11

Optimal Liquidation and Acquisition with Limit and Market Orders

11.1 Introduction

This chapter addresses the optimal liquidation problem when the agent trades, first using limit orders and then with both limit and market orders. As in the approach taken in chapter **??**, here, the asset's mid-price S_t is assumed to be a Brownian motion $S_t = S_0 + \sigma W_t$. However, the agent is now allowed to trade using limit orders. In this manner, the trader is no longer a taker of liquidity and instead will be providing liquidity to the market. Hence, they will receive prices that are better than the mid-price, although execution is not guaranteed and the trader now faces additionally execution risk.

In this chapter, four cases are investigated: (i) in section 11.2, the agent is only allowed to use LOs, (ii) in section 11.3, the agent is allowed to trade with both limit orders (LOs) as well as market orders (MOs) (iii) in section 11.4, the agent aims to track the **time weighted average price** (TWAP), and (iv) in section 11.5, the agent aims to track the **volume weighted average price** (VWAP) . In all cases, when the agent posts LOs, she posts a limit sell order for a fixed volume (e.g., some percentage of the average size of a market order, or even more simply a fixed amount of say 10 shares) at a price of $S_t + \delta_t$. Hence, δ_t acts as a premium the agent demands for posting liquidity. The larger δ, the larger the profit given that a trader occurs, however, the probability that an order arrives which eats into the book at the posted depth decreases with δ.

The probability of being filled when posting at a given depth δ, conditional on the arrival of a market order, is called the **fill probability** $h(\delta)$. Naturally, h must be decreasing in δ, it will be dynamic throughout the day, and it is sensitive the current status of the limit order book. To see why this true, consider the left panel of Figure 11.1 which shows a block-shaped LOB together with (i) a post at $\delta = 10$ (the dashed-line) (ii) the depth to which a MO of volume 700 lifts orders (dark green region), and (iii) the depth to which a MO of vol-

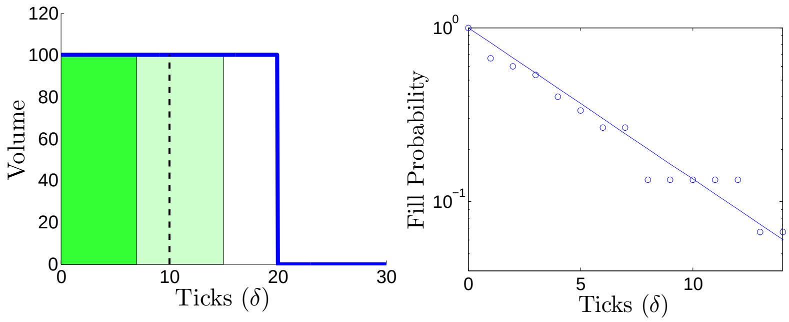

Figure 11.1 (Left) A flat (or block shaped) limit order book. (Right) Empirical fill probabilities for NFLX on June 21, 2011 for the time interval 12:55pm to 1:00pm using 500 millisecond resting times. The straight line shows the fit to an exponential function.

ume 1500 lifts orders (dark plus light green region). The deeper the order is posted, the less likely it is that market order large enough eats into the book to that price level. Hence, the probability of being filled decreases as δ increases. In fact, under the assumption that the volume of individual market orders V is exponentially distributed (with mean volume of η) and that the limit order book is block shaped with level A, i.e. the posted volume at a price of $S + \delta$ is equal to a constant A out to a maximum price level of $S + \overline{\delta}$, the probability of fill will be exponential. To see why, note that if conditional on the arrival of a market order of volume V, the probability that the order is lifted/hit is given by

$$\mathbb{P}(\text{order posted at depth } \delta \text{ is hit / lifted}) = \mathbb{P}(V > A\delta) = e^{-\frac{A}{\eta}\delta}. \tag{11.1}$$

This exponential fill probability is a reasonable approximation to the true fill probability, however, some argue that power law decay better captures the tails of the distribution. The right panel of Figure 11.1 shows empirical fill probabilities together with an exponential fit. The empirical fill probabilities are obtained through the following procedure:

(i) Pick a time interval in which on average 10 market orders arrive. Denote this time interval by ΔT.
(ii) Record the best bid (S_n^-) and ask (S_n^+) at fixed times $n\Delta T$ from market open.
(iii) Place levels at $S_n^- - m\Delta S$ and $S_n^+ + m\Delta S$, where ΔS is the tick sizes, with $m = 1, \ldots, 20$ for the first 20 ticks.
(iv) If a buy market order arrives above the price level $S_n^+ + m\Delta S$ within the time window ΔT, then level-m receives a $+1$ indicator, otherwise it receives a 0, for this time interval. Similarly, if a sell market order arrives below the

