future premium payments cease. In this setup, the wealth process has the dynamics

$$d\widetilde{W}_t = \begin{cases} \left[(\boldsymbol{\mu} - \boldsymbol{r})^T \boldsymbol{\pi}_t + r\ \widetilde{W}_t + \epsilon A F\right]\ dt + \pi_t^{(1)}\sigma_1\ dB_t^{(1)} + \pi_t^{(2)}\sigma_2\ dB_t^{(2)}, & t < \tau_1, \\ \left[(\mu_1 - r)\pi_t^{(1)} + r\ \widetilde{W}_t + \epsilon A F\right]\ dt + \pi_t^{(1)}\sigma_1 dB_t^{(1)}, & \tau_1 < t < \tau_2, \\ \left[(\mu_1 - r)\pi_t^{(1)} + r\ \widetilde{W}_t\right]\ dt + \pi_t^{(1)}\sigma_1 dB_t^{(1)}, & t > \tau_2, \end{cases}$$

subject to $\widetilde{W}_{\tau_1} = \widetilde{W}_{\tau_1^-}, \quad \widetilde{W}_{\tau_2} = \widetilde{W}_{\tau_2^-} - \epsilon(1 - R)F \cdot \mathbb{I}\{\tau_2 < T\}$. Here $\epsilon = +1$ for the seller and $\epsilon = -1$ for the buyer of protection.

To determine the *indifference rate* A, the value functions in the two regimes must be determined separately, very much like for the defaultable bond.

5.1. Valuation in the Distressed Regime

The value function $\widetilde{V}$ in the distressed regime is defined as

$$\widetilde{V}(w, P, t; A) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[u(\widetilde{W}_T)\ \middle|\ \widetilde{W}_t = w,\ P_t = P,\ \tau_h \leq t < \tau_d\right] .$$

The rate A is written here to emphasis that the solution explicitly depends on the CDS rate. Through the dynamic programming principle, an HJB equation for the investor exposed to the CDS risk can be written as

$$\begin{cases} \partial_t \widetilde{V} + \sup_{\pi \in \mathbb{R}} \Big\{ [rw + \epsilon A F + (\mu_1 - r)\pi]\ \partial_w \widetilde{V} + \frac{1}{2}\sigma_1^2 \pi^2 \partial_{ww} \widetilde{V} + \\ \qquad + \kappa \left[V(w - \epsilon(1 - \widetilde{R}_t)F,\ t) - \widetilde{V}\right] \Big\} = 0 \ , \\ \qquad \widetilde{V}(w, T) = u(w), \quad w \in \mathbb{R} \ . \end{cases} \tag{13}$$

[4] Again assumed independent of all other stochastic factors.

Verification Theorem 6 once again shows that any solution to the HJB equation above is indeed a solution to the original optimal control problem.

Letting $\widetilde{V}(w,t) = u(we^{r(T-t)})\, \widetilde{g}(t)$ leads to the following ODE for $\widetilde{g}$:

$$\begin{cases} \partial_t \widetilde{g} - (\kappa - \epsilon AF)\, a_t \widetilde{g} - \inf_{\pi\in\mathbb{R}} \left\{ (\mu_1 - r)a_t + \frac{1}{2}\pi^2\sigma_1^2 a_t^2 \right\} \widetilde{g} = -\kappa e^{-\frac{1}{2}\lambda^2(T-t)-\epsilon(1-\widetilde{R}_t)a_t} \\ \widetilde{g}(T) = 1. \end{cases} \tag{14}$$

The infimum is attained at $\pi^* = -\lambda^2/(2a_t)$, and on resubstituting leads to the equation

$$\begin{cases} \widetilde{g}' - \left(\kappa + \frac{1}{2}\lambda^2 - \epsilon AF a_t\right)\, \widetilde{g} = -\kappa\; e^{-\frac{1}{2}\lambda^2(T-t)-\epsilon(1-\widetilde{R}_t)Fa_t} \\ \overline{g}(T) = 1\ . \end{cases} \tag{15}$$

This ODE has the solution

$$\widetilde{g}(t) = e^{-\frac{\lambda^2}{2}(T-t)} \left\{ e^{-\kappa(T-t)} \cdot e^{\epsilon AF \int_t^T a_u du} + \int_t^T e^{\epsilon F\left(A\int_t^s a_u du - (1-\widetilde{R}_s)a_s\right)}\; \kappa\; e^{-\kappa(s-t)}\; ds \right\}\ .$$

This can be simplified slightly by noticing that $\int_t^s a_u du = \frac{1}{r}(a_s - a_t)$; however, in its current form a natural interpretation arises akin to the result for the risky bond's value function in the distress region (see Eq.(9)). In particular, it is easy to see that

$$\widetilde{g}(t) = e^{-\frac{\lambda^2}{2}(T-t)}\mathbb{E}\left[\exp\left\{ -\gamma\left(\epsilon FA \int_t^{\tau_d\wedge T} e^{r(T-u)}du - \epsilon F(1-R)e^{r(T-\tau_d)}\mathbb{I}_{\tau_d\leq T} \right)\right\} \middle|\, \tau_h < t < \tau_d \right]\ .$$

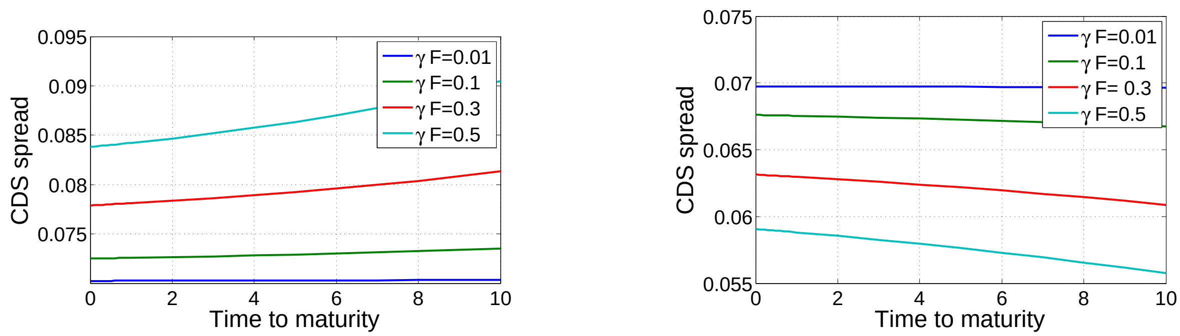
Once again this is a specific realization of the general duality results of Delbane, Grandits, Rheinlnder, Samperi, Schweizer, and Stricker (2002).

Similar to the price of the defaultable bond, the *indifference CDS spread* is defined as the rate A which renders the investor indifferent to taking on the risk or not, i.e. A such that $\widetilde{V}(w,t;A) = V(w,t)$, implying

$$e^{-\kappa(T-t)} \cdot e^{\epsilon AF\int_t^T a_u du} + \int_t^T e^{\epsilon F\left(A\int_t^s a_u du - (1-\widetilde{R}_s)a_s\right)}\; \kappa\; e^{-\kappa(s-t)}\; ds = 1.$$

It is not possible to obtain an analytical expression for A in terms of the remaining model parameters; however, it is possible to carry out an asymptotic expansion for A in powers of the risk-aversion parameter γ. Expansions of this type have been explored in Davis (1998) in the options context. He demonstrates that the zeroth order term is equivalent to price of an infinitesimal position in the option - the so called marginal price. Rather than exploring this direction, we will opt to solve for the spread numerically.

In Figure 3, we illustrate the seller's and buyer's CDS rates in the distressed regime using the same parameter values as in the risky bond (see Figure 1). As expected, for any given level of risk-aversion and term, the seller's rate is indeed higher than the buyer's rate. Also, for the seller, the required premium increases with increasing risk aversion, while for the buyer, the indifference premium decreases with increasing risk aversion. Furthermore, as maturity increases, the seller requires a higher premium, while the buyer's indifference premium decreases.



(a) Seller's CDS spreads

(b) Buyer's CDS spreads

Figure 3. The indifference CDS rate term structure for the buyer and seller in the distressed regime. See Figure 1 for the model parameters.

5.2. Valuation in the Healthy Regime

Now that the distressed regime value function has been obtained, we move onto the more interesting task of the healthy regime. In this case, the value function is defined as

$$\widetilde{U}(w,P,S,C,t;A) \triangleq \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[u(\widetilde{W}_T) \;\middle|\; \widetilde{W}_t = w,\ P_t = P,\ S_t = S,\ C_t = C,\ t < \tau_h\right] ,$$

which has the corresponding HJB equation

$$\begin{cases} \partial_t \widetilde{U} + \epsilon AF \partial_w \widetilde{U} + \sup_{\boldsymbol{\pi}\in\mathbb{R}^2} \mathcal{L}^{\boldsymbol{\pi}} \widetilde{U} = 0, \\ \qquad\qquad\qquad \widetilde{U}(w,D,t) = \widetilde{V}(w,t), \\ \qquad\qquad\qquad \widetilde{U}(w,C,T) = u(w), \quad C > D. \end{cases} \tag{16}$$

This differs from equation (2) by the boundary condition along $C = D$ and the inclusion of the term $\epsilon AF\partial_w \widetilde{U}$ representing the accumulation of premium payments. Once gain, $\widetilde{U}$ is independent of P and S and assuming that $\widetilde{U}$ has the form

$$\widetilde{U}(w,C,t) = u\left(w\ e^{r(T-t)}\right)\ \widetilde{G}^{\beta}\left(\ln\frac{C}{D}, T-t\right)\cdot e^{\psi(T-t)}$$

with

$$\beta = \left(1 - \tfrac{1}{\eta^2}(\boldsymbol{\omega}^T \boldsymbol{\Omega}^{-1} \boldsymbol{\omega})\right)^{-1} \;, \quad \text{and} \qquad \psi(\tau) = -\frac{1}{2}\Lambda^2 \tau - \epsilon\, \gamma\, \frac{AF}{r}\, e^{r\tau}$$

linearizes equation (16) resulting in

$$\begin{cases} -\partial_\tau \widetilde{G} + \widetilde{\nu}\partial_x \widetilde{G} + \frac{1}{2}\eta^2 \partial_{xx}\widetilde{G} = 0 \;, \\ \qquad\qquad \widetilde{G}(0,\tau) = e^{-\psi(\tau)/\beta} \cdot \widetilde{g}(T-\tau)^{1/\beta} \;, \\ \qquad\qquad \widetilde{G}(x,0) = 1 \;. \end{cases} \tag{17}$$

This can be solved as before to find

$$\widetilde{G}(x,\tau) = q_t(T;\widetilde{\nu}) + \frac{x}{\eta\sqrt{2\pi}} \int_0^\tau \frac{e^{-(x+\widetilde{\nu}u)^2/(2\eta^2 u)}}{u^{3/2}}\, e^{-\psi(\tau-u)/\beta}\, \left(\widetilde{g}(T-\tau+u)\right)^{1/\beta} du \;. \tag{18}$$

Armed with the solutions (4) and (18) the indifference CDS rate $A = A(C,t)$ makes the two value functions $U(w,C,t)$ and $\widetilde{U}(w,C,t;A)$ equal and requires solving the highly non-linear equation

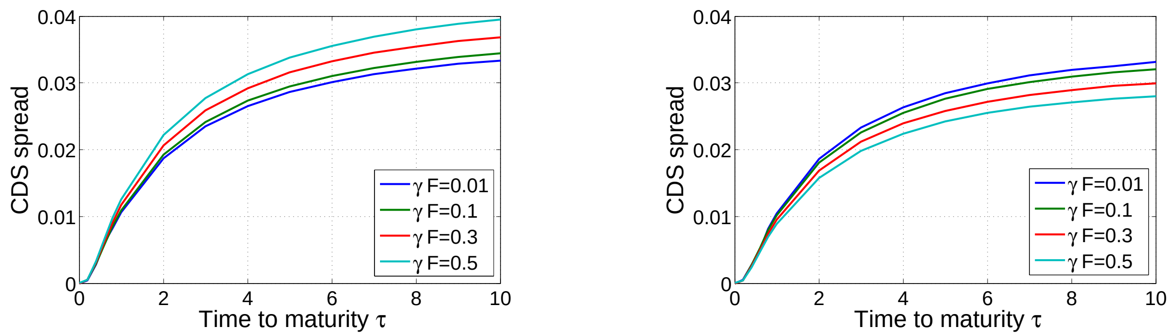
$$\epsilon\gamma\, F\; A = \frac{\beta\, r}{e^{r\tau}} \ln\left(\widetilde{G}\left(\ln \tfrac{C}{D}, \tau; A\right) \Big/ G\left(\ln \tfrac{C}{D}, \tau\right) \right) \;. \tag{19}$$

The dependence of $\widetilde{G}(x,\tau;A)$ on A is explicitly shown to emphasis the embedded non-linearity. In Figure 4, we numerically solve this equation for the model parameters in Figure 2 and illustrate the seller's and buyer's CDS spreads for several levels of initial perceived health C_0. As expected, as the perceived health approaches the distress barrier, the CDS spread increases, while at every level of perceived health, the seller's rate is higher than the buyer's rate. Unlike in the distressed regime, the spreads do indeed tend to zero for very short maturities; however, this occurs only at very short maturities. Once uncertainty in model parameters is accounted for, this steepening can be controlled not only by the proximity to the distress barrier, but also by the amount of model uncertainty. The next section addresses this issue.

6. Indifference Pricing With Model Misspecification

6.1. Motivation

Following ideas of Anderson, Hansen, and Sargent (2000) and Maenhout (2004), we now incorporate model uncertainty into our default model. Suppose that in a market with a money market account and n risky, tradable and default free assets $S_t^{(1)}, \ldots, S_t^{(n)}$ the investor seeks to maximize expected utility of terminal wealth over all admissible trading strategies, i.e. wants



(a) Seller's CDS spreads

(b) Buyer's CDS spreads

Figure 4. The indifference CDS rate term structure for the buyer and seller in the distressed regime. See Figure 1 for the model parameters.

to determine $\sup_{\boldsymbol{\pi}\in\mathcal{A}} \mathbb{E}[u(W_T)]$. The dynamics of the economy is first estimated to be described by the measure $\mathbb{P}$, called the *reference measure*, under which $S_t^{(1)}, \ldots, S_t^{(n)}$ have the dynamics

$$dS_t^{(i)} = S_t^{(i)} \left(\mu_t^{(i)} dt + \sigma_t^{(i)} dB_t^{(i)} \right),$$

the $B_t^{(i)}$ are correlated Wiener processes. Since the investor is uncertain whether or not $\mathbb{P}$ is indeed the correct measure, he is willing to consider other *candidate measures* $Q \sim \mathbb{P}$ as well. However, a measure change comes at the cost of a penalty for his value function, and his new goal is to find

$$\sup_{\boldsymbol{\pi}\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \left\{ \mathbb{E}^Q[u(W_T)] + h(Q) \right\} \tag{20}$$

In equation (20) the freedom of choice of the candidate measure can be interpreted as a second control (apart from the trading strategy). The penalty term $h(Q)$ controls the distance between the candidate measure Q and the reference measure $\mathbb{P}$ which the investor still considers reasonable.

A popular choice for h (see e.g. Anderson, Hansen, and Sargent (2000)) is the entropic penalty function

$$h(Q) = k\ \mathcal{H}(Q|\mathbb{P}) = k\ \mathbb{E}^Q \left[\log \frac{dQ}{d\mathbb{P}} \right] = k\ \mathbb{E} \left[\frac{dQ}{d\mathbb{P}} \log \frac{dQ}{d\mathbb{P}} \right],$$

where k is a positive constant.

