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performance criteria:

terminal reward/penalty — running reward/penalty

$$H^{\nu}(t,x) = \mathbb{E}_{t,x}\Big[\, G(X_T^{\nu}) + \int_t^T g(X_u^{\nu})\,du \Big]$$

$$dX^{\nu} = \mu_t^{\nu}\,dt + \sigma_t^{\nu}\,dW_t$$

$\sigma_t^{\nu} = \sigma(t, X_t, \nu_t)$

$\mu_t^{\nu} \to \mu(t, X_t, \nu_t)$

ν – control

X – controlled process

Value function:

$$H(t,x) = \sup_{\nu \in \mathcal{A}} H^{\nu}(t,x)$$

Find the value function — Find ν which attains the sup

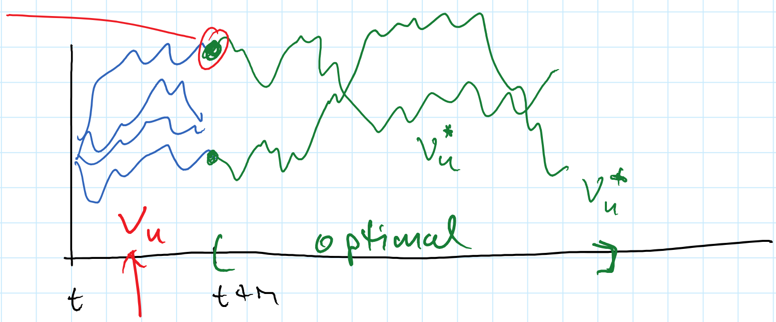
Bellman (HJB)

Hamilton Jacobi Bellman equation

Dynamic Programming Equation

$$\nu_t = \begin{cases} \nu_u, & u \in [t, t+h] \\ \nu_u^*, & u \in (t+h, T] \end{cases}$$

X_{t+h}^{ν}



"arbitrary"

$$H^{\nu}(t,x) = \mathbb{E}_{t,x}\Big[\, G(X_T^{\nu}) + \int_t^T g(X_u^{\nu})\,du \Big]$$

$$= \mathbb{E}_{t,x}\Big[\, G(X_T^{\nu}) + \int_t^{t+h} g(X_u^{\nu})\,du + \int_{t+h}^T g(X_u^{\nu^*})\,du \Big]$$

$$= \mathbb{E}_{t,x}\Big[\int_t^{t+h} g(X_u^{\nu})\,du + \mathbb{E}\Big[G(X_T^{\nu}) + \int^T g(X_u^{\nu^*})\,du \Big]\Big]$$

$$= \mathbb{E}_{t,x}\Big[\int_t^{t+h} g(X_u^{\nu})\,du + \underbrace{\mathbb{E}_{t+h, X_{t+h}^{\nu}}\Big[G(X_T^{\nu}) + \int_{t+h}^{T} g(X_u^{\nu^*})\,du \Big]}_{\substack{H^{\nu^*}(t+h,\, X_{t+h}^{\nu}) \\ = H(t+h,\, X_{t+h}^{\nu})}} \Big]$$

u^x

$$H(t+h, X_{t+h}^{\nu}) = H(t, X_t^{\nu}) + \int_t^{t+h} (\partial_t + \mathcal{L}^{\nu}) H(u, X_u^{\nu})\,du$$
$$+ \int_t^{t+h} \sigma_u^{\nu}\, \partial_x H(u, X_u^{\nu})\,dW_u$$

$$\mathcal{L}^{\nu} = \mu(t,x,\nu)\,\partial_x + \tfrac{1}{2}\sigma^2(t,x,\nu)\,\partial_{xx}$$

$$\Rightarrow \quad H^{\nu}(t,x) = \mathbb{E}_{t,x}\Big[\int_t^{t+h} \big[g(X_u^{\nu}) + (\partial_t + \mathcal{L}^{\nu}) H(u, X_u^{\nu}) \big]\,du \Big] + H(t,x)$$