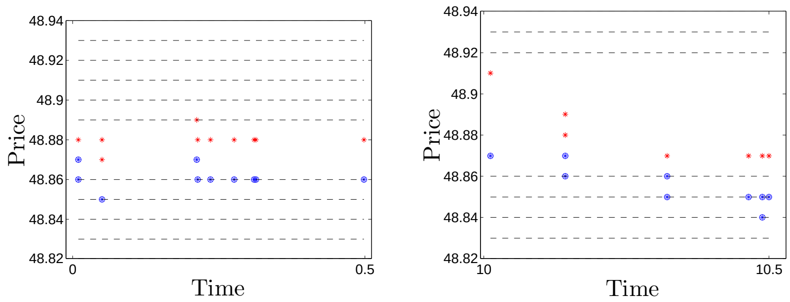

Figure 11.2 Estimating fill probabilities by placing levels one tick apart from the best bid/ask at equal time intervals and count how many times a level at depth m is hit/lifted by market orders.

price level $S_n^- - m\Delta S$ within the time window ΔT, then level-m receives a +1 indicator, otherwise it receives a 0, for this time interval.

(v) The percentage of +1 indicators for level-m represents the probability that limit orders placed at level-m would be filled – i.e., it provides as estimate of the function $h(\delta)$ at the points $\delta = m\Delta S$.

The procedure described above can be visualized as in Figure 11.2 where the red crosses and blue circles indicate the time and prices at which a market buy and sell orders arrive, respectively. In the left panel, the lowest price which a market order hit a bid was 48.85 and appears two ticks into the LOB. Hence, for this time period both levels 1 and 2 of the buy side of the book were filled. Further, the highest price which a market order lifted an offer was 48.89 and only level 1 of the sell side of the book was filled. Similarly, In the right panel, the buy side of the book was filled all the way to level 3, while the sell side was completely unfilled. Notice that the bid-ask spread in the left and right panels are 3 cents and 6 cents, respectively.

11.2 Liquidation With Only Limit Orders

In the previous chapter, the optimal liquidation problem for an agent who places only market orders was considered. Here, the agent will post only limit orders. To this end, let the asset's mid-price be denoted by $S = (S_t)_{0\le t\le T}$ with $S_t = S_0 + \sigma W_t$ and let $\delta = (\delta)_{0\le t\le T}$ denote the spread at which the agent posts limit sell orders (T represents the terminal time by which the liquidation should be complete). That is, the agent posts LOs at a price of $S_t + \delta_t$ at time t. Next, let $M = (M_t)_{0\le t\le T}$ denote a Poisson process (with intensity λ) corresponding to the number of market buy orders that have arrived. Finally, let $N = (N_t)_{0\le t\le T}$ denote the counting process corresponding the number of market buy orders which lifted the agent's offer, i.e., market orders which eat into the sell side of book to a price less than or equal to $S_t - \delta_t$. Note that whenever the process N jumps, the process M must also jump, but when M jumps, N will jump only if the market order was large enough. Setting $h(x) = e^{-\kappa x}$ as the probability that the agent's LO will be lifted when a buy market order arrives, then N jumps with probability $h(\delta_t)$ whenever M jumps and N is not a Poisson process.

The filtration $\mathcal{F}$ on which the problem is setup will be the natural one generated by S, N and M. Moreover, the agent's postings (or strategy) δ will be $\mathcal{F}$-predictable and in particular will be left continuous with right limits (LCRL). The agent is allowed to control δ and the agents actions affects the intensity of arrivals of N. Moreover, the agents choice of δ_t affects their profits from the trade. The profit relative to the mid-price of a sale is δ_t. Since an increase in δ leads not only to large profits but also small probability of fill, the agent must intelligently balance these two factors. To see how this is achieved, one more object is required – the cash-process corresponding to the agents wealth from trading $X = (X_t)_{0\le t\le T}$. After a trade, the agent gains $S_t + \delta_t$ and therefore, X_t satisfies the SDE

$$dX_t = (S_t + \delta_t)\, dN_t \tag{11.2}$$

and the agent's inventory $Q_t = \mathfrak{N} - N_t$ where $\mathfrak{N}$ is the number of shares the agent wishes to liquidate.

