Finally, it is easy to see that the optimal control is

$$\tilde{\nu}_t^* = \frac{\tilde{Q}_t^\nu}{T - t + \alpha^{-1} k} \qquad \text{or} \qquad \nu_t^* = \frac{Q - Q_t^\nu}{T - t + \alpha^{-1} k}\,.$$

10.4 The General Case with Terminal and Running Penalty, Temporary and Permanent Impact

Example: The Optimal Liquidation Problem

This section addresses the optimal liquidation problem introduced in section 4.2.2. The optimisation problem was introduced in (4.3) and has the associated time dependent performance criteria

$$H^\nu(t,x,S,q) = \mathbb{E}\left[X_T^\nu + Q_T^\nu(S_T^\nu - \alpha Q_T^\nu) - \phi \int_t^T (Q_u^\nu)^2\, du\right] \tag{10.23}$$

and time dependent value function $H(t,x,S,q) = \sup_{\nu \in \mathcal{A}_{t,T}} H^\nu(t,x,S,q)$ where the processes corresponding to the investor's wealth X, the fundamental asset price S, and the investor's inventory Q all satisfy the system of SDEs in (4.4) which in turn are all controlled by the speed ν at which the agent trades. The infinitesimal generator associated with (X_t, S_t, Q_t) is

$$\mathcal{L}^\nu = \nu\,(S - h(\nu))\,\partial_x - g(\nu)\,\partial_S + \tfrac{1}{2}\sigma^2\,\partial_{SS} - \nu\,\partial_q\,.$$

The DPE then implies that the value function should satisfy the equation

$$\begin{aligned} 0 &= \left(\partial_t + \tfrac{1}{2}\sigma^2\,\partial_{SS}\right) H(t,x,S,q) - \phi\, q^2 \\ &\quad + \sup_\nu \left\{\left(\nu\,(S - h(\nu))\,\partial_x - g(\nu)\,\partial_S - \nu\,\partial_q\right) H(t,x,S,q)\right\} \end{aligned}$$

subject to the terminal condition $H(T,x,S,q) = x + S\,q - \alpha q^2$. Under the simplifying assumption that permanent and temporary price impact functions are linear in the speed of trading, i.e., $h(\nu) = a\,\nu$ and $g(\nu) = b\,\nu$ for constants $a > 0$ and $b > 0$, the sup attains a maximum. By completing the square

$$\begin{aligned} &\left(\nu\,(S - a\,\nu)\,\partial_x - b\,\nu\,\partial_S - \nu\,\partial_q\right) H(t,x,S,q) \\ &\quad = -a\,\nu^2 \partial_x H(t,x,S,q) + \nu\,(S\,\partial_x - b\,\partial_S - \partial_q) H(t,x,S,q) \\ &\quad = -a\,\partial_x H(t,x,S,q)\,\left\{(\nu - \nu^*)^2 - (\nu^*)^2\right\} \end{aligned}$$

where

$$\nu^*(t,x,S,q) = \frac{(S\,\partial_x - b\,\partial_S - \partial_q) H(t,x,S,q)}{2a\,\partial_x H(t,x,S,q)}$$

is the optimal control in feedback form. Upon substituting the optimal feedback control into the DPE it reduces to

$$0 = \left(\partial_t + \tfrac{1}{2}\sigma^2\,\partial_{SS}\right) H(t,x,S,q) - \phi\, q^2 + \frac{\left[(S\,\partial_x - b\,\partial_S - \partial_q) H(t,x,S,q)\right]^2}{4a\,\partial_x H(t,x,S,q)}\,.$$

The terminal condition $H(T,x,S,q) = x + S\,q - \alpha\, q^2$ suggests the ansatz $H(t,x,S,q) = x + S\,q + h(t,S,q)$ leading to the simpler equation

$$0 = \left(\partial_t + \tfrac{1}{2}\sigma^2\,\partial_{SS}\right) h(t,S,q) - \phi\, q^2 + \frac{1}{4a}\left[b\,(q + \partial_S h(t,S,q)) - \partial_q h(t,S,q)\right]^2$$

with terminal condition $h(T,S,q) = -\alpha\, q^2$. Since the above PDE contains no explicit dependence on S and the terminal condition is independent of S, it is clear that $\partial_S h(t,S,q) = 0$ so that $h(t,S,q) = h(t,q)$ and the equation reduces even further to

$$0 = \partial_t h(t,q) - \phi\, q^2 + \frac{1}{4a}\left[b\,q - \partial_q h(t,q)\right]^2\ .$$

In this form, it appears that the solution admits a separation of variables $h(t,q) = \psi(t)\, q^2$ where $\psi(t)$ satisfies

$$0 = \partial_t \psi(t) - \phi + \frac{1}{4a}\left[b - 2\psi(t)\right]^2\,,$$

subject to $\psi(T) = -\alpha$. This is now an ODE and can be integrated exactly. First, let $\psi(t) = \frac{1}{2}b + \chi(t)$, then rearranging the ODE implies that

$$\frac{\partial_t \chi(t)}{a\phi - \chi(t)^2} = \frac{1}{a}\,,$$

subject to $\chi(T) = -\frac{1}{2}b - \alpha$. Next, integrating both sides of the above over $[t,T]$ yields

$$\begin{aligned}\log\frac{\sqrt{a\phi} + \chi(T)}{\sqrt{a\phi} - \chi(T)} - \log\frac{\sqrt{a\phi} + \chi(t)}{\sqrt{a\phi} - \chi(t)} &= 2\gamma\,(T - t)\\ \Rightarrow\qquad \chi(t) &= \sqrt{a\,\phi}\,\frac{1 + \xi\, e^{2\gamma\,(T-t)}}{1 - \xi\, e^{2\gamma\,(T-t)}}\,,\end{aligned}$$

where

$$\gamma = \sqrt{\frac{\phi}{a}} \quad\text{and}\quad \xi = \frac{\alpha + \frac{1}{2}b + \sqrt{a\,\phi}}{\alpha + \frac{1}{2}b - \sqrt{a\,\phi}}\,.$$