If we work with the standard filtration generated by the driving Brownian motions $B_t^{(1)}, \ldots, B_t^{(n)}$, then an equivalent measure change corresponds to an adjustment of the drifts in the dynamics of the $S_t^{(i)}$, i.e. under a measure $Q \sim \mathbb{P}$ the risky assets have the dynamics

$$dS_t^{(i)} = S_t^{(i)} \left[(\mu_t^{(i)} + v_t^{(i)}) \; dt + \sigma_t^{(i)} \; dB_t^{Q,(i)} \right] \tag{21}$$

for some $\mathcal{F}_t$-adapted processes $v_t^{(1)}, \ldots, v_t^{(n)}$. Here the $B_t^{Q,(i)}$ are correlated Brownian motions under Q.

A short calculation shows that

$$\mathbb{E}^Q \left[\log \frac{dQ}{d\mathbb{P}} \right] = \frac{1}{2} \; \mathbb{E}^Q \left[\; \int_0^T \boldsymbol{v}_s^T \boldsymbol{\Omega}^{-1} \boldsymbol{v}_s \; ds \; \right],$$

where $\boldsymbol{v}_t = (v_t^{(1)}, \ldots, v_t^{(n)})^T$ and $\boldsymbol{\Omega}$ is the variance-covariance matrix of $S_t^{(1)}, \ldots, S_t^{(n)}$. It follows, that if U is the value function defined by (20), then U satisfies the HJB equation

$$\partial_t U + \sup_{\boldsymbol{\pi} \in \mathbb{R}^n} \inf_{\boldsymbol{v} \in \mathbb{R}^n} \left\{ \mathcal{L}^{\boldsymbol{\pi}, \boldsymbol{v}} U + \frac{1}{2} k \; \boldsymbol{v}^T \boldsymbol{\Omega}^{-1} \boldsymbol{v} \right\} = 0, \tag{22}$$

subject to the appropriate terminal condition. Here $\mathcal{L}^{\boldsymbol{\pi}, \boldsymbol{v}}$ is the infinitesimal generator of the price processes.

Unfortunately, due to the last term on the left hand side not containing U or any of its derivatives, equation (22) cannot be solved analitically. In fact, for the case of power or exponential utility, we are even unable to factor wealth out of the solution. As a resolution, Maenhout (2004) suggests the following approach: he modifies the HJB equation by scaling the penalty term and defines U to be the solution of the equation

$$\partial_t U + \sup_{\boldsymbol{\pi} \in \mathbb{R}^n} \inf_{\boldsymbol{v} \in \mathbb{R}^n} \left\{ \mathcal{L}^{\boldsymbol{\pi}, \boldsymbol{v}} U + \frac{1}{2} k U \; \boldsymbol{v}^T \boldsymbol{\Omega}^{-1} \boldsymbol{v} \right\} = 0. \tag{23}$$

subject to the natural boundary condition at maturity $t = T$. Maenhout then shows that this modified equation can be solved and uses it to optimize portfolios under power utility.

Alternatively one can also modify the optimization problem and define the value function as the solution of

$$U(w,t) = \sup_{\boldsymbol{\pi} \in \mathcal{A}} \inf_{Q \sim \mathbb{P}} \mathbb{E}^Q \left[u(W_T) + \frac{1}{2} k \int_t^T U(W_s, s) \; \boldsymbol{v}_s^T \boldsymbol{\Omega}^{-1} \boldsymbol{v}_s \; ds \; \middle| \; W_t = w \right]. \tag{24}$$

The penalty term equation (23) can therefore be interpreted as a scaled version of the penalty term corresponding to relative entropy. However the scaling factor is not constant, but dependent on future utility. The definition of U would normally also depend on initial conditions corresponding to the risky assets. However, as in the case with complete certainty, it turns out that e.g. for power and exponential utiliy, U depends only on wealth and time. The HJB-equation corresponding to the modification (24) is in fact equation (23).

In the following sections we adopt this approach to generalize our results from the previous sections on the investment problem as well as on pricing the defaultable bond and the credit default swap. Our work differs from Maenhout (2004) firstly in the choice of the utility function. More importantly, due to the default risk of one of the assets, the arising optimization problems are different and lead to more complicated HJB equations. Nevertheless, we show that we can solve them analytically, at least to the same extent as for the case of complete model specification.

In a related paper, Uppal and Wang (2003) generalize the setting in Maenhout (2004) and introduce different levels of ambiguity for different assets. Applying their idea to our setting, we define the value function U as solution of the equation

$$U(w,t) = \sup_{\boldsymbol{\pi}\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \mathbb{E}^Q \left[u(W_T) + \frac{1}{2}\int_t^T U(W_s,s)\ \boldsymbol{v}_s^T \boldsymbol{\Phi} \boldsymbol{v}_s\ ds \ \middle|\ W_t = w \right]. \tag{25}$$

where the matrix $\boldsymbol{\Phi}$ arises as a weighted sum of the levels of model uncertainty corresponding to different subsets of the risky assets. The construction of $\boldsymbol{\Phi}$ is discussed in detail in appendix C. This generalization is particularly useful for our setting, since the model for the tradable assets can usually be estimated well from past data, whereas the dynamics of the health of the defaultable firm are rather uncertain.

6.2. The Investment Problem

For the remainder of the paper we assume that under a measure $Q \sim \mathbb{P}$ the dynamics of P_t, S_t, C_t are given by

$$\begin{cases} dP_t = P_t\ \left[(\mu_1 + v_t^{(1)})\ dt + \sigma_1\ dB_t^{Q,(1)} \right]\ , \\ dS_t = S_t\ \left[(\mu_2 + v_t^{(2)})\ dt + \sigma_2\ dB_t^{Q,(2)} \right]\ , \\ dC_t = C_t\ \left[(\nu + v_t^{(3)})\ dt + \eta\ dB_t^{Q,(3)} \right] \end{cases} \tag{26}$$

with correlated Q-Brownian motions $B_t^{Q,(1)}$, $B_t^{Q,(2)}$, $B_t^{Q,(3)}$. Normally we will write $B_t^{(1)}$, $B_t^{(2)}$, $B_t^{(3)}$ only for notational purposes. In addition to the same notation as in section 3 we let $\boldsymbol{v}_t = (v_t^{(1)}, v_t^{(2)}, v_t^{(3)})^T$. As in the completely specified case, we assume that the investor liquidates their position in S at time τ_h. Consequently, in the absence of default risk, the corresponding wealth process has the Q-dynamics

$$dW_t = \begin{cases} \left[((\boldsymbol{\mu} - \boldsymbol{r})^T + (v_t^{(1)}, v_t^{(2)}))\boldsymbol{\pi}_t + r\ W_t \right]\ dt + \pi_t^{(1)}\sigma_1\ dB_t^{(1)} + \pi_t^{(2)}\sigma_2\ dB_t^{(2)}, & t < \tau_h, \\ \left[(\mu_1 - r + v_t^{(1)})\pi_t^{(1)} + r\ W_t \right]\ dt + \pi_t^{(1)}\sigma_1 dB_t^{(1)}, & t > \tau_h, \end{cases} \tag{27}$$

subject to

$$W_{\tau_h} = W_{\tau_h^-}.$$

Since we are working with several different measures, we have to modify the definition of the set of admissible trading strategies. We now require that in order for a trading strategy to be admissible, the conditions from section 3.2 have to hold for every candidate measure Q.

6.2.1. The Value Function in the Distressed Regime

We start with computing the value function V corresponding to an optimal investment in the distressed regime. For $t > \tau_h$ the wealth process has the Q dynamics

$$dW_t = \left[rW_t + (\mu_1 - r + v_t^{(1)})\pi_t^{(1)} \right] \; dt + \pi_t^{(1)}\sigma_1 dB_t^{(1)} \; .$$

From this point forward, we will write v and π instead of $v^{(1)}$ and $\pi^{(1)}$, when there is no confusion.

We define V to be the solution of the optimization problem

$$V(w,P,t) = \sup_{\pi\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \mathbb{E}^Q \left[u(W_T) + \frac{1}{2}\int_t^T V(W_s,P_s,s) \; \phi \; v_s^2 \; ds \; \middle| \; W_s = w, \; P_s = P \right]. \quad (28)$$

Here ϕ is a *negative scalar*. The penalty term in (28) is a scaled version of the relative entropy of a measure change induced by adjusting the drift of the index P_t only. We choose this penalty, because in the distressed regime the only model uncertainty relevant for the wealth process is through $v_t^{(1)}$, the drift adjustment of P_t. The scalar ϕ is negative, because V is negative.

Assuming V is independent of P, the corresponding HJB equation is

$$\begin{cases} \partial_t V + \sup\limits_{\pi\in\mathbb{R}} \inf\limits_{v\in\mathbb{R}} \left\{ \mathcal{L}^{\pi,v} \; V + \frac{1}{2} \; V\phi v^2 \right\} = 0 \; , \\ \qquad V(w,T) = u(w) \; , \quad w \in \mathbb{R} \; , \end{cases} \quad (29)$$

where

$$\mathcal{L}^{\pi,v}V = (rw + (\mu_1 - r + v)\pi) \; \partial_w V + \frac{1}{2}\pi^2\sigma_1^2 \; \partial_{ww}V.$$

In contrast to the completely specified case, it is by far not obvious that V is indeed independent of P. However verification Theorem 7 from the appendix shows that a solution of equation (29) indeed coincides with the value function V in equation (28).

As in the completely specified case, we make the ansatz $V(w,t) = u(we^{r(T-t)}) \; g(t)$, which results in the following ODE for g:

$$\begin{cases} g' + \inf\limits_{\pi\in\mathbb{R}} \sup\limits_{v\in\mathbb{R}} \left\{ g(\mu_1 - r + v) \; a_t\pi + \frac{1}{2}\pi^2\sigma_1^2 a_t^2 \; g + \frac{1}{2}g \; \phi \; v^2 \right\} = 0, \\ \qquad g(T) = 1. \end{cases} \quad (30)$$

To find the saddle point, it is convenient to write the PDE in the form

$$g' + \inf_{\pi\in\mathbb{R}} \sup_{v\in\mathbb{R}} F(\boldsymbol{h}) = 0, \tag{31}$$

where

$$F(\boldsymbol{h}) = \frac{1}{2}\boldsymbol{h}^T\boldsymbol{K}\boldsymbol{h} + \boldsymbol{d}^T\boldsymbol{h}$$

and

$$\boldsymbol{h} = \begin{pmatrix} v \\ \tilde{\pi} \end{pmatrix}, \quad \tilde{\pi} = a_t\pi, \quad \boldsymbol{K} = g\begin{pmatrix} \phi & 1 \\ 1 & \sigma_1^2 \end{pmatrix}, \quad \boldsymbol{d} = \begin{pmatrix} 0 \\ (\mu_1 - r)g \end{pmatrix}.$$

The first order condition for a saddle point is $\boldsymbol{K}\boldsymbol{h} = -\boldsymbol{d}$ which has the unique solution

$$\boldsymbol{h}^* = -\boldsymbol{K}^{-1}\boldsymbol{d} = \frac{\mu_1 - r}{\phi\sigma_1^2 - 1}\begin{pmatrix} -1 \\ \phi \end{pmatrix}.$$

The corresponding critical value is $F(\boldsymbol{h}^*) = -\frac{1}{2}\boldsymbol{d}^T\boldsymbol{K}^{-1}\boldsymbol{d}$, and leads to the ODE

$$\begin{cases} g' - \dfrac{1}{2}\dfrac{(\mu_1 - r)^2\phi}{\phi\sigma_1^2 - 1}\, g = 0, \\ \qquad\qquad g(T) = 1. \end{cases} \tag{32}$$

This ODE has the solution $g(t) = e^{-\frac{1}{2}\overline{\lambda}^2\,(T-t)}$ with $\overline{\lambda}^2 = (\mu_1 - r)\phi/(\phi\sigma_1^2 - 1)$.

The above result is quite interesting in the two extreme cases $\phi = 0$ and $\phi \to \infty$. In the limit $\phi \to \infty$ the solution reduces to the standard Merton problem with complete certainty. This is expected, since $\phi \to \infty$ penalizes any deviation from the reference measure $\mathbb{P}$ heavily. In contrast, $\phi = 0$ corresponds a complete lack of confidence in the reference measure $\mathbb{P}$. It is interesting to observe that in this case, $\pi^* \to 0$, i.e. the investor invests less and less money in the risky asset. Furthermore, $v_t = r - \mu_1$, and the corresponding measure Q is measure under which the risky asset grows at the risk-free rate.

6.2.2. The Value Function in the Healthy Regime

Prior to τ_h the corresponding wealth process has the Q-dynamics

$$dW_t = \left[\left((\boldsymbol{\mu} - \boldsymbol{r})^T + (v_t^{(1)}, v_t^{(2)})\right)\boldsymbol{\pi}_t + r\;W_t\right]\;dt + \pi_t^{(1)}\sigma_1\;dB_t^{(1)} + \pi_t^{(2)}\sigma_2\;dB_t^{(2)},$$

The value function $U(w, P, S, C, t)$ in the healthy regime is defined to be the solution of the optimization problem

$$\begin{aligned} U(w,P,S,C,t) = \sup_{\boldsymbol{\pi}\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \mathbb{E}^Q \Bigg[u(W_T) + \frac{1}{2}\int_t^{\tau_h\wedge T} U(W_s,P_s,S_s,C_s,s)\;\boldsymbol{v}_s^T\boldsymbol{\Phi}\boldsymbol{v}_s\;ds\; + \\ + \frac{1}{2}\int_{\tau_h\wedge T}^{T} V(W_s,s)\;\phi\;v_s^2\;ds\;\Bigg|\; W_t = w,\; P_t = P,\; S_t = S,\; C_t = C \Bigg]. \end{aligned} \tag{33}$$

Here $\boldsymbol{\Phi}$ is a *negative semidefinite* matrix. Its construction is explained in detail in appendix C.

The choice of the penalty terms in equation (33) are motivated in a similar way as in equation (28). Up to time τ_h, the penalty is a scaled version of the relative entropy of the measure change from $\mathbb{P}$ to Q, since the wealth process is affected by model uncertainty in all three processes P_t, Q_t, C_t. After time τ_h only model uncertainty in P_t affects W_t, so we choose the same penalty term as in the value function for the distressed regime.

Assuming that U is dependent on w, C, and t only, the corresponding HJB equation is

$$\begin{cases} \partial_t U + \sup_{\boldsymbol{\pi}\in\mathbb{R}^2} \inf_{\boldsymbol{v}\in\mathbb{R}^3} \mathcal{L}^{\boldsymbol{\pi},\boldsymbol{v}} U = 0, \\ \qquad U(w,C,T) = u(w) \\ \qquad U(w,D,t) = V(w,t), \end{cases} \tag{34}$$

with

$$\begin{aligned} \mathcal{L}^{\boldsymbol{\pi},\boldsymbol{v}} U = &\left[rw + \boldsymbol{\pi}^T \left((\boldsymbol{\mu} - \boldsymbol{r}) + (v^{(1)}, v^{(2)})^T \right) \right] \partial_w \, U + \frac{1}{2} \boldsymbol{\pi}^T \boldsymbol{\Omega} \boldsymbol{\pi} \; \partial_{ww} U + \\ &+ (\nu + v^{(3)}) C \; \partial_C U + \boldsymbol{\pi}^T \boldsymbol{\omega} C \; \partial_{wC} + \frac{1}{2} \eta^2 C^2 \; \partial_{CC} U + \frac{1}{2} \; U \; \boldsymbol{v}^T \boldsymbol{\Phi} \boldsymbol{v} \end{aligned} \tag{35}$$

Once again, it is not obvious that U should depend only on w, C and t; however, Verification Theorem 7c demonstrates that any solution to (34) indeed coincides with the value function U in (33).