Now that all of the fundamental objects have been defined, the optimization problem can now be posed. Intuitively, the agent wishes to maximize their profit from the sales of the assets, but also wants to ensure that most, if not all, of the assets are sold by the terminal time T. If the agent has inventory remaining at the end of the trading horizon, they must post a market order and therefore take liquidity and obtain worse prices for those shares. As argued in the previous chapter, a linear impact function on market orders is a reasonable first order approximation of market impact, hence the agent's optimization problem is to find

$$H(x, S) = \sup_{\delta\in\mathcal{A}} \mathbb{E}\left[\, X_\tau + Q_\tau\, (S_\tau - \alpha\, Q_\tau) \mid X_{0^-} = x,\; S_0 = S, Q_{0^-} = \mathfrak{N}\,\right] \tag{11.3}$$

where the admissible set $\mathcal{A}$ consists of strategies δ which are bounded from below, and the stopping time

$$\tau = T \wedge \min\{\, t \,:\, Q_t = 0\}$$

is the minimum of T or the first time that the inventory hits zero, because then no more trading is necessary.

The corresponding value function is therefore

$$H(t,x,S,q) = \sup_{\delta\in\mathcal{A}} \mathbb{E}_{t,x,S,q}\left[\, X_\tau + Q_\tau\, (\, S_\tau - \alpha\, Q_\tau)\right] \tag{11.4}$$

where the notation $\mathbb{E}_{t,x,S,q}[\cdot]$ represents expectation conditional on $X_{t^-} = x$, $S_t = S$, and $Q_{t^-} = q$. The corresponding DPE is

$$\begin{cases} \partial_t H + \tfrac{1}{2}\sigma^2 \partial_{SS} H & \\ \quad + \sup_\delta \left\{ \lambda e^{-\kappa\delta} \left[H(t, x + (S+\delta), S, q-1) - H(t,x,S,q) \right] \right\} & = 0, \\ \qquad\qquad H(t,x,S,0) & = x, \\ \qquad\qquad H(T,x,S,q) & = x + q\,(S - \alpha\, q). \end{cases}$$

The various terms in the PIDE can be interpreted as follows:

(i) The operator ∂_{SS} corresponds to the generator of the Brownian motion which drives the mid-price.
(ii) The supremum takes into account the agent's ability to control the spread.
(iii) The term $\lambda e^{-\kappa\delta}$ represents the rate of arrival of market orders which fill the agents posted limit order at price $S+\delta$.
(iv) The difference term $H(t, x + (S+\delta), S, q-1) - H(t,x,S,q)$ represents the change in the valuation when a market order which fills the agent's post arrives – the agent's wealth increases by $S+\delta$ and their inventory decreases by 1.

The terminal and boundary conditions suggest the natural ansatz for the value function $H(t,x,S,q) = x + q\,S + h(t,q)$. In which case $h(t,q)$ satisfies the coupled system of ODEs

$$\begin{cases} \partial_t h + \sup_\delta \left\{ \lambda e^{-\kappa\delta} \left[\delta + h(t,q-1) - h(t,q) \right] \right\} & = 0, \\ h(t,0) & = 0, \\ h(T,q) & = -\alpha\, q^2. \end{cases} \tag{11.5}$$

The first order condition for the supremum imply

$$\begin{aligned} 0 &= \partial_\delta \left\{ \lambda e^{-\kappa\delta} \left[\delta + h(t,q-1) - h(t,q) \right] \right\} \\ &= \lambda \left(-\kappa\, e^{-\kappa\delta} \left[\delta + h(t,q-1) - h(t,q) \right] + e^{-\kappa\delta} \right) \\ &= \lambda\, e^{-\kappa\delta} \left(-\kappa \left[\delta + h(t,q-1) - h(t,q) \right] + 1 \right) \end{aligned}$$

and so the optimal strategy δ^* in feedback control form is given by

$$\delta^* = \frac{1}{\kappa} + \left[h(t,q) - h(t,q-1) \right]\,. \tag{11.6}$$

Inserting the optimal feedback control into (11.5) provides a non-linear coupled system of ODEs for $h(t,q)$

$$\partial_t h + \frac{\tilde{\lambda}}{\kappa} \exp\{-\kappa [h(t,q) - h(t,q-1)]\} = 0 \tag{11.7}$$

where $\tilde{\lambda} = \lambda e^{-1}$ and the same terminal and boundary conditions as (11.5) apply. This ODE can be solved exactly by making the substitution $h(t,q) = \frac{1}{\kappa} \log \omega(t,q)$ and writing a new equation for $\omega(t,q)$, in which case,

$$\begin{aligned}
0 &= \partial_t h + \frac{\tilde{\lambda}}{\kappa} \exp\{-\kappa [h(t,q) - h(t,q-1)]\} \\
&= \frac{1}{\kappa} \frac{\partial_t \omega(t,q)}{\omega(t,q)} + \frac{\tilde{\lambda}}{\kappa} \frac{\omega(t,q-1)}{\omega(t,q)} \\
\Rightarrow \quad 0 &= \partial_t \omega(t,q) + \tilde{\lambda}\, \omega(t,q-1)
\end{aligned} \tag{11.8}$$

and the terminal and boundary conditions are now $\omega(T,q) = e^{-\kappa\alpha q^2}$ and $\omega(t,0) = 1$, respectively. This coupled system can be easily solved (see **Exercise I.**) resulting in

$$\omega(t,q) = \sum_{n=0}^{q} \frac{\tilde{\lambda}^n}{n!}\, e^{-\kappa\alpha\,(q-n)^2}\, (T-t)^n . \tag{11.9}$$