Given the solution to the DPE, the optimal speed of trading can now be explicitly stated in terms of the state variables rather than in feedback form. Specifically, from (10.4), the optimal speed to trade at is

$$\nu_t^* = -\frac{\chi(t)}{a}\, q_t\ .$$

Interestingly, in this case, the optimal speed to trade is simply proportional to the investor's current inventory level while the proportionality factor depends on time. The explicit formula for $\chi(t)$ shows that the proportionality is an increasing function of time as can be seen in Figure

It is in fact also possible to obtain the investor's inventory $Q_t^{\nu^*}$ given that the investor follows this strategy. Recall that the investor's inventory satisfies $dQ_t^\nu = -\nu_t\, dt$, so then

$$dQ_t^{\nu^*} = \frac{\chi(t)}{a}\, Q_t\, dt \quad \Rightarrow \quad Q_t^{\nu^*} = Q_0 \exp\left\{ \int_0^t \frac{\chi(s)}{a}\, ds \right\} .$$

By direct computation, and after some manipulation,

$$\int_0^t \frac{\chi(s)}{a}\, ds = \log \frac{\xi\, e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\xi\, e^{\gamma T} - e^{-\gamma T}} ,$$

so finally, the inventory along the optimal strategy is given by

$$Q_t^{\nu^*} = Q_0 \frac{\xi\, e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\xi\, e^{\gamma T} - e^{-\gamma T}} .$$

Note that in the limit in which the terminal quadratic penalty goes to infinity, i.e. $\alpha \to +\infty$, then $\xi \to 1$ and so

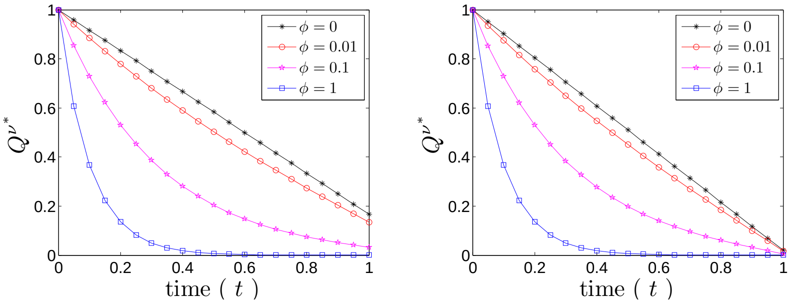
$$Q_t^{\nu^*} \underset{\alpha \to +\infty}{\longrightarrow} Q_0 \frac{\sinh(\gamma(T-t))}{\sinh(\gamma T)} .$$

Figure 10.1 contains plots of the inventory level under the optimal strategy for two levels of the terminal penalty α and several levels of the running penalty ϕ. Note that with no running penalty, the strategies are straight lines and in particular, with larger α the strategy is equivalent to one which would give a cost equal to the time weighted average price (TWAP). As the running penalty increases, the trading curves become more convex and the optimal strategy aims to sell more assets sooner. Naturally, as the terminal penalty increases, the terminal inventory is pushed to zero.

10.5 Further Readings / Extensions

10.6 Exercises

Exercise 1. The set-up of the agent's liquidation problem is similar to that above. The only difference is that the agent does not require $q_T = 0$, i.e. he is



(a) $\alpha = 0.05$ (b) $\alpha = 0.5$

Figure 10.1 The investor's inventory along the optimal path for various levels of the running penalty ϕ. The remaining model parameters are $a = 10^{-2}$, $b = 10^{-4}$.

not required to liquidate all shares by T. Instead, the agent picks up a penalty for all remaining shares q_T. Thus, the expected revenues from sales is

$$R = \mathbb{E}\left[\int_0^T \hat{S}_t \nu_t dt + Q_T^\nu S_T - \alpha\,(Q_T^\nu)^2\right], \tag{10.24}$$

where $-\alpha\left(Q_T^\nu\right)^2$ is the penalty received from not liquidating all the shares, $\alpha > 0$ is a penalty parameter,

$$\hat{S}_t = S_t - k\nu_t \tag{10.25}$$

with $k > 0$,

$$\nu_t = -\frac{dQ_t^\nu}{dt}, \tag{10.26}$$

and Q_t^ν is the amount of outstanding shares at time t.

Moreover, the stock price satisfies the SDE

$$dS_t = \sigma dW_t$$

where $\sigma > 0$ and W_t is a standard Brownian motion.

(i) Show that the value function H satisfies

$$0 = -\left(\partial_q H - S_t\right)^2 - 4k\partial_t H - 2k\sigma^2 \partial_{SS} H\,.$$

(ii) Assume that the solution depends linearly on S. Make the change of variable $V(t,q) = H(t,q) - qS$ and write

$$-4kV_t = \left(\partial_q V\right)^2, \tag{10.27}$$