The method for solving this equation is similar to the one in section 3. Our goal is to write U in the form $U(w,C,t) = u(we^{r(T-t)}) \; e^{-\frac{1}{2}\overline{\Lambda}^2 (T-t)} \; G(\ln \frac{C}{D}, T-t)^{\beta}$. Then we determine $\overline{\Lambda}$ and β such that the PDE for G is linear. Firstly, letting $U(w,C,t) = u(we^{r(T-t)}) \; g(C,t)$ leads to the PDE

$$\begin{cases} \partial_t g + \inf_{\boldsymbol{\pi}\in\mathbb{R}^2} \sup_{\boldsymbol{v}\in\mathbb{R}^3} \Big\{ \boldsymbol{\pi}^T \left(\boldsymbol{\mu} - \boldsymbol{r} + (v^{(1)}, v^{(2)})^T \right) a_t g + \frac{1}{2} \boldsymbol{\pi}^T \boldsymbol{\Omega} \boldsymbol{\pi} \; a_t^2 g + \\ \quad + (\nu + v^{(3)}) C \; \partial_C g + \boldsymbol{\pi}^T \boldsymbol{\omega} \; a_t C \; \partial_C g + \frac{1}{2} \eta^2 C^2 \; \partial_{CC} g + \frac{1}{2} g \; \boldsymbol{v}^T \boldsymbol{\Phi} \boldsymbol{v} \Big\} = 0, \\ \qquad g(C,T) = 1 \\ \qquad g(D,t) = e^{-\frac{1}{2}\overline{\Lambda}^2 (T-t)}. \end{cases} \tag{36}$$

As in the case of the distressed regime, the notation for this equation can be simplified by writing it in the form

$$\partial_t g + \nu \; C \partial_C g + \frac{1}{2} \eta^2 C^2 \; \partial_{CC} g + \inf_{\boldsymbol{\pi}\in\mathbb{R}^2} \sup_{\boldsymbol{v}\in\mathbb{R}^3} F(\boldsymbol{h}) = 0,$$

where

$$F(\boldsymbol{h}) = \frac{1}{2}\ \boldsymbol{h}^T \boldsymbol{K} \boldsymbol{h} + \boldsymbol{d}^T \boldsymbol{h}$$

and

$$\boldsymbol{h} = \begin{pmatrix} \boldsymbol{v} \\ \tilde{\boldsymbol{\pi}} \end{pmatrix}, \quad \tilde{\boldsymbol{\pi}} = a_t \boldsymbol{\pi}, \quad \boldsymbol{K} = g \begin{pmatrix} & & & 1 & 0 \\ & \boldsymbol{\Phi} & & 0 & 1 \\ & & & 0 & 0 \\ 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \boldsymbol{\Omega} \end{pmatrix}, \quad \boldsymbol{d} = \begin{pmatrix} 0 \\ 0 \\ C\ \partial_C g \\ (\boldsymbol{\mu} - \boldsymbol{r}) g + \boldsymbol{\omega} C\ \partial_C g \end{pmatrix}.$$

As before, the optimization problem has the unique solution $\boldsymbol{h}^* = -\boldsymbol{K}^{-1}\boldsymbol{d}$, the corresponding critical value is $F(\boldsymbol{h}^*) = -\frac{1}{2}\ \boldsymbol{d}^T \boldsymbol{K}^{-1} \boldsymbol{d}$, and the PDE reduces to

$$\partial_t g + \nu C \partial_C g + \frac{1}{2}\ \eta^2 C^2 \partial_{CC} g - \frac{1}{2}\ \boldsymbol{d}^T \boldsymbol{K}^{-1} \boldsymbol{d} = 0. \tag{37}$$

To solve this equation, we first compute $\boldsymbol{K}^{-1}$. Letting $\boldsymbol{E} \triangleq \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and noting that $\boldsymbol{\Phi}$ and $\boldsymbol{\Omega}$ are symmetric, we have

$$\boldsymbol{K}^{-1} = \frac{1}{g} \left(\begin{array}{c|c} (\boldsymbol{\Phi} - \boldsymbol{E}\boldsymbol{\Omega}^{-1}\boldsymbol{E}^T)^{-1} & -\boldsymbol{\Phi}^{-1}\boldsymbol{E}(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E} + \boldsymbol{\Omega})^{-1} \\ \hline -(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E} + \boldsymbol{\Omega})^{-1}\boldsymbol{E}^T\boldsymbol{\Phi}^{-1} & (-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E} + \boldsymbol{\Omega})^{-1} \end{array} \right) \tag{38}$$

Since the term $-\frac{1}{2}\boldsymbol{d}^T\boldsymbol{K}^{-1}\boldsymbol{d}$ contributes the term $-\frac{1}{2}g\ (\boldsymbol{\mu} - \boldsymbol{r})^T(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E} + \boldsymbol{\Omega})^{-1}(\boldsymbol{\mu} - \boldsymbol{r})$ to equation (37), we make the substitution

$$g(C,t) = G^\beta(\ln \frac{C}{D}, T-t))\ e^{-\frac{1}{2}\overline{\Lambda}^2(T-t)},$$

where

$$\overline{\Lambda}^2 = (\boldsymbol{\mu} - \boldsymbol{r})^T(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E} + \boldsymbol{\Omega})^{-1}(\boldsymbol{\mu} - \boldsymbol{r}).$$

To find the resulting equation for G it is convenient to write

$$\boldsymbol{d} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \boldsymbol{\mu} - \boldsymbol{r} \end{pmatrix} g + \begin{pmatrix} 0 \\ 0 \\ 1 \\ \boldsymbol{\omega} \end{pmatrix} C \partial_C g,$$

Using the notation $\boldsymbol{e} = (0,0,1)^T$, G satisfies the PDE

$$-\partial_\tau G + \left(\nu - \frac{1}{2}\eta^2 - (\boldsymbol{\mu} - \boldsymbol{r})^T \left(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E} + \boldsymbol{\Omega}\right)^{-1} \left(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{e} + \boldsymbol{\omega}\right)\right)\ \partial_x G + \frac{1}{2}\eta^2\ \partial_{xx}G - \\ + \frac{1}{2}\frac{(\partial_x G)^2}{G}\left[(\beta-1)\eta^2 - \beta\ \widetilde{d}^T\boldsymbol{K}^{-1}\widetilde{d}\right] = 0,$$

where $\widetilde{d} = \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\omega} \end{pmatrix}$, and hence the appropriate choice of β is

$$\beta = \frac{1}{1 - \frac{1}{\eta^2}\ \widetilde{d}^T\boldsymbol{K}^{-1}\widetilde{d}}.$$

With this choice of β the PDE for G is linear:

$$\begin{cases} -\partial_\tau G + \widetilde{\nu}\ \partial_x G + \dfrac{1}{2}\eta^2\ \partial_{xx}G = 0, \\ G(0,\tau) = e^{-\frac{1}{2\beta}(\overline{\Lambda}^2 - \overline{\lambda}^2)\tau}, \\ G(x,0) = 1, \end{cases} \tag{39}$$

where this time $\widetilde{\nu} = \nu - \frac{1}{2}\eta^2 - (\boldsymbol{\mu}-\boldsymbol{r})^T\left(-E^T\boldsymbol{\Phi}^{-1}E + \boldsymbol{\Omega}\right)^{-1}\left(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{e} + \boldsymbol{\omega}\right)$. It is pleasing that the same PDE that arises in the case of full certainty (equation (3), section 3) appears here as well – except with λ, Λ replaced by $\overline{\lambda}$, $\overline{\Lambda}$ and with a modified $\tilde{\nu}$ (no longer the MEMM adjusted drift).

As for the distressed regime, we consider the limiting behaviour of the value function for the cases $\boldsymbol{\Phi} \to \infty$ and $\boldsymbol{\Phi} \to 0$. For convenience we restate the values of the parameters in equation (39):

$$\begin{aligned} \widetilde{\nu} &= \nu - \frac{1}{2}\eta^2 - (\boldsymbol{\mu}-\boldsymbol{r})^T\left(-E^T\boldsymbol{\Phi}^{-1}E + \boldsymbol{\Omega}\right)^{-1}\left(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{e} + \boldsymbol{\omega}\right), \\ \beta &= \frac{1}{1 - \frac{1}{\eta^2}\ \widetilde{d}^T\boldsymbol{K}^{-1}\widetilde{d}}, \\ \overline{\Lambda}^2 &= (\boldsymbol{\mu}-\boldsymbol{r})^T(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E} + \boldsymbol{\Omega})^{-1}(\boldsymbol{\mu}-\boldsymbol{r}), \\ \overline{\lambda}^2 &= \frac{(\mu_1 - r)^2\phi}{\phi\sigma_1^2 - 1}. \end{aligned}$$

For $\boldsymbol{\Phi} \to \infty$, $\phi \to \infty$ it is easy to see that these parameters converge to the corresponding parameters in the completely specified case, and hence the same applies for the value function.

For the other extreme case we first explain what we mean by $\boldsymbol{\Phi} \to 0$. Let $\boldsymbol{\Phi} = \varepsilon\boldsymbol{\Phi}_0$, $\phi = \varepsilon\phi_0$, for some fixed $\boldsymbol{\Phi}_0$ and ϕ_0. We will examine the behaviour of the value function as $\varepsilon \to 0$. It is convenient to write $\boldsymbol{\Phi}_0$ and $\boldsymbol{\Phi}_0^{-1}$ in the form

$$\boldsymbol{\Phi}_0 = \left(\begin{array}{c|c} \boldsymbol{\Psi}_0 & \boldsymbol{a} \\ \hline \boldsymbol{a}^T & \varphi_0 \end{array}\right), \quad \boldsymbol{\Phi}_0^{-1} = \left(\begin{array}{c|c} \overline{\boldsymbol{\Psi}}_0^{-1} & \overline{\boldsymbol{a}} \\ \hline \overline{\boldsymbol{a}}^T & \overline{\varphi}_0 \end{array}\right).$$

If we assume that $\boldsymbol{\Phi}_0$ is strictly negative definite, then $\varphi_0 < 0$, $\overline{\varphi}_0 < 0$ and $\boldsymbol{\Psi}_0$, $\overline{\boldsymbol{\Psi}}_0^{-1}$ are strictly negative definite.

Since the entries of $\boldsymbol{K}^{-1}$ frequently appear in the parameters above, we first examine their behaviour for $\varepsilon \to 0$. We immediately see that

$$(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E} + \boldsymbol{\Omega})^{-1} = (-\varepsilon\overline{\boldsymbol{\Psi}}_0^{-1} + \boldsymbol{\Omega})^{-1} \;=\; -\varepsilon\overline{\boldsymbol{\Psi}}_0(I - \varepsilon\boldsymbol{\Omega}\overline{\boldsymbol{\Psi}}_0)^{-1} \;=\; -\varepsilon\overline{\boldsymbol{\Psi}}_0 + O(\varepsilon^2), \quad (40)$$

and

$$\boldsymbol{\Phi}^{-1}\boldsymbol{E}(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E} + \boldsymbol{\Omega}^{-1})^{-1} = \frac{1}{\varepsilon}\begin{pmatrix} \overline{\boldsymbol{\Psi}}_0^{-1} \\ \hline \overline{\boldsymbol{a}}^T \end{pmatrix} \cdot \left[-\varepsilon\overline{\boldsymbol{\Psi}}_0 + O(\varepsilon^2)\right] = \begin{pmatrix} -\boldsymbol{I} \\ \hline -\overline{\boldsymbol{a}}^T\overline{\boldsymbol{\Psi}}_0 \end{pmatrix} + O(\varepsilon). \quad (41)$$

Now we examine the behaviour of the entries of $(\boldsymbol{\Phi} - \boldsymbol{E}\boldsymbol{\Omega}\boldsymbol{E}^T)^{-1}$.

Lemma 2. *The (3,3) entry of* $(\boldsymbol{\Phi} - \boldsymbol{E}\boldsymbol{\Omega}\boldsymbol{E}^T)^{-1}$ *is*

$$\frac{1}{\varepsilon\varphi_0} + O(1), \quad (42)$$

and all other entries of $(\boldsymbol{\Phi} - \boldsymbol{E}\boldsymbol{\Omega}^{-1}\boldsymbol{E}^T)^{-1}$ *approach finite values as* $\varepsilon \to 0$.

Proof. Recall that for a regular square matrix $\boldsymbol{A} = (a_{ij})$, the elements of the inverse matrix $\boldsymbol{A}^{-1} = (\overline{a}_{ij})$ can be computed as follows:

$$\overline{a}_{ij} = \frac{(-1)^{i+j} \cdot [\text{(j,i) minor of } \boldsymbol{A}]}{\det \boldsymbol{A}}.$$

By (i,j) minor of $\boldsymbol{A}$ we mean the determinant of the submatrix obtained from $\boldsymbol{A}$ by deleting the ith row and the jth column.

We have

$$\boldsymbol{\Phi} - \boldsymbol{E}\boldsymbol{\Omega}\boldsymbol{E}^T = \left(\begin{array}{c|c} \varepsilon\boldsymbol{\Psi}_0 - \boldsymbol{\Omega}^{-1} & \varepsilon\boldsymbol{a} \\ \hline \varepsilon\boldsymbol{a}^T & \varepsilon\varphi_0 \end{array}\right),$$

and hence

$$\det(\boldsymbol{\Phi} - \boldsymbol{E}\boldsymbol{\Omega}\boldsymbol{E}^T) = \varepsilon\varphi_0 \cdot \det\boldsymbol{\Omega}^{-1} + O(\varepsilon^2).$$

The (3,3) minor of $\boldsymbol{\Phi} - \boldsymbol{E}\boldsymbol{\Omega}^{-1}\boldsymbol{E}^T$ is

$$\det(\varepsilon\boldsymbol{\Psi}_0 - \boldsymbol{\Omega}^{-1}) = \det\boldsymbol{\Omega}^{-1} + O(\varepsilon),$$

and therefore the (3,3) entry of $(\boldsymbol{\Phi} - \boldsymbol{E}\boldsymbol{\Omega}\boldsymbol{E}^T)^{-1}$ is

$$\frac{1}{\varepsilon\varphi_0} + O(1). \quad (43)$$

All other minors of $\boldsymbol{\Phi} - \boldsymbol{E}\boldsymbol{\Omega}^{-1}\boldsymbol{E}^T$ are at least of order $O(\varepsilon)$, and therefore all other entries of $(\boldsymbol{\Phi} - \boldsymbol{E}\boldsymbol{\Omega}^{-1}\boldsymbol{E}^T)^{-1}$ approach finite values as $\varepsilon \to 0$. □

From the results of (40), (41) it follows that as $\varepsilon \to 0$,

$$\begin{aligned}\widetilde{\nu} &= \nu - \frac{1}{2}\eta^2 - (\boldsymbol{\mu}-\boldsymbol{r})^T\left(-\varepsilon\overline{\boldsymbol{\Psi}}_0 + O(\varepsilon^2)\right)\left(\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{e}+\boldsymbol{\omega}\right)\\ &= \nu - \frac{1}{2}\eta^2 - (\boldsymbol{\mu}-\boldsymbol{r})^T\left(-\varepsilon\overline{\boldsymbol{\Psi}}_0 + O(\varepsilon^2)\right)(\varepsilon^{-1}\overline{\boldsymbol{a}}+\boldsymbol{\omega}) \quad \to \quad \nu - \frac{1}{2}\eta^2 + (\boldsymbol{\mu}-\boldsymbol{r})^T\overline{\boldsymbol{\Psi}}_0\overline{\boldsymbol{a}}\end{aligned}$$

and

$$\begin{aligned}\overline{\Lambda}^2 &= -\varepsilon\ (\boldsymbol{\mu}-\boldsymbol{r})^T\overline{\boldsymbol{\Psi}}_0(\boldsymbol{\mu}-\boldsymbol{r}) + O(\varepsilon^2),\\ \overline{\lambda}^2 &= -\varepsilon\ (\mu_1 - r)^2\ \phi_0 + O(\varepsilon^2).\end{aligned}$$

Furthermore, the only entry of $\boldsymbol{K}^{-1}$ of order $O(\varepsilon^{-1})$ is the $(3,3)$ entry. Hence $\beta \to 0$ $(\varepsilon \to 0)$, and more precisely by lemma 2, we have

$$\beta = -\varepsilon\eta^2\varphi_0 + O(\varepsilon^2).$$

It follows that

$$\frac{1}{\beta}(\overline{\Lambda}^2 - \overline{\lambda}^2) \to \frac{1}{\eta^2\varphi_0}\cdot\left[(\boldsymbol{\mu}-\boldsymbol{r})^T\overline{\boldsymbol{\Psi}}_0(\boldsymbol{\mu}-\boldsymbol{r}) - (\mu_1-r)^2\ \phi_0\right].$$

Therefore the boundary conditions in equation (39) imply that G remains bounded as well as bounded away from 0 (as $\varepsilon \to 0$). Since $\beta \to 0$, it follows that $g \to 1$ and hence $U \to u(we^{r(T-t)})$.