$$\sup_{(\nu)_{[t,t+h]}} \Big(\qquad \Big)$$

$$\Rightarrow \quad \cancel{H(t,x)} = \sup_{(\nu)\in[t,t+h]} \mathbb{E}_{t,x}\Big[\int_t^{t+h} \Big(g(X_u^{\nu}) + (\partial_t + \mathcal{L}^{\nu}) H(u, X_u^{\nu}) \Big)\,du \Big] + \cancel{H(t,x)}$$

$$\left(\times \tfrac{1}{h}\right)$$

$$\Rightarrow \quad 0 = \sup_{(\nu)_{[t,t+h]}} \mathbb{E}_{t,x}\Big[\underbrace{\frac{1}{h}\int_t^{t+h} \Big(g(X_u^{\nu}) + (\partial_t + \mathcal{L}^{\nu}) H(u, X_u^{\nu}) \Big)\,du}_{\lim_{h\downarrow 0}(\ ) \;=\; g(X_t^{\nu}) + (\partial_t + \mathcal{L}^{\nu}) H(t, X_t^{\nu})} \Big]$$

$$0 = g(x) + \partial_t H(t,x) + \sup_{\nu_t} \mathcal{L}^{\nu} H(t,x)$$

$$\boxed{\begin{array}{c} \partial_t H + \sup_{\nu} \mathcal{L}^{\nu} H + g(x) = 0 \\ \\ H(T,x) = G(x) \end{array}}$$

HJB equation

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W_t - B-mtn

$F(W_t)$

$$dF = \tfrac{1}{2}\partial_{xx}F(W_t)\,dt + \partial_x F(W_t)\,dW_t$$

$$F(W_t + dW_t) - F(W_t)$$

$$= \partial_x F(W_t)\,dW_t + \tfrac{1}{2}\partial_{xx}F(W_t)\underbrace{(dW_t)^2}_{= dt}$$

$$+ \dots$$

$$\Delta W_t \sim \mathcal{N}(0, \Delta t)$$

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$

$F(t, X_t)$

$$dF = \left(\partial_t F_t + \mu(t, X_t)\,\partial_x F_t + \tfrac{1}{2}\sigma^2(t, X_t)\,\partial_{xx} F_t\right)dt$$
$$+ \sigma(t, X_t)\,\partial_x F_t\,dW_t$$

$F(t, X_t)$

$$dX_t = \mu_t^X\,dt + \sigma_t^X\,dW_t^X$$
$$dY_t = \mu_t^Y\,dt + \sigma_t^Y\,dW_t^Y$$
ρ

$F(X_t, Y_t)$

$$dF_t = \Big(\mu_t^X\,\partial_x F_t + \tfrac{1}{2}(\sigma_t^X)^2\,\partial_{xx}F_t$$
$$+ \mu_t^Y\,\partial_y F_t + \tfrac{1}{2}(\sigma_t^Y)^2\,\partial_{yy}F_t + \rho\,\sigma_t^X\sigma_t^Y\,\partial_{xx}F_t\Big)\,dt$$

$$+ \ \sigma_t^x \, \partial_x f_t \, dW_t^x + \sigma_t^y \, \partial_y f_t \, dW_t^y$$

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Optimal Liquidation

$$dF_t = \sigma \, dW_t$$

$$S_t^\nu = F_t - a \nu_t$$

$$dX_t^\nu = S_t^\nu \cdot \nu_t \, dt = (F_t - a\nu_t)\,\nu_t \, dt$$

$$dq_t^\nu = -\nu_t \, dt$$

terminal liquidation

$$H^\nu(t,x,q,f) = \mathbb{E}_{t,x,q,f}\left[X_T^\nu + \left(q_T^\nu F_T - \varphi \left(q_T^\nu\right)^2 \right) - \phi \int_t^T (q_s^\nu)^2 \, ds \right]$$

terminal penalty

$$H(t,x,q,f) = \sup_\nu H^\nu(t,x,q,f)$$

$$\boxed{\partial_t H + \sup_\nu \mathcal{L}^\nu H - \phi q^2 = 0 \,, \qquad H(T,\cdot) = x + \left(qf - \varphi q^2\right)}$$