This solution then provides the function $h(t,q)$ which can then be plugged into the equation for the optimal spread (11.6) to find

$$\delta^*(t,q) = \frac{1}{\kappa} \left[1 + \log \frac{\sum_{n=0}^{q} \frac{\tilde{\lambda}^n}{n!}\, e^{-\kappa\alpha\,(q-n)^2}\, (T-t)^n}{\sum_{n=0}^{q-1} \frac{\tilde{\lambda}^n}{n!}\, e^{-\kappa\alpha\,(q-1-n)^2}\, (T-t)^n} \right]$$

for $q = 1, 2, \ldots$. In Figure 11.3, the optimal spreads are shown as a function of time for several inventory levels as well as penalty parameter α. There are several interesting features of the spreads:

(i) The spreads are decreasing in inventory. This is natural, since if the agent holds a lot of inventory, they are willing to give up profit in exchange to increase the probability that their orders are filled so that they may complete liquidation by end of the time horizon and avoid crossing the spread and paying a terminal penalty. However, if inventories are low, they are willing to hold on to it in exchange for large profits, because now the terminal penalty will be moderate.

(ii) For fixed inventory level, the spreads all decrease in time. Once again, this

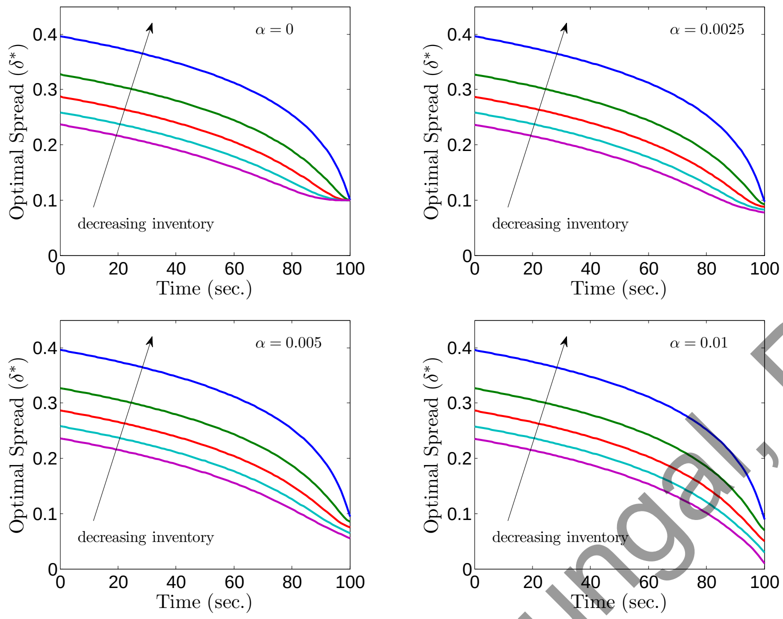

Figure 11.3 The optimal spreads δ^* at which an agent posts limit orders as a function of time and current inventory. The parameters are $\lambda = 0.5$, $\kappa = 10$, and $N = 5$ with the penalty α shown each panel. The lowest spreads correspond to $q = 5$ and the highest spread to $q = 0$.

is due to the agent becoming more averse to holding inventories as the terminal time approaches due to the penalty they will receive from crossing the spread.

(iii) As the penalty parameter α increases, all spreads decrease and is a natural consequence of since increasing the penalty induces the trader to liquidate their position faster, but at lower prices.

(iv) The spreads do not become constant far from maturity. The reason is that the agent is only being penalized by their terminal inventory, so far from terminal time, there is no incentive to liquidate their position. If the agent instead penalizes inventories through time, the strategies will become asymptotically constant far from maturity. For this case see **Exercise III.**.

Far from the terminal time, i.e., $T - t \gg 1$, the ratio appearing in the logarithm above (i.e., $\omega(t,q)/\omega(t,q-1)$) is to $o\left((T-t)^{-1}\right)$ given by the ratio of the two terms $n = q-1$ and $n = q$ in the numerator to the term $n = q-1$ in the