Now we examine how the optimal trading strategy $\boldsymbol{\pi}^*$ and the optimal measure Q^* behave as $\varepsilon \to 0$. Recall that $\begin{pmatrix}\boldsymbol{v}^*\\ \boldsymbol{\pi}^*\end{pmatrix} = -\boldsymbol{K}^{-1}\boldsymbol{d}$. Firstly, it is helpful to notice that

$$\frac{C\partial_C g}{g} = \beta\ \frac{\partial_x G}{G} \to 0 \quad (\varepsilon \to 0),$$

since in the limit as $\epsilon \to 0$: $\partial_x G$ is bounded, G is bounded away from 0 and $\beta \to 0$.

Theorem 3. *For* $\varepsilon \to 0$,

$$\pi_t^{(1),*} \to 0,\ \ \pi_t^{(2),*} \to 0$$

pointwise for all $w,\ P,\ S,\ C,\ t.$

In other words, when there is complete uncertainty, no risky investments are made. This is the analog to the distressed regime's result.

Proof. For $\boldsymbol{\pi}^*$ we get from equation (38)

$$\boldsymbol{\pi}^* = (-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E}+\boldsymbol{\Omega})^{-1}(\boldsymbol{\mu}-\boldsymbol{r}) - (-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E}+\boldsymbol{\Omega})^{-1}\left(\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{e}+\boldsymbol{\omega}\right)\frac{C\partial_C g}{g}$$

and hence

$$\boldsymbol{\pi}^* = -\varepsilon\overline{\boldsymbol{\Psi}}_0(\boldsymbol{\mu} - \boldsymbol{r}) + \varepsilon\overline{\boldsymbol{\Psi}}_0(\varepsilon^{-1}\overline{\boldsymbol{a}} + \boldsymbol{\omega})\frac{C\;\partial_C g}{g} + O(\varepsilon) \quad \to \quad \mathbf{0}.$$

□

For the optimal measure we first focus on the limiting behaviour of $v_t^{(1)}$ and $v_t^{(2)}$.

Theorem 4. *As* $\varepsilon \to 0$*,*

$$v_t^{(1),*} \to r - \mu_1, \quad v_t^{(2),*} \to r - \mu_2$$

pointwise for all w*,* P*,* S*,* C*,* t*.*

Hence under the optimal measure, the drifts of P_t and S_t tend to r as uncertainty increases. This result is also the analog to the distressed regime.

Proof. For $\boldsymbol{v}^*$ we get from equation (38)

$$\begin{aligned}\boldsymbol{v}^* \;&=\; -(\boldsymbol{\Phi} - \boldsymbol{E}\boldsymbol{\Omega}^{-1}\boldsymbol{E}^T)^{-1}\boldsymbol{e}\frac{C\partial_C g}{g} + \boldsymbol{\Phi}^{-1}\boldsymbol{E}(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E} + \boldsymbol{\Omega})^{-1}\,(\boldsymbol{\mu} - \boldsymbol{r}) + \\ &\quad + \boldsymbol{\Phi}^{-1}\boldsymbol{E}(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E} + \boldsymbol{\Omega})^{-1}\,\boldsymbol{\omega}\frac{C\partial_C g}{g}. \end{aligned} \tag{44}$$

Since $\boldsymbol{\Phi}^{-1} = \varepsilon^{-1}\boldsymbol{\Phi}_0^{-1}$, $(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E}+\boldsymbol{\Omega})^{-1} = -\varepsilon\overline{\boldsymbol{\Psi}}_0 + O(\varepsilon^2)$, $\frac{C\partial_C g}{g} \to 0$, the third term approaches 0 as $\varepsilon \to 0$. Furthermore, since by (41),

$$\boldsymbol{\Phi}^{-1}\boldsymbol{E}(-\boldsymbol{E}^T\boldsymbol{\Phi}^{-1}\boldsymbol{E} + \boldsymbol{\Omega}^{-1})^{-1} = \left(\begin{array}{c} -\boldsymbol{I} \\ \hline -\overline{\boldsymbol{a}}^T\overline{\boldsymbol{\Psi}}_0 \end{array}\right) + O(\varepsilon),$$

it is easy to see that the first two components of the second term tend to $r - \mu_1$ and $r - \mu_2$. Therefore we still have to show that the first two components of the first term approach 0. Recalling that $\boldsymbol{e} = (0,0,1)^T$, this follows from the fact that the (1,3) and (2,3) entries of $(\boldsymbol{\Phi} - \boldsymbol{E}\boldsymbol{\Omega}^{-1}\boldsymbol{E}^T)^{-1}$ approach finite values for $\varepsilon \to 0$. □

Theorem 5. *Let* $L(x,t) \triangleq \frac{\partial_x G}{G}$*. Then for* $\varepsilon \to 0$*,*

$$v_t^{(3),*} \to \eta^2 L(x,t) - \overline{\boldsymbol{a}}^T\overline{\boldsymbol{\Psi}}_0(\boldsymbol{\mu} - \boldsymbol{r}).$$

In particular, $v^{(3),*}$ approaches a finite limit for all w, P, S, C, t and not $\pm\infty$, as one might expect.

Proof. We start from equation (44). The third component of the third term on the right hand side is $O(\varepsilon)$, whereas the third component of the second term is $-\overline{\boldsymbol{a}}^T\overline{\boldsymbol{\Psi}}_0(\boldsymbol{\mu}-\boldsymbol{r})+O(\varepsilon)$. Noticing that $\frac{C\partial_C g}{g} = \beta\cdot\frac{\partial_x G}{G}$ and $\beta = -\varepsilon\eta^2\varphi_0 + O(\varepsilon^2)$ and furthermore using the result from lemma 2, we can see that the third component of the first term is

$$\eta^2 L(x,t) + O(\varepsilon).$$

This proves the lemma. □

6.3. Valuation of Credit Derivatives

In this section we examine how prices of defaultable bond and CDS rates change under model misspecification. Since the computations are similar to those in sections 4 and 5, we treat the two kinds of credit derivatives simultaneously.

Under a measure $Q\sim\mathbb{P}$ we assume that the wealth process has the dynamics

$$dW_t = \begin{cases} \left[((\boldsymbol{\mu}-\boldsymbol{r})^T + (v_t^{(1)}, v_t^{(2)}))\boldsymbol{\pi}_t + r\ W_t + \epsilon AF\right]\ dt + \\ \qquad\qquad +\pi_t^{(1)}\sigma_1\ dB_t^{(1)} + \pi_t^{(2)}\sigma_2\ dB_t^{(2)}, & t<\tau_1, \\ \left[(\mu_1 - r + v_t^{(1)})\pi_t^{(1)} + r\ W_t + \epsilon AF\right]\ dt + \pi_t^{(1)}\sigma_1 dB_t^{(1)}, & \tau_1 < t < \tau_2, \\ \left[(\mu_1 - r + v_t^{(1)})\pi_t^{(1)} + r\ W_t\right]\ dt + \pi_t^{(1)}\sigma_1 dB_t^{(1)}, & t > \tau_2, \end{cases}$$

subject to

$$\begin{aligned} W_{\tau_1} &= W_{\tau_1^-} + R_2\cdot\mathbb{I}\{\tau_1 = T\}, \\ W_{\tau_2} &= W_{\tau_2^-} - R_1\cdot\mathbb{I}\{\tau_2 < T\} + R_2\cdot\mathbb{I}\{\tau_2 = T\}. \end{aligned}$$

Here R_1 is a random payment independent of the driving Brownian motions, and R_2 is a deterministic payment. The choice $\epsilon = 0$, $R_1 = RF$ (R=recovery), $R_2 = F$ corresponds to an investment in the defaultable bond, whereas $\epsilon = \pm 1$, $R_1 = \epsilon(1-R)F$, $R_2 = 0$ corresponds to the investment of the seller/buyer of credit protection.

6.3.1. Valuation in the Distressed Regime

In the distressed regime we assume that the only model uncertainty comes from the drift of the tradable asset P. While it would be both desirable and realistic to incorporate uncertainty in the hazard rate κ as well, we choose not do so here due to analytical tractability. In certain cases this can be somewhat justified by assuming that the default probability of the firm can be estimated fairly well from past data (e.g. from firms within the same market sector and of similar size). Furthermore we also assume complete model certainty in the distribution of the recovery rate.

We define the value function $\overline{V}$ as the solution of the optimization problem

$$\overline{V}(w,P,t) = \sup_{\pi\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \mathbb{E}^Q \left[u(W_T) + \frac{1}{2}\int_t^{\tau_d\wedge T} \overline{V}(W_s,P_s,s)\ \phi\ v_s^2\ ds\ + \right.$$
$$\left. +\ \frac{1}{2}\int_{\tau_d\wedge T}^{T} V(W_s,s)\ \phi\ v_s^2\ ds\ \middle|\ W_s = w,\ P_s = P \right].$$

This definition is the analog of the definition of V in section 6.2.1. The corresponding HJB equation for $\overline{V}$ is

$$\begin{cases} \partial_t \overline{V} + \epsilon A\ \partial_w \overline{V} + \sup_{\pi\in\mathbb{R}} \inf_{v\in\mathbb{R}} \left\{ \mathcal{L}^{\pi,v}\ \overline{V} + \frac{1}{2}\ \overline{V}\phi v^2 \right\} + \\ \qquad +\kappa \left[V(w-\epsilon\widetilde{R}_1,\ t) - \overline{V} \right] = 0\ , \\ \qquad V(w,T) = u(w+R_2)\ , \quad w\in\mathbb{R}\ , \end{cases} \tag{45}$$

where

$$\mathcal{L}^{\pi,v}\overline{V} = (rw + (\mu_1 - r + v)\pi)\ \partial_w \overline{V} + \frac{1}{2}\pi^2\sigma_1^2\ \partial_{ww}\overline{V}.$$

Factoring out wealth by writing $\overline{V}(w,t) = u(we^{r(T-t)})\ \overline{g}(t)$ leads to the following ODE for $\overline{g}$:

$$\begin{cases} \overline{g}' + \inf_{\pi\in\mathbb{R}} \sup_{v\in\mathbb{R}} \left\{ [\epsilon A + (\mu_1 - r + v)\pi]\ a_t\overline{g} + \frac{1}{2}\sigma_1^2\pi^2 a_t^2 \overline{g} + \frac{1}{2}\overline{g}\ \phi v^2 \right\} + \\ \qquad +\kappa\ \left[e^{-\frac{1}{2}\overline{\lambda}^2(T-t) - \epsilon\widetilde{R}_1 a_t} - \overline{g} \right] = 0, \\ \qquad \overline{g}(T) = e^{-\gamma R_2}, \end{cases}$$

Carrying out the optimization on the left hand side like in section 6.2.1 leads to

$$\begin{cases} \overline{g}' - \left(\kappa + \frac{1}{2}\overline{\lambda}^2 - \epsilon A a_t \right)\ \overline{g} + \kappa e^{-\frac{1}{2}\overline{\lambda}^2(T-t) - \epsilon\widetilde{R}_1 a_t} = 0, \\ \qquad \overline{g}(T) = e^{-\gamma R_2}, \end{cases}$$

which can be solved in the usual way. Things become easy by noticing that for the defaultable bond ($\epsilon = 0$, $R_1 = RF$, $R_2 = F$) the equation above is the same as equation (7), its analog for the completely specified case, only with λ replaced by $\overline{\lambda}$. Similarly, for the CDS we get the same equation as (15), where again λ is replaced by $\overline{\lambda}$.

It is interesting to see that since the indifference price of the bond as well as the indifference CDS rates do not depend on $\overline{\lambda}$, they are exactly the same as in the case with complete model certainty. This fact makes it even more interesting to check whether and how uncertainty on κ influences the prices of credit derivatives. This however is material for future research.

6.3.2. Valuation in the Healthy Regime

In analogy to the previous sections we define the value function $\overline{U}(w, P, S, C, t)$ to be the solution of the optimization problem

$$\overline{U}(w,P,S,C,t) = \sup_{\boldsymbol{\pi}\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \mathbb{E}^Q \Bigg[u(W_T) + \frac{1}{2}\int_t^{\tau_h\wedge T} \overline{U}(W_s,P_s,S_s,C_s,s)\ \boldsymbol{v}_s^T \boldsymbol{\Phi} \boldsymbol{v}_s\ ds + \\ + \frac{1}{2}\int_{\tau_h\wedge T}^{\tau_d\wedge T} \overline{V}(W_s,s)\ \phi v_s^2\ ds\ + \frac{1}{2}\int_{\tau_d\wedge T}^{T} V(W_s,s)\ \phi v_s^2\ ds\ \Bigg| \\ \Bigg|\ W_t = w,\ P_t = P,\ S_t = S,\ C_t = C \Bigg]. \quad (46)$$

Here $\boldsymbol{\Phi}$ is the same matrix as in the investment problem in section 6.2.2.