$$\mathcal{L}^\nu = \tfrac{1}{2}\sigma^2 \partial_{ff} + (f - a\nu)\nu \, \partial_x - \nu \, \partial_q$$

$$\sup_\nu \mathcal{L}^\nu H = \tfrac{1}{2}\sigma^2 \partial_{ff} H + \sup_\nu \underbrace{\left((f-a\nu)\nu \, \partial_x H - \nu \, \partial_q H \right)}_{A}$$

$$A = -a\,\partial_x H \, \nu^2 + (f\,\partial_x H - \partial_q H)\,\nu$$

$$= -a\,\partial_x H \left[\left(\nu - \left(\frac{f\,\partial_x H - \partial_q H}{2a\,\partial_x H} \right) \right)^2 - \left(\frac{f\,\partial_x H - \partial_q H}{2a\,\partial_x H} \right)^2 \right]$$

$$\boxed{\nu^* = \frac{f\,\partial_x H - \partial_q H}{2a\,\partial_x H}}$$

optimal control in "feedback" control form.

$$\Rightarrow \quad \sup_\nu \mathcal{L}^\nu H = \frac{(f\,\partial_x H - \partial_q H)^2}{4a\,\partial_x H} + \tfrac{1}{2}\sigma^2 \partial_{ff} H$$

$$\boxed{\begin{cases} \partial_t H + \dfrac{(f\,\partial_x H - \partial_q H)^2}{4a\,\partial_x H} + \frac{1}{2}\sigma^2 \partial_{ff} H - \phi q^2 = 0 \\ \qquad\qquad\qquad H(T,\cdot) = x + qf - \varphi q^2 \end{cases}}$$

$$H(t,x,q,f) = x + qf + h(t,q) \quad , \quad h(T,q) = -\varphi q^2$$

$$\partial_t H = \partial_t h, \ \partial_x H = 1, \ \partial_f H = q, \ \partial_{ff} H = 0, \ \partial_q H = f + \partial_q h$$

$$\Rightarrow \quad \partial_t h + \frac{(f \cdot 1 - (f + \partial_q h))^2}{4a \cdot 1} + 0 - \phi q^2 = 0$$

$$\boxed{\begin{cases} \partial_t h + \dfrac{(\partial_q h)^2}{4a} - \phi q^2 = 0 \\ \qquad\qquad h(T,q) = -\varphi q^2 \end{cases}}$$

$$h(t,q) = \alpha(t)\,\beta(q) \quad , \quad \alpha(T) = 1 \ , \ \beta(q) = -\varphi q^2$$

$$\Rightarrow (-\varphi q^2) \cdot \partial_t \alpha + 4\frac{\varphi^2 q^2}{4a} \cdot \alpha^2 - \phi q^2 = 0$$

$$\Rightarrow \quad \boxed{\partial_t \alpha - \frac{\varphi}{a}\alpha^2 + \frac{\phi}{\varphi} = 0} \quad , \quad \alpha(T) = 1$$

put all together:

$$\boxed{H = x + qf - \varphi q^2 \alpha(t)}$$

$$\nu^* = \frac{f\,\partial_x H - \partial_q H}{2a\,\partial_x H} = \frac{f - (f + \partial_q h)}{2a}$$

$$= -\frac{\partial_q \beta \cdot \alpha}{2a} = \frac{\varphi}{a} \cdot q^{\nu^*} \cdot \alpha(t)$$

$$d\,q^{\nu^*} = \ \ \nu^* \, dt$$

$$dq^{v^*} = -v^* \, dt$$

$$= -\frac{c}{a}\,\alpha(t)\; q^{v^*}\; dt$$

$$\ln\left(\frac{q(t)}{q(0)}\right) = -\frac{c}{a}\int_0^t \alpha(u)\,du$$

$$q(t) = q(0)\,.\; e^{-\frac{c}{a}\int_0^t \alpha(u)\,du}$$

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