Assuming that $\overline{U}$ is independent of P and S, the corresponding HJB equation is

$$\begin{cases} \partial_t \overline{U} + \epsilon A\ \partial_w \overline{U} + \sup\limits_{\boldsymbol{\pi}\in\mathbb{R}^2} \inf\limits_{\boldsymbol{v}\in\mathbb{R}^3} \mathcal{L}^{\boldsymbol{\pi},\boldsymbol{v}} \overline{U} = 0, \\ \overline{U}(w,C,T) = u(w+R_2) \\ \overline{U}(w,D,t) = \overline{V}(w,t). \end{cases}$$

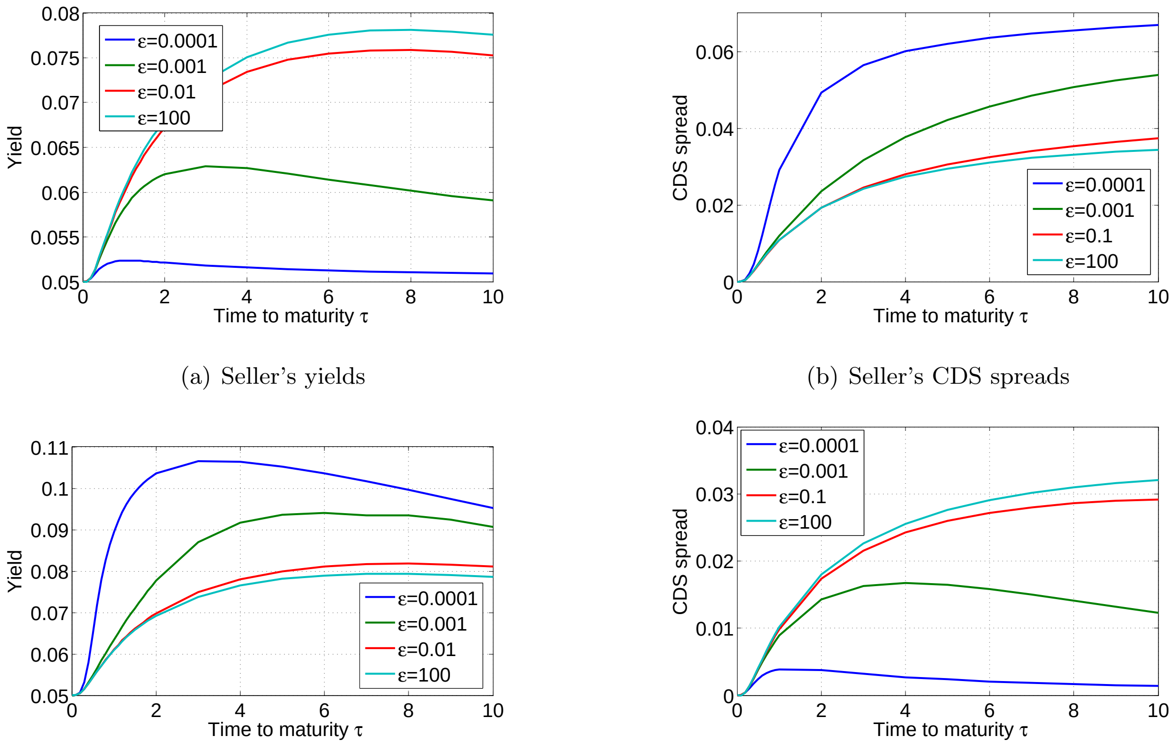
The operator $\mathcal{L}^{\boldsymbol{\pi},\boldsymbol{v}}$ is the same as in (35). As in section 5, we make an ansatz of the form $\overline{U}(w,C,t) = u(we^{r(T-t)})\ e^{\psi(T-t)}\ \overline{h}(C,t)$ with $\psi(\tau) = -\epsilon\gamma\frac{A}{r}e^{r\tau}$ leading to the PDE

$$\begin{cases} \partial_t \overline{h} + \inf\limits_{\boldsymbol{\pi}\in\mathbb{R}^2} \sup\limits_{\boldsymbol{v}\in\mathbb{R}^3} \Big\{ \boldsymbol{\pi}^T \left(\boldsymbol{\mu} - \boldsymbol{r} + (v^{(1)}, v^{(2)})^T \right) a_t \overline{h} + \frac{1}{2}\boldsymbol{\pi}^T \boldsymbol{\Omega} \boldsymbol{\pi}\ a_t^2 \overline{h} + \\ \quad +(\nu + v^{(3)})C\ \partial_C \overline{h} + \boldsymbol{\pi}^T \boldsymbol{\omega}\ a_t C\ \partial_C \overline{h} + \frac{1}{2}\eta^2 C^2\ \partial_{CC}\overline{h} + \frac{1}{2}\overline{h}\ \boldsymbol{v}^T \boldsymbol{\Phi} \boldsymbol{v} \Big\} = 0, \\ \overline{h}(C,T) = e^{-\gamma R_2}, \\ \overline{h}(D,t) = e^{-\psi(T-\tau)} \cdot \overline{g}(t). \end{cases}$$

Note that this equation is the same as (36), only with different boundary conditions. To solve it, we can therefore make an analogous substitution to get a linear equation. More specifically, we let $\overline{h}(C,t) = \overline{G}^{\beta}(\ln\frac{C}{D},\ T-t) \cdot e^{-\frac{1}{2}\overline{\Lambda}^2(T-t)}$ with $\overline{\Lambda}$, β as in equation (40) to get

$$\begin{cases} -\partial_\tau \overline{G} + \widetilde{\nu}\ \partial_x \overline{G} + \frac{1}{2}\eta^2\ \partial_{xx}\overline{G} = 0, \\ \overline{G}(0,\tau) = e^{-\frac{1}{2\beta}(\overline{\Lambda}^2 - \overline{\lambda}^2)\tau - \frac{\psi(\tau)}{\beta}}, \\ \overline{G}(x,0) = e^{-\gamma R_2/\beta} \end{cases}$$

with $\widetilde{\nu}$ as in (40). As in the distressed regime, we find the same equation as in the fully specified cases in sections 4 and 5, only now with the new parameters $\widetilde{\nu}$ and β, and with λ and Λ replaced by $\overline{\lambda}$ and $\overline{\Lambda}$. Once again the MEMM adjusted drift that appeared in the fully specified case disappears and is replaced by a model uncertainty version.



(a) Seller's yields

(b) Seller's CDS spreads

(c) Buyer's Yields

(d) Buyer's CDS spreads

Figure 5. The effect of model misspecification on yields and CDS spreads. The parameters for the measure reference measure $\mathbb{P}$ are as in Figure 2. Initial CWI was set at $C_0 = 1.05$. The uncertainty scalar $\phi = \varepsilon\phi_0$ and matrix $\mathbf{\Phi} = \varepsilon\mathbf{\Phi}_0$ with ϕ_0 and $\mathbf{\Phi}_0$ reported in Appendix .

As for the completely specified case, we plot the bond yields as well as the seller's and buyer's CDS spreads. For the following plots we have made specific choices for the negative scalar ϕ_0 and the negative definite matrix $\mathbf{\Phi}_0$. This choice is explained in detail at the end of Appendix C. We then let $\phi = \varepsilon\phi_0$, $\mathbf{\Phi} = \varepsilon\mathbf{\Phi}_0$ for different values of ε. In Figure 5 we plot the resulting yields and buyer/seller CDS spreads as the uncertainty varies. The case $\varepsilon = 100$ corresponds to almost complete model certainty. For this case we get almost the same yields and CDS rates as in the completely specified case (for $C_0 = 1.05$). With increasing model uncertainty we observe that the bond yields and seller's CDS rates increase, while the buyer's CDS rates decrease. This is what we intuitively expect. Furthermore, there appears to be more flexibility in the shapes of the resulting CDS spreads when compared with those in the fully specified case.

7. Conclusions

In this article we introduced a new hybrid model for default occurring in two stages. Firstly, the perceived health of the company, modeled as a GBM, must drop below a critical level leaving the firm in a state of distress – this is the structural part of the model. Once distressed, the firm defaults at an exponential time, viewed as the first arrival of an independent Poisson process – providing the intensity base of the model. The perceived health is not a traded asset, however, it is correlated to the firm's equity and a wide-base (non-defaultable) index. Since the market is incomplete, we utilize certainty equivalence to value credit derivatives written on the firm. When the intensity of the Poisson process driving default in the distressed regime tends to infinity, the barrier for the perceived health behaves as a default barrier and our model reduces to that of Leung, Sircar, and Zariphopoulou (2008). However, in real world settings default will not occur instantly at this point. We succeed in deriving closed form, classical, solutions to the optimization in the absence and presence of the credit risk and hence are able to determine the certainty equivalent risky yields and CDS spreads.

Given that estimating model parameters from limited data, particularly for the perceived health process, we also develop an uncertain parameter formulation of our model and valuation framework. Motivated by Maenhout (2004) and the robust optimization literature, we introduce a value function which maximizes over admissible trading strategies while minimizing over equivalent measures subject to a scaled entropic penalty. We succeed in obtaining classical solutions to this problem and determine risky yields and CDS spreads subject to parameter uncertainty. All of the observed behaviour is consistent with intuition and we find that parameter uncertainty allows for a wider range of term structures.

There are several doors remaining open for further study. One clear direction is to incorporate multiple firms. The main difficulty here is that a high dimensional first passage time problem must be solved. However, if the portfolio has enough symmetry and if the perceived health factors are viewed as uncorrelated the dimensionality reduces considerably. Such an approach, in the purely intensity based model, was explored by Sircar and Zariphopoulou (2008) where the authors demonstrate that the effective correlation can be introduced through risk-aversion alone – without the necessity of correlating the underlying intensity processes. Another interesting direction, which we have already begun exploring, is to randomizing the boundary below which the firm becomes distressed. This will allow us to introduce a gap in the very term spreads in the healthy regime (recall that a gad appears in the distressed since this regime corresponds to an intensity model). In the complete market setting this has already been addressed, and it is well known that introducing randomness in the default boundary allows the structural model to inherit intensity model features. A third direction is to allow the firm to recover from the distressed regime. This is a more difficult problem than the one studied here, since now the healthy and distressed regimes will be coupled not only through the boundary condition along the boundary but also through the source terms in the HJB equations. In all, this arena of combining structural and intensity models and incorporating risk-aversion together with parameter uncertainty is a rich area full of interesting and worthwhile problems.

A. Verification Theorems

A.1. For the Completely Specified Case

We assume the same model as in section 2. Let $\tau_1 \triangleq \tau_h \wedge T$ and $\tau_2 \triangleq \tau_d \wedge T$. Furthermore we assume that the wealth process W_t has the following dynamics:

$$dW_t = \begin{cases} \Big[(\boldsymbol{\mu} - \boldsymbol{r})^T \boldsymbol{\pi}_t + r\ W_t + \epsilon A\Big]\ dt + \pi_t^{(1)}\sigma_1\ dB_t^{(1)} + \pi_t^{(2)}\sigma_2\ dB_t^{(2)}, & t < \tau_1, \\ \Big[(\mu_1 - r)\pi_t^{(1)} + r\ W_t + \epsilon A\Big]\ dt + \pi_t^{(1)}\sigma_1 dB_t^{(1)}, & \tau_1 < t < \tau_2, \\ \Big[(\mu_1 - r)\pi_t^{(1)} + r\ W_t\Big]\ dt + \pi_t^{(1)}\sigma_1 dB_t^{(1)}, & t > \tau_2, \end{cases}$$

subject to

$$\begin{aligned} W_{\tau_1} &= W_{\tau_1^-} + R_2 \cdot \mathbb{I}\{\tau_1 = T\}, \\ W_{\tau_2} &= W_{\tau_2^-} - \epsilon R_1 \cdot \mathbb{I}\{\tau_2 < T\} + R_2 \cdot \mathbb{I}\{\tau_2 = T\}. \end{aligned}$$

A is a constant and corresponds to a continuous payment made ($\epsilon = -1$) or received ($\epsilon = +1$) up to time τ_d, or making/receiving no continuous payments at all ($\epsilon = 0$). R_1 is a random variable independent of the processes driving the wealth process and corresponds to a payment made/received at time τ_d. Finally, R_2 is a constant and corresponds to a potential payoff at maturity T. For $t \in [0, T]$ we define

$$\begin{aligned} U_1(w, P, S, C, t) &\triangleq \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[\ u(W_T) \mid W_t = w,\ P_t = P,\ S_t = S,\ C_t = C,\ \tau_h \le t < \tau_d\ \right] \\ U_2(w, P, S, C, t) &\triangleq \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[\ u(W_T) \mid W_t = w,\ P_t = P,\ S_t = S,\ C_t = C,\ t < \tau_h\ \right]. \end{aligned}$$

Wealth is independent of P and S for $t \in [0, \tau_h)$ and independent of P, S and C for $t \ge \tau_h$. Hence $U_2 = U_2(w, C, t)$ and $U_1 = U_1(w, t)$. We consider the corresponding HJB equation for U_1,

$$\begin{cases} \partial_t U_1 + \sup_{\pi \in \mathbb{R}} \mathcal{L}_1^\pi U_1 + \kappa \left[V(w + \widetilde{R}_1 F,\ t) - U_1\right] = 0, \\ \qquad\qquad U_1(w, T) = u(w + R_2), \quad w \in \mathbb{R} \end{cases} \tag{47}$$

with

$$\mathcal{L}_1^\pi U_1 = [rw + \epsilon A + (\mu_1 - r)\pi]\ \partial_w U_1 + \frac{1}{2}\sigma_1^2 \pi^2\ \partial_{ww} U_1$$

and

$$\widetilde{R}_1 = \frac{1}{\gamma\epsilon e^{r(T-t)}} \log \mathbb{E}\ e^{\gamma\epsilon R_1 e^{r(T-t)}},$$

as well as the HJB equation for U_2,

$$\begin{cases} \partial_t U_2 + \sup_{\boldsymbol{\pi} \in \mathbb{R}^2} \mathcal{L}_2^{\boldsymbol{\pi}} U_2 = 0, \\ \qquad U_2(w, C, T) = u(w + R_2), \quad w \in \mathbb{R},\ \ C > D, \\ \qquad U_2(w, D, t) = U_1(w, t), \quad w \in \mathbb{R},\ \ t \in [0, T], \end{cases} \tag{48}$$

with

$$\begin{aligned} \mathcal{L}_2^{\boldsymbol{\pi}} U_2 \;\; &= \;\; (rw + \epsilon A)\ \partial_w U_2 + \nu C\ \partial_C U_2 + \frac{1}{2}\eta^2 C^2\ \partial_{CC} U_2 + \\ &\quad + \frac{1}{2}\boldsymbol{\pi}^T \boldsymbol{\Omega} \boldsymbol{\pi}\ \partial_{ww} U_2 + \boldsymbol{\pi}^T \left[(\boldsymbol{\mu} - \boldsymbol{r})\ \partial_w U_2 + \boldsymbol{\omega} C\ \partial_{wC} U_2 \right]\ , \end{aligned}$$

As in previous sections, V is the value function for the standard Merton investment problem and π_t^M is the corresponding optimal investment strategy. We demand that the functions in the following verification theorem be *sufficiently integrable* in the sense that all the stochastic integrals in the proofs exist and that we can exchange the order of taking expectations and limits with respect to time, where necessary.

Theorem 6. *(a) Suppose there exists a function $\widetilde{U}_1 = \widetilde{U}_1(w, t)$ which is a solution of (47) and which is sufficiently integrable. Furthermore, suppose that for each $(w, t) \in \mathbb{R} \times [0, T]$ there exists $\pi^* = \pi^*(w, t) \in \mathbb{R}$ such that*

$$\mathcal{L}_1^{\pi^*} \widetilde{U}_1 = \sup_{\pi \in \mathbb{R}} \mathcal{L}_1^{\pi} \widetilde{U}_1. \tag{49}$$

Assume that the trading strategy $\overline{\pi}_t$ defined by

$$\overline{\pi}_t = \begin{cases} \pi^*(w_t, t), & \tau_h \le t < \tau_d, \\ \pi_t^M, & t \ge \tau_d \end{cases}$$

is admissible. Then $U_1 = \widetilde{U}_1$ for $(w, t) \in \mathbb{R} \times [0, T]$, and $\overline{\pi}$ is an optimal strategy, i.e. $U_1(w, t) = \mathbb{E}_t \left[u\left(W_T^{\overline{\pi}} \right) \right]$.

(b) Suppose there exists a function $\widetilde{U}_2 = \widetilde{U}_2(w, C, t)$ which solves (48) and which is sufficiently integrable. Suppose that for each $(w, C, t) \in \mathbb{R} \times (D, \infty) \times [0, T]$ there exists $\boldsymbol{\pi}^{**} = \boldsymbol{\pi}^{**}(w, C, t) \in \mathbb{R}^2$ such that

$$\mathcal{L}_2^{\boldsymbol{\pi}^{**}} \widetilde{U}_2 = \sup_{\boldsymbol{\pi} \in \mathbb{R}^2} \mathcal{L}_2^{\boldsymbol{\pi}} \widetilde{U}_2. \tag{50}$$

Assume that the trading strategy defined by

$$\overline{\boldsymbol{\pi}}_t = \begin{cases} \boldsymbol{\pi}^{**}(w_t, C_t, t), & t < \tau_h, \\ (\overline{\pi}_t, 0), & t \ge \tau_h, \end{cases}$$

is admissible. Then $U_2 = \widetilde{U}_2$ *for* $(w, C, t) \in \mathbb{R} \times (D, \infty) \times [0, T]$*, and* $\overline{\boldsymbol{\pi}}$ *is an optimal strategy, i.e.* $U_2(w, C, t) = \mathbb{E}_t \left[u \left(W_T^{\overline{\boldsymbol{\pi}}} \right) \right]$.

Proof. We begin with proving part (a). Let $\widetilde{U}_1$ be as in the theorem, and let π be any admissible trading strategy. Writing τ instead of τ_2 for convenience, Ito's lemma yields

$$\widetilde{U}_1(W_{\tau^-}, \tau^-) = \widetilde{U}_1(w, t) + \int_t^\tau (\partial_t \widetilde{U}_1 + \mathcal{L}_1^\pi \widetilde{U}_1)\ ds + \int_t^\tau \pi_s \sigma_1\ \partial_w \widetilde{U}_1\ dB_s^{(1)}.$$

Since π is an arbitrary admissible strategy and noting that $\widetilde{U}_1$ solves the HJB equation, we always have $\partial_t \widetilde{U}_1 + \mathcal{L}_1^\pi \widetilde{U}_1 \leq -\kappa \left[V(w + \widetilde{R}_1, t) - \widetilde{U}_1 \right]$. Taking expectations on both sides makes the stochastic integral on the right hand side vanish and therefore yields

$$\widetilde{U}_1(w, t) \geq \mathbb{E}_t\ \widetilde{U}_1(W_{\tau^-}, \tau^-) + \mathbb{E}_t\ \int_t^\tau \kappa \left[V(W_s + \widetilde{R}_1, s) - \widetilde{U}_1(W_s, s) \right]\ ds.$$

Here we use the notation $\mathbb{E}_t$ to abreviate the conditioning $W_t = w$. Assuming that we may interchange the order of taking limits in time and taking expectations, a short calculation shows that

$$\mathbb{E}_t \int_t^\tau \kappa \left[V(W_s + \widetilde{R}_1, s) - \widetilde{U}_1(W_s, s) \right]\ ds = \mathbb{E}_t \left[\left(V(W_{\tau^-} + \widetilde{R}_1, \tau) - \widetilde{U}_1(W_{\tau^-}, \tau^-) \right) \cdot \mathbb{I}\{\tau_d \leq T\} \right],$$

so we get

$$\widetilde{U}_1(w, t) \geq \mathbb{E}_t \left[\widetilde{U}_1(W_{\tau^-}, \tau^-) \cdot \mathbb{I}\{\tau_d > T\} \right] + \mathbb{E}_t \left[V(W_{\tau^-} + \widetilde{R}_1, \tau^-) \cdot \mathbb{I}\{\tau_d \leq T\} \right]. \qquad (51)$$

If $\tau_d > T$, then

$$\widetilde{U}_1(W_{\tau^-}, \tau^-) = \widetilde{U}_1(W_{T^-}, T^-) = u(W_{T^-} + R_2) = u(W_T),$$

and if $\tau_d \leq T$, then obviously $V(W_{\tau^-} + \widetilde{R}_1, \tau^-) = V(W_\tau, \tau) \geq \mathbb{E}_\tau u(W_T)$. Therefore, (51) implies $\widetilde{U}_1(w, t) \geq \mathbb{E}_t\ [u(W_T)]$. Since this holds for any admissible strategy, it follows that

$$\widetilde{U}_1(w, t) \geq U_1(w, T).$$

On the other hand, for $\pi = \overline{\pi}$ we get equality everywhere, and hence $\widetilde{U}_1(w, t) = U_1(w, t)$.

Now we prove part (b). Let $\overline{\boldsymbol{\pi}}_t$ and $\widetilde{U}_2$ as in the theorem, and let $\boldsymbol{\pi} \in \mathcal{A}$ be an arbitrary admissible strategy. Writing τ instead of τ_1 this time, we get from Ito's lemma

$$\widetilde{U}_2(W_\tau, C_\tau, \tau) = \widetilde{U}_2(w, C, t) + \int_t^\tau \left(\partial_t \widetilde{U}_2 + \mathcal{L}_2^{\boldsymbol{\pi}} \widetilde{U}_2 \right)\ ds + \int_t^\tau \nabla_{(w,C)} \widetilde{U}_2(W_s, C_s, s)\ d\boldsymbol{B}_s.$$

Since $\boldsymbol{\pi} \in \mathcal{A}$ is an arbitrary strategy, we always have $\partial_t \widetilde{U}_2 + \mathcal{L}_2^{\boldsymbol{\pi}} \widetilde{U}_2 \leq 0$, so that taking expectations on both sides yields $\widetilde{U}_2(w, C, t) \geq \mathbb{E}_t \, \widetilde{U}_2(W_\tau, C_\tau, \tau)$. Making use of the fact that $\widetilde{U}_2$ is a solution of (48), we get

$$
\begin{aligned}
\widetilde{U}_2(W_\tau, C_\tau, \tau) &= \widetilde{U}_2(W_T, C_T, T) \cdot \mathbb{I}\{\tau_h > T\} + \widetilde{U}_2(W_\tau, C_\tau, \tau) \cdot \mathbb{I}\{\tau_h \leq T\} \\
&= u(W_T) \cdot \mathbb{I}\{\tau_h > T\} + \widetilde{U}_2(W_\tau, D, \tau) \cdot \mathbb{I}\{\tau_h \leq T\} \\
&= u(W_T) \cdot \mathbb{I}\{\tau_h > T\} + U_1(W_\tau, \tau) \cdot \mathbb{I}\{\tau_h \leq T\}.
\end{aligned}
$$

Taking expectations on both sides and using the definition of U_1 leads to

$$
\begin{aligned}
\mathbb{E}_t \, \widetilde{U}_2(W_\tau, C_\tau, \tau) &\geq \mathbb{E}_t \left[u(W_T) \cdot \mathbb{I}\{\tau_h > T\} \right] + \mathbb{E}_t \left[u(W_T) \cdot \mathbb{I}\{\tau_h \leq T\} \right] \\
&= \mathbb{E}_t u(W_T),
\end{aligned}
$$

and hence $\widetilde{U}_2(w, C, t) \geq \mathbb{E}_t u(W_T)$. Since $\boldsymbol{\pi}$ is an arbitrary admissible strategy, this implies $\widetilde{U}_2(w, C, t) \geq U_2(w, C, t)$. Now let $\boldsymbol{\pi} = \overline{\boldsymbol{\pi}}$. By the same argument we get equality in all the steps above, and therefore $\widetilde{U}_2 = U_2$.

□

A.2. For the Misspecified Case

We assume the same model as in sections 6.2 and 6.3. Let $\tau_1 \triangleq \tau_h \wedge T$ and $\tau_2 \triangleq \tau_d \wedge T$. Furthermore we assume that under a measure $Q \sim \mathbb{P}$, the dynamics of P_t, S_t, C_t are

$$
\begin{aligned}
dP_t &= P_t \left[\left(\mu_1 + v_t^{(1)} \right) \, dt + \sigma_1 \, dB_t^{(1)} \right], \\
dS_t &= S_t \left[\left(\mu_2 + v_t^{(2)} \right) \, dt + \sigma_2 \, dB_t^{(2)} \right], \\
dC_t &= C_t \left[\left(\nu + v_t^{(3)} \right) \, dt + \eta \, dB_t^{(3)} \right],
\end{aligned}
$$

and the wealth process W_t has the dynamics

$$
dW_t = \begin{cases}
\left[\left((\boldsymbol{\mu} - \boldsymbol{r})^T + (v_t^{(1)}, v_t^{(2)}) \right) \boldsymbol{\pi}_t + r \, W_t + \epsilon A \right] \, dt + \pi_t^{(1)} \sigma_1 \, dB_t^{(1)} + \pi_t^{(2)} \sigma_2 \, dB_t^{(2)}, & t < \tau_1, \\
\left[\left(\mu_1 - r + v_t^{(1)} \right) \pi_t^{(1)} + r \, W_t + \epsilon A \right] \, dt + \pi_t^{(1)} \sigma_1 \, dB_t^{(1)}, & \tau_1 < t < \tau_2, \\
\left[\left(\mu_1 - r + v_t^{(1)} \right) \pi_t^{(1)} + r \, W_t \right] \, dt + \pi_t^{(1)} \sigma_1 \, dB_t^{(1)}, & t > \tau_2,
\end{cases}
$$

subject to

$$
\begin{aligned}
W_{\tau_1} &= W_{\tau_1^-} + R_2 \cdot \mathbb{I}\{\tau_1 = T\}, \\
W_{\tau_2} &= W_{\tau_2^-} - \epsilon R_1 \cdot \mathbb{I}\{\tau_2 < T\} + R_2 \cdot \mathbb{I}\{\tau_2 = T\}.
\end{aligned}
$$

As in previous sections, we often write v_t instead of $v_t^{(1)}$. The interpretation of A, R_1, R_2 is the same as section 6.3. The value functions U_1, U_2 and U_3 are defined as solutions of the equations

$$U_1(w,P,t) = \sup_{\pi\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \mathbb{E}^Q \left[u(W_T) + \frac{1}{2}\int_t^T U_1(W_s,P_s,s)\ \phi\ v_s^2\ ds \ \middle|\ W_t = w,\ P_t = P,\ t > \tau_d \right], \tag{52}$$

$$U_2(w,P,t) = \sup_{\pi\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \mathbb{E}^Q \left[u(W_T) + \frac{1}{2}\int_t^{\tau_2} U_2(W_s,P_s,s)\ \phi\ v_s^2\ ds\ + \frac{1}{2}\int_{\tau_2}^T U_1(w_s,P_s,s)\ \phi\ v_s^2\ ds \ \middle|\ W_s = w,\ P_s = P,\ \tau_h \le t < \tau_d \right], \tag{53}$$

$$U_3(w,P,S,C,t) = \sup_{\boldsymbol{\pi}\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \mathbb{E}^Q \left[u(W_T) + \frac{1}{2}\int_t^{\tau_1} U_3(W_s,P_s,S_s,C_s,s)\ \boldsymbol{v}_s^T\boldsymbol{\Phi}\boldsymbol{v}_s\ ds + \frac{1}{2}\int_{\tau_1}^{\tau_2} U_2(W_s,P_s,s)\ \phi\ v_s^2\ ds\ + \frac{1}{2}\int_{\tau_2}^T U_1(w_s,P_s,s)\ \phi\ v_s^2\ ds \ \middle|\ W_t = w,\ P_t = P,\ S_t = S,\ C_t = C,\ t < \tau_h \right]. \tag{54}$$

Here $\phi < 0$ is a constant and $\boldsymbol{\Phi} \in \mathbb{R}_{3\times 3}$ is a negative definite matrix.

Corresponding to U_1, U_2, U_3 we consider the HJB equations

$$\begin{cases} \partial_t U_1 + \sup_{\pi\in\mathbb{R}} \inf_{v\in\mathbb{R}} \left\{ \mathcal{L}_1^{\pi,v}\ U_1 + \frac{1}{2}U_1\ \phi\ v^2 \right\} = 0\ , \\ U_1(w,T) = u(w)\ , \quad w\in\mathbb{R}\ , \end{cases} \tag{55}$$

where

$$\mathcal{L}_1^{\pi,v}U_1 = (rw + (\mu_1 - r + v)\pi)\ \partial_w U_1 + \frac{1}{2}\pi^2\sigma_1^2\ \partial_{ww}U_1,$$

$$\begin{cases} \partial_t U_2 + \epsilon A\ \partial_w U_2 + \sup_{\pi\in\mathbb{R}} \inf_{v\in\mathbb{R}} \left\{ \mathcal{L}_2^{\pi,v}\ U_2 + \frac{1}{2}U_2\ \phi\ v^2 \right\} + \\ \qquad + \kappa \left[U_1(w+\widetilde{R}_1,t) - U_2 \right] = 0\ , \\ U_2(w,T) = u(w+R_2)\ , \quad w\in\mathbb{R}\ , \end{cases} \tag{56}$$

where

$$\mathcal{L}_2^{\pi,v}U_2 = (rw + (\mu_1 - r + v)\pi)\ \partial_w U_2 + \frac{1}{2}\pi^2\sigma_1^2\ \partial_{ww}U_2, \quad \widetilde{R}_1 = \frac{1}{\gamma\epsilon e^{r(T-t)}}\log\mathbb{E}\ e^{\gamma\epsilon R_1 e^{r(T-t)}},$$

and

$$\begin{cases} \partial_t U_3 + \epsilon A \partial_w U_3 + \sup_{\boldsymbol{\pi}\in\mathbb{R}^2} \inf_{\boldsymbol{v}\in\mathbb{R}^3} \mathcal{L}_3^{\boldsymbol{\pi},\boldsymbol{v}} U_3 = 0, \\ \qquad U_3(w,C,T) = u(w+R_2), \quad w\in\mathbb{R},\ C>D, \\ \qquad U_3(w,D,t) = U_2(w,t), \quad w\in\mathbb{R},\ t\in[0,T], \end{cases} \tag{57}$$

with

$$\begin{aligned} \mathcal{L}_3^{\boldsymbol{\pi},\boldsymbol{v}} U_3 \;\; = \;\; & \left[rw + \boldsymbol{\pi}^T \left((\boldsymbol{\mu} - \boldsymbol{r}) + (v^{(1)}, v^{(2)})^T \right) \right] \partial_w U_3 + \frac{1}{2} \boldsymbol{\pi}^T \boldsymbol{\Omega} \boldsymbol{\pi}\ \partial_{ww} U_3 + (\nu_1 + v^{(3)}) C\ \partial_C U_3 + \\ & + \boldsymbol{\pi}^T \boldsymbol{\omega} C\ \partial_{wC} U_3 + \frac{1}{2} \eta^2 C^2\ \partial_{CC} U_3 + \frac{1}{2}\ U_3\ \boldsymbol{v}^T \boldsymbol{\Phi} \boldsymbol{v}. \end{aligned}$$

Let $\overline{\boldsymbol{\Omega}}$ be the variance-covariance matrix of P_t, S_t, C_t (in contrast to $\boldsymbol{\Omega}$, the variance-covariance matrix of P_t, S_t). In analogy to the completely specified case, the following verification theorem holds:

Theorem 7. *(a) Suppose that there exists a function $\widetilde{U}_1 = \widetilde{U}_1(w,t)$ that is a solution of (55). Furthermore, suppose that for each $(w,t) \in \mathbb{R}\times[0,T]$ there exist $\pi^M = \pi^M(w,t) \in \mathbb{R}$, $v^M = v^M(w,t) \in \mathbb{R}$ such that*

$$\mathcal{L}_1^{\pi^M, v^M} \widetilde{U}_1 = \sup_{\pi\in\mathbb{R}} \inf_{v\in\mathbb{R}} \mathcal{L}_1^{\pi,v} \widetilde{U}_1. \tag{58}$$

Assume that the trading strategy defined by (58) is admissible and that there exists a measure $Q^M \sim \mathbb{P}$ under which P has the dynamics

$$dP_t = P_t\ \left[(\mu_1 + v_t^M)\ dt + \sigma_1\ dB_t^{(1)} \right].$$

Then $\widetilde{U}_1(w,t)$ is a solution of equation (52) for $(w,P,t) \in \mathbb{R}\times[0,\infty)\times[0,T]$.

(b) Suppose there exists a function $\widetilde{U}_2 = \widetilde{U}_2(w,t)$ that is a solution of (56). Furthermore, suppose that for each $(w,t) \in \mathbb{R}\times[0,T]$ there exist $\pi^* = \pi^*(w,t) \in \mathbb{R}$, $v^* = v^*(w,t) \in \mathbb{R}$ such that

$$\mathcal{L}_2^{\pi^*, v^*} \widetilde{U}_2 = \sup_{\pi\in\mathbb{R}} \inf_{v\in\mathbb{R}} \mathcal{L}_2^{\pi,v} \widetilde{U}_2. \tag{59}$$

Assume that the trading strategy $\overline{\pi_t}$ defined by

$$\overline{\pi}_t = \begin{cases} \pi^*(w_t,t), & \tau_h \le t < \tau_d, \\ \pi_t^M, & t \ge \tau_d \end{cases}$$

is admissible and that there exists a measure $Q^* \sim \mathbb{P}$ under which P_t has the dynamics

$$dP_t = P_t \left[(\mu_1 + \overline{v}_t)\ dt + \sigma_1\ dB_t^{(1)} \right],$$

where

$$\overline{v}_t = \begin{cases} v^*, & t < \tau_d, \\ v^M, & t \geq \tau_d. \end{cases}$$

Then $\widetilde{U}_2$ is a solution of (53).

(c) Suppose there exists a function $\widetilde{U}_3 = \widetilde{U}_3(w, C, t)$ which solves (48), and suppose that for each $(w, C, t) \in \mathbb{R} \times (D, \infty) \times [0, T]$ there exist $\boldsymbol{\pi}^{**} = \boldsymbol{\pi}^{**}(w, C, t) \in \mathbb{R}^2$ and $\boldsymbol{v}^{**} = \boldsymbol{v}^{**}(w, C, t)$ such that

$$\mathcal{L}_3^{\boldsymbol{\pi}^{**}, \boldsymbol{v}^{**}} \widetilde{U}_3 = \sup_{\boldsymbol{\pi} \in \mathbb{R}^2} \inf_{\boldsymbol{v} \in \mathbb{R}^3} \mathcal{L}_3^{\boldsymbol{\pi}, \boldsymbol{v}} \widetilde{U}_3. \tag{60}$$

Assume that the trading strategy defined by

$$\overline{\boldsymbol{\pi}}_t = \begin{cases} \boldsymbol{\pi}^{**}(w_t, C_t, t), & t < \tau_h, \\ (\overline{\pi}_t, 0), & t \geq \tau_h, \end{cases}$$

is admissible and that there exists a measure $Q^{**} \sim \mathbb{P}$ under which P, S, C have the drift adjustments $\boldsymbol{v}_t^{**}$ up to time $\tau_h \wedge T$, and furthermore P has drift adjustment $\overline{v_t}$ between $\tau_h \wedge T$ and T. Then $\widetilde{U}_3$ is a solution of (54).

Proof. We start with part (a). Note that this part is very similar to the verification theorem of the standard Merton investment problem. Since our optimization problem is somewhat different and non-standard, we give the proof anyway.

Consider the measure Q^M and let $\pi \in \mathcal{A}$ be any admissible strategy. Working under Q^M, we get from Ito's lemma

$$\widetilde{U}_1(W_T, T) = \widetilde{U}_1(w, t) + \int_t^T (\partial_t \widetilde{U}_1 + \mathcal{L}_1^{\pi, v^M} \widetilde{U}_1) \; ds + \int_t^T \pi_s \sigma_1 \; \partial_w \widetilde{U}_1 \; dB_s^{(1)}.$$

Using the facts that $\widetilde{U}_1$ is a solution of (55) and that $\pi \in \mathcal{A}$ is an arbitrary strategy, we always have $\partial_t \widetilde{U}_1 + \mathcal{L}^{\pi, v^M} \widetilde{U}_1 \leq -\frac{1}{2} \widetilde{U}_1 \phi (v^M)^2$. Taking expectations therefore leads to

$$\mathbb{E}_t^{Q^M} \; \widetilde{U}_1(W_T, T) \leq \widetilde{U}_1(w, t) - \mathbb{E}_t^{Q^M} \left[\frac{1}{2} \int_t^T \widetilde{U}_1(W_s, s) \; \phi \; (v_s^M)^2 \; ds \right].$$

Using $\widetilde{U}_1(W_T, T) = u(W_t)$ we get

$$\widetilde{U}_1(w, t) \geq \mathbb{E}_t^{Q^M} \left[u(W_T) + \frac{1}{2} \int_t^T \widetilde{U}_1(W_s, s) \; \phi \; (v_s^M)^2 \; ds \right].$$

Since this inequality holds for all admissible trading strategies, it follows that

$$\begin{aligned}\widetilde{U}_1(w,t) &\geq \sup_{\pi\in\mathcal{A}} \mathbb{E}_t^{Q^M}\left[u(W_T)+\frac{1}{2}\int_t^T \widetilde{U}_1(W_s,s)\ \phi\ (v_s^M)^2\ ds\right] \\ &\geq \sup_{\pi\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \mathbb{E}_t^{Q}\left[u(W_T)+\frac{1}{2}\int_t^T \widetilde{U}_1(W_s,s)\ \phi\ v_s^2\ ds\right]. \qquad (61)\end{aligned}$$

Now fix the strategy π_t^M and let $Q\sim\mathbb{P}$ be any equivalent measure. Let P_t have drift μ_1+v_t under Q. Since we always have $\partial_t\widetilde{U}_1+\mathcal{L}^{\pi^M,v}\widetilde{U}_1\geq -\frac{1}{2}\widetilde{U}_1\phi v_s^2$, a similar argument as above leads to

$$\mathbb{E}_t^Q\ \widetilde{U}_1(W_T,T)\geq \widetilde{U}_1(w,t)-\mathbb{E}_t^Q\left[\frac{1}{2}\int_t^T \widetilde{U}_1(W_s,s)\ \phi\ v_s^2\ ds\right],$$

and hence

$$\widetilde{U}_1(w,t)\leq \mathbb{E}_t^Q\left[u(W_T)+\frac{1}{2}\int_t^T \widetilde{U}_1(W_s,s)\ \phi\ v_s^2\ ds\right].$$

Since this relation holds for any $Q\sim\mathbb{P}$, we get

$$\begin{aligned}\widetilde{U}_1(w,t) &\leq \inf_{Q\sim\mathbb{P}} \mathbb{E}_t^{Q}\left[u(W_T)+\frac{1}{2}\int_t^T \widetilde{U}_1(W_s,s)\ \phi\ v_s^2\ ds\right] \\ &\leq \sup_{\pi\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \mathbb{E}_t^{Q}\left[u(W_T)+\frac{1}{2}\int_t^T \widetilde{U}_1(W_s,s)\ \phi\ v_s^2\ ds\right]. \qquad (62)\end{aligned}$$

The theorem then follows from inequalities (61) and (62). For later and in analogy to the verification theorem for the completely specified case we note that for the strategy π^M and the measure Q^M we get equality everywhere, i.e. $\widetilde{U}_1(w,t)=\mathbb{E}^{Q^M}\left[u(W_T^{\pi^M}+\frac{1}{2}\int_t^T \widetilde{U}_1(W_s^{\pi^M},s)\ \phi\ (v_s^M)^2\ ds\ \right]$.

Now we prove part (b). Let π_t be an arbitrary trading strategy and consider the measure Q^*. Then as in part (a), Ito's lemma yields

$$\widetilde{U}_2(W_\tau,\tau)=\widetilde{U}_2(w,t)+\int_t^\tau(\partial_t\widetilde{U}_2+\epsilon A\partial_w\widetilde{U}_2+\mathcal{L}_2^{\pi,v^*}\widetilde{U}_2)\ ds+\int_t^\tau \pi_s\sigma_1\ \partial_w\widetilde{U}_2\ dB_s^{(1)},$$

where we have written τ instead of τ_2. Using the fact that $\widetilde{U}_2$ solves equation (56), we have $\partial_t\widetilde{U}_2+\epsilon A\ \partial_w\widetilde{U}_2+\mathcal{L}_2^{\pi,v^*}\widetilde{U}_2\leq -\frac{1}{2}U_2\phi v^2-\kappa\left[U_1(w+\widetilde{R}_1,\ t)-\widetilde{U}_2\right]$. Taking expectations we therefore get

$$\begin{aligned}\widetilde{U}_2(w,t)\geq&\mathbb{E}_t^{Q^*}\left[\widetilde{U}_2(W_{\tau^-},\tau^-)\right]+\\ &+\mathbb{E}_t^{Q^*}\left[\frac{1}{2}\int_t^\tau \widetilde{U}_2(W_s,s)\ \phi\ (v_s^*)^2\ ds+\int_t^\tau \kappa\left[U_1(W_s+\widetilde{R}_1,\ t)-\widetilde{U}_2(W_s,s)\right]\ ds\right].\end{aligned}$$

As in appendix A.1, we have

$$\mathbb{E}_t^{Q^*}\left[\int_t^\tau \kappa\left[U_1(W_s+\widetilde{R}_1,s)-\widetilde{U}_2(W_s,s)\right]\ ds\right]=\mathbb{E}_t^{Q^*}\left[\left(U_1(W_{\tau^-}+\widetilde{R}_1,\tau)-\widetilde{U}_2(W_{\tau^-},\tau^-)\right)\cdot\mathbb{I}\{\tau_d\leq T\}\right].$$

Using $\mathbb{E}_t^{Q^*}\left[U_1(W_{\tau^-}-\epsilon\widetilde{R}_1,\tau^-)\cdot\mathbb{I}\{\tau_d\le T\}\right]=\mathbb{E}_t^{Q^*}\left[\,U_1(W_\tau,\tau)\cdot\mathbb{I}\{\tau_d\le T\}\right]$, the inequality above becomes

$$\widetilde{U}_2(w,t)\ge\mathbb{E}_t^{Q^*}\left[U(W_T,T)\cdot\mathbb{I}\{\tau_d>T\}+U_1(W_\tau,\tau)\cdot\mathbb{I}\{\tau_d\le T\}+\frac{1}{2}\int_t^\tau\widetilde{U}_2(W_s,s)\;\phi\;(v_s^*)^2\;ds\right].$$

From the proof of part (a) we know that

$$U_1(W_\tau,\tau)\ge\mathbb{E}_\tau^{Q^*}\left[u(W_T)+\frac{1}{2}\int_\tau^T U_1(W_s,s)\;\phi\;(v_s^*)^2\;ds\right],$$

so we get

$$\widetilde{U}_2(w,t)\ge\mathbb{E}_t^{Q^*}\left[u(W_T)+\frac{1}{2}\int_t^\tau\widetilde{U}_2(W_s,s)\;\phi\;(v_s^*)^2\;ds\;+\frac{1}{2}\int_\tau^T U_1(w_s,s)\;\phi\;(v_s^*)^2\;ds\right],$$

hence, since π is an arbitrary admissible strategy,

$$\begin{aligned}\widetilde{U}_2(w,t)&\ge\sup_{\pi\in\mathcal{A}}\mathbb{E}_t^{Q^*}\left[u(W_T)+\frac{1}{2}\int_t^\tau\widetilde{U}_2(W_s,s)\;\phi\;(v_s^*)^2\;ds\;+\frac{1}{2}\int_\tau^T U_1(W_s,s)\;\phi\;(v_s^*)^2\;ds\right]\\&\ge\sup_{\pi\in\mathcal{A}}\inf_{Q\sim\mathbb{P}}\mathbb{E}_t^{Q}\left[u(W_T)+\frac{1}{2}\int_t^\tau\widetilde{U}_2(W_s,s)\;\phi\;v_s^2\;ds\;+\frac{1}{2}\int_\tau^T U_1(W_s,s)\;\phi\;v_s^2\;ds\right].\end{aligned}\tag{63}$$

Now fix the strategy $\overline{\pi}_t$ and let $Q\sim\mathbb{P}$ be any equivalent measure. Then from arguments analog to those above and in part (a), it follows that

$$\begin{aligned}\widetilde{U}_2(w,t)&\le\inf_{Q\sim\mathbb{P}}\mathbb{E}_t^{Q}\left[u(W_T)+\frac{1}{2}\int_t^\tau\widetilde{U}_2(W_s,s)\;\phi\;v_s^2\;ds\;+\frac{1}{2}\int_\tau^T U_1(W_s,s)\;\phi\;v_s^2\;ds\right]\\&\le\sup_{\pi\in\mathcal{A}}\inf_{Q\sim\mathbb{P}}\mathbb{E}_t^{Q}\left[u(W_T)+\frac{1}{2}\int_t^\tau\widetilde{U}_2(W_s,s)\;\phi\;v_s^2\;ds\;+\frac{1}{2}\int_\tau^T U_1(W_s,s)\;\phi\;v_s^2\;ds\right].\end{aligned}\tag{64}$$

Them the claim follows from inequalities (63) and (64). Furthermore, for $\pi=\overline{\pi}$ and $Q=Q^*$ we get equality everywhere, i.e. we have

$$\widetilde{U}_2(w,t)=\mathbb{E}_t^{Q^*}\left[u(W_T^{\overline{\pi}})+\frac{1}{2}\int_t^{\tau_2}\widetilde{U}_2(W_s^{\overline{\pi}},s)\;\phi\;(v_s^*)^2\;ds+\frac{1}{2}\int_{\tau_2}^T U_1(W_s^{\overline{\pi}},s)\;\phi\;(v_s^*)^2\;ds\;\right].$$

The proof of part (c) is analogous.

□

B. The Heat Equation on the Half Plane

We would like to find a solution u to the heat equation

$$\begin{cases}\partial_t u+\nu\;\partial_x u+\dfrac{1}{2}\eta^2\;\partial_{xx}u=0,\\ \qquad u(0,t)=g(t),\\ \qquad u(x,T)=f(x).\end{cases}\tag{65}$$

We assume that f and g are chosen such that a solution exists. Alternatively we can solve the equation

$$\begin{cases} -\partial_t \overline{u} + \nu\ \partial_x \overline{u} + \frac{1}{2}\eta^2\ \partial_{xx}\overline{u} = 0, \\ \qquad\qquad\qquad \overline{u}(0,t) = g(T-t), \\ \qquad\qquad\qquad \overline{u}(x,0) = f(x), \end{cases} \tag{66}$$

and then let $u(x,t) = \overline{u}(x, T-t)$.

Assume that u is a solution of (65) and fix x and t. As introduced in section 3, for $\theta \in \mathbb{R}$ let $\mathbb{Q}^\theta$ be a measure under which a certain stochastic process has the dynamics $X_s \triangleq x + \theta(s-t) + \eta B^\theta_{s-t}$ where B^θ_s is a standard Brownian motion under $\mathbb{Q}^\theta$. Furthermore let $\tau \triangleq \inf\{s \geq t :\ X_s = 0\} \wedge T$.

Working under the measure $\mathbb{Q}^\nu$, we get from Ito's lemma

$$u(X_\tau, \tau) = u(x,t) + \int_t^\tau \left(\partial_t u + \nu\ \partial_x u + \frac{1}{2}\eta^2\ \partial_{xx} u\right)\ dt + \int_t^\tau \eta\ \partial_x u\ dB^\nu_s.$$

Taking expectations on both sides and using the fact that u solves the given heat equation yields

$$u(x,t) = \mathbb{E}^{\mathbb{Q}^\nu}\left[u(X_\tau, \tau)\right] = \mathbb{E}^{\mathbb{Q}^\nu}\left[g(\tau)\cdot \mathbb{I}\{\tau \leq T\} + f(X_T)\cdot \mathbb{I}\{\tau > T\}\right]. \tag{67}$$

If f is a constant K as in this paper, then $\mathbb{E}^{\mathbb{Q}^\nu}[f(X_T)\cdot \mathbb{I}\{\tau > T\}]$ obviously simplifies to $K \cdot q_t(T;\nu)$, where

$$q_t(s;\theta) \triangleq \mathbb{Q}^\theta(\tau > s)$$

Under certain circumstances we can also simplify the first term on the right hand side of equation (67). Switching to the measure $\mathbb{Q}^0$ under which X_s has the dynamics $X_s = x + \eta B^0_{s-t}$ and applying the reflection principle, we get the well-known result that

$$q_t(s;\theta) = \Phi\left(\frac{x + \theta(s-t)}{\eta\sqrt{s-t}}\right) - e^{-2\theta x/\eta^2}\Phi\left(\frac{-x + \theta(s-t)}{\eta\sqrt{s-t}}\right).$$

We can compute the corresponding density $d_t(s;\theta)$ of τ by differentiating $1 - q_t(s;\theta)$ with respect to s to get

$$\begin{aligned} d_t(s;\theta) &= \frac{-1}{\sqrt{2\pi}}\left[\exp\left\{-\frac{1}{2}\left(\frac{x+\theta(s-t)}{\eta\sqrt{s-t}}\right)^2\right\}\cdot\left(-\frac{x}{2\eta(s-t)^{3/2}} + \frac{\theta}{2\eta\sqrt{s-t}}\right)\right. \\ &\quad \left. - e^{-2\theta x/\eta^2}\cdot \exp\left\{-\frac{1}{2}\left(\frac{-x+\theta(s-t)}{\eta\sqrt{s-t}}\right)^2\right\}\cdot\left(\frac{x}{2\eta(s-t)^{3/2}} + \frac{\theta}{2\eta\sqrt{s-t}}\right)\right] \\ &= \frac{1}{\sqrt{2\pi}}\frac{x}{\eta(s-t)^{3/2}}\cdot \exp\left(-\frac{[x+\theta(s-t)]^2}{2\eta^2(s-t)}\right) \end{aligned}$$

We then have

$$\mathbb{E}^{\mathbb{Q}^\nu}\left[g(\tau)\cdot\mathbb{I}\{\tau\le T\}\right]=\int_t^T g(s)\ d_t(s;\nu)\ ds.$$

If the boundary condition at $x=0$ is of the form $u(0,t)=e^{L(T-t)}$ for some constant L, it follows from equation (68) that

$$\begin{aligned}\mathbb{E}\left[g(\tau)\cdot\mathbb{I}\{\tau\le T\}\right] &= \frac{1}{\sqrt{2\pi}}\ e^{L(T-t)}\cdot\int_t^T\frac{x}{\eta(s-t)^{3/2}}\ e^{-L(s-t)}\cdot\exp\left\{-\frac{(x+\theta(s-t))^2}{2\eta^2(s-t)}\right\}\ ds\\ &= e^{L(T-t)}\cdot e^{\frac{1}{\eta^2}(x\sqrt{\theta^2+2\eta^2L}-x\theta)}\cdot\\ &\quad\cdot\frac{1}{\sqrt{2\pi}}\int_t^T\frac{x}{\eta(s-t)^{3/2}}\ \exp\left\{-\frac{(x+\sqrt{\theta^2+2\eta^2L}\ (s-t))^2}{2\eta^2(s-t)}\right\}\ ds\\ &= e^{L(T-t)+\frac{(\widehat{\theta}-\theta)x}{\eta^2}}\left(1-q_t(T;\widehat{\theta})\right)\end{aligned}$$

with $\widehat{\theta}=\sqrt{\theta^2+2\eta^2L}$.

C. Motivation for the Definition of $\boldsymbol{\Phi}$

In this section we give the defintion and its motivation of the matrix $\boldsymbol{\Phi}$ in section 6.2. Suppose that instead of equation (**??**), the definition for U is

$$\begin{aligned}U(w,P,S,C,t) &\triangleq \sup_{\boldsymbol{\pi}\in\mathcal{A}}\inf_{Q\sim\mathbb{P}}\mathbb{E}^Q\left[u(W_T)+k\log\frac{dQ}{d\mathbb{P}}\right] \qquad (68)\\ &= \sup_{\boldsymbol{\pi}\in\mathcal{A}}\inf_{Q\sim\mathbb{P}}\mathbb{E}^Q\left[u(W_T)+\frac{1}{2}k\int_t^T\boldsymbol{v}_s\overline{\boldsymbol{\Omega}^{-1}}\boldsymbol{v}_s\ ds\right],\quad k>0,\end{aligned}$$

where $\overline{\boldsymbol{\Omega}}$ is the covariance matrix of P, S and C. This is the analog of the original definition in Anderson, Hansen, and Sargent (2000). Following the idea in Uppal and Wang (2003), we would like to modify this definition to take into account different levels of uncertainty corresponding to the marginal distributions of different subsets of $\{P_t,\ S_t,\ C_t\}$.

Let $A_1,\ \ldots,\ A_7$ be the different non-empty subsets of $\{P,\ S,\ C\}$. Corresponding to a subset A_i, we start with the matrix $\widetilde{\boldsymbol{\Omega}}_i^{-1}\in\mathbb{R}^{|A_i|\times|A_i|}$, the inverse of the variance-covariance matrix of the elements contained in A_i. Then we define the matrix $\boldsymbol{\Omega}_i^{-1}=\begin{pmatrix}\omega_{PP}^{(i)} & \omega_{PS}^{(i)} & \omega_{PC}^{(i)}\\ \omega_{SP}^{(i)} & \omega_{SS}^{(i)} & \omega_{SC}^{(i)}\\ \omega_{CP}^{(i)} & \omega_{CS}^{(i)} & \omega_{CC}^{(i)}\end{pmatrix}$ as follows (following Uppal and Wang (2003)):

- Let $X,Y\in\{P,\ S,\ C\}$. If $X\notin A_i$ or $Y\notin A_i$, then $\omega_{XY}^{(i)}=0$.

- If we delete all rows and columns of $\boldsymbol{\Omega}_i^{-1}$ indexed by elements not contained in A_i, the resulting matrix is $\widetilde{\boldsymbol{\Omega}}_i^{-1}$. [5]

Then $\boldsymbol{\Phi}$ is defined as linear combination

$$\boldsymbol{\Phi} \triangleq \sum_{i=1}^{7} \alpha_i \boldsymbol{\Omega}_i^{-1}, \tag{69}$$

where the α_i are *non-positive* weights. A large $|\alpha_i|$ corresponds to a high level of certainty for the marginal distribution of the corresponding subset, whereas a small $|\alpha_i|$ corresponds to a high level of uncertainty. Note that as a negative linear combination of positive semi-definite matrices, $\boldsymbol{\Phi}$ is negative semi-definite. If the weight corresponding to the uncertainty in the joint distribution of P_t, S_t, C_t is negative, then $\boldsymbol{\Phi}$ is strictly negative definite.

Given a measure $Q \sim \mathbb{P}$ and a non-empty subset $A_i \subseteq \{P_t,\ S_t,\ C_t\}$, let $\mathbb{P}_i$, Q_i be the induced measures for the marginal distribution of the elements in A_i. Then an appropriate modification of (68) is

$$U(w,P,S,C,t) \;\triangleq\; \sup_{\boldsymbol{\pi}\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \left\{ \mathbb{E}^Q\left[u(W_T)\right] + \sum_{i=1}^{7} \alpha_i \mathbb{E}^{Q_i}\left[\log \frac{dQ_i}{d\mathbb{P}_i}\right] \right\}, \quad w_i < 0. \tag{70}$$

For a given subset A_i, let $\boldsymbol{v}_t^{A_i} \in \mathbb{R}^{|A_i|}$ be the vector obtained from $\boldsymbol{v}_t \in \mathbb{R}^3$ by deleting the components that correspond to elements not contained in A_i. Then it is easy to see that

$$\mathbb{E}^{Q_i}\left[\log \frac{dQ_i}{d\mathbb{P}_i}\right] = \mathbb{E}^{Q_i}\left[\frac{1}{2}\int_t^T (\boldsymbol{v}_s^{A_i})^T \widetilde{\boldsymbol{\Omega}}_i^{-1} \boldsymbol{v}_s^{A_i}\ ds\right] = \mathbb{E}^{Q}\left[\frac{1}{2}\int_t^T \boldsymbol{v}_s^T \boldsymbol{\Omega}_i^{-1} \boldsymbol{v}_s\ ds\right],$$

so that (70) becomes

$$U(w,P,S,C,t) = \sup_{\boldsymbol{\pi}\in\mathcal{A}} \inf_{Q\sim\mathbb{P}} \left[u(W_T) + \frac{1}{2}\int_t^T \boldsymbol{v}_s^T \boldsymbol{\Phi} \boldsymbol{v}_s\ ds\right].$$

Applying the same modification to this equation as in Maenhout (2004) then leads to the definition in (**??**).

To conclude this section we explain the choices of the matrix $\boldsymbol{\Phi}_0$ and the scalar ϕ_0 for the plots in section 6.3. For $\boldsymbol{\Phi}_0$ we chose all the weights α_i to equal -1, i.e.

$$\boldsymbol{\Phi}_0 = -\sum_{i=1}^{7} \boldsymbol{\Omega}_i^{-1}.$$

---

[5] Example: Let $A_i = \{P_t,\ C_t\}$. If $\widetilde{\boldsymbol{\Omega}}_i = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\eta \\ \rho\sigma_1\eta & \eta^2 \end{pmatrix}$ is the variance-covariance matrix of P_t and C_t, then $\widetilde{\boldsymbol{\Omega}}_i^{-1} = \frac{1}{\sigma_1^2\eta^2(1-\rho^2)} \begin{pmatrix} \eta^2 & -\rho\sigma_1\eta \\ -\rho\sigma_1\eta & \sigma_1^2 \end{pmatrix}$ and hence $\boldsymbol{\Omega}_i^{-1} = \frac{1}{\sigma_1^2\eta^2(1-\rho^2)} \begin{pmatrix} \eta^2 & 0 & -\rho\sigma_1\eta \\ 0 & 0 & 0 \\ -\rho\sigma_1\eta & 0 & \sigma_1^2 \end{pmatrix}$.

Furthermore, it is reasonable to assume that the level of uncertainty in the marginal distribution of P does not change upon switching from the healthy to the distressed regime. Consequently, if we assume that

$$\mathbf{\Omega}_1^{-1} = \begin{pmatrix} \sigma_1^{-2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then a suitable choice for the scalar ϕ is

$$\phi = \alpha_1 \sigma_1^{-2}.$$

Therefore we have chosen $\phi_0 = -\sigma_1^{-2}$ for the plots in section 6.3.

References

Anderson, Hansen, and Sargent (2000). Robustness, detection and the price of risk.

Black, F. and J. Cox (1976). Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance 31*, 351–367.

Davis, M. (1998). *Mathematics of Derivative Securities*, Chapter Option pricing in incomplete markets, pp. 216–226. The Newton Institute.

Davis, M., V. Panas, and T. Zariphopoulou (1993). *SIAM J. Control and Optimization 31*, 470–493.

Delbane, F., P. Grandits, T. Rheinlnder, D. Samperi, M. Schweizer, and C. Stricker (2002). Exponential hedging and entropic penalties. *Mathematical Finance 12(2)*, 99–123.

Duffie, D. and D. Lando (2001). Term structure of credit spreads with incomplete accounting information. *Econometrica 69*, 633–664.

Fouque, J., R. Sircar, and K. Solna (2006). Stochastic volatility effects on defaultable bonds. *Applied Mathematical Finance 13(3)*, 215–244.

Henderson, V. and D. Hobson (2002a). Real options with constant relative risk aversion. *Journal of Economic Dynamics and Control 27*, 329–355.

Henderson, V. and D. Hobson (2002b). Substitute hedging. *RISK 15(5)*, 71–75.

Hodges, S. and A. Neuberger (1989). Optimal replication of contingent claims under transaction costs. *Review of Futures Markets 8*, 222–239.

Kim, I., K. Ramaswamy, and S. Sundaresan (1993). Does default risk in coupons affect the valuation of corporate bonds?: A contingent claims model. *Financial Management 22*, 117–131.

Lando, D. (1994). *On Cox Processes and Credit Risky Securities*. Ph. D. thesis, Cornell University.

Leland, H. E. (1977). The valuation of corporate liabilities as compound options. *Journal of Financial and Quantitative Analysis 12*, 541–552.

Leland, H. E. (1994). Risky debt, bond covenants and optimal capital structure. *Journal of Finance 49*, 1213–1252.

Leland, H. E. and K. B. Toft (1996). Optimal capital structure, endogenous bankruptcy and the term structure of credit spreads. *Journal of Finance 50*, 789–819.

Leung, Sircar, and Zariphopoulou (2008). Credit derivatives and risk aversion. *to be published*.

Longstaff, F. and E. Schwartz (1995). A simple approach to valuing risky fixed and floating rate debt. *Journal of Finance 50*, 789–819.

Ludkovski, M. and V. Young (2008). Indifference pricing of pure endowments and life annuities under stochastic hazard and interest rates. *Insurance: Mathematics and Economics 42(1)*, 14–30.

Maenhout (2004). Robust portfolio rules and asset pricing. *The Review of Financial Studies 17*(4).

Merton, R. (1974). On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance 29*, 449–470.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics 51*(3), 247–257.

Schönbucher, P. J. (2003). *Credit Derivatives Pricing Models*. Wiley Finance.

Schweizer, M. (1999). A minimality property of the minimal martingale measure. *Statistics and Probability Letters 42(1)*, 27–31.

Sircar, R. and T. Zariphopoulou (2007). *Advances in Mathematical Finance*, Chapter Utility Valuation of Credit Derivatives: Single and Two-Name Cases, pp. 279–301. Birkhauser.

Sircar, R. and T. Zariphopoulou (2008). Utility valuation of credit derivatives and application to cdos. *Quantitative Finance, to appear*.

Stoikov, S. (2005). Princing options from the point of view of a trader. *preprint*.

Uppal and Wang (2003). Model misspecification and underdiversification. *The Journal of Finance 58*(6), 2465–2486.

Young, V. (2004). Pricing in an incomplete market with an affine term structure. *Mathematical Finance 14(3)*, 359–381.

Young, V. and T. Zariphopoulou (2002). Pricing dynamic insurance risks using the principle of equivalent utility. *Scandinavian Actuarial Journal 4*, 246279.

Zariphopoulou, T. (2001). A solution approach to valuation with unhedgeable risks. *Finance and Stochastics 5(1)*, 61–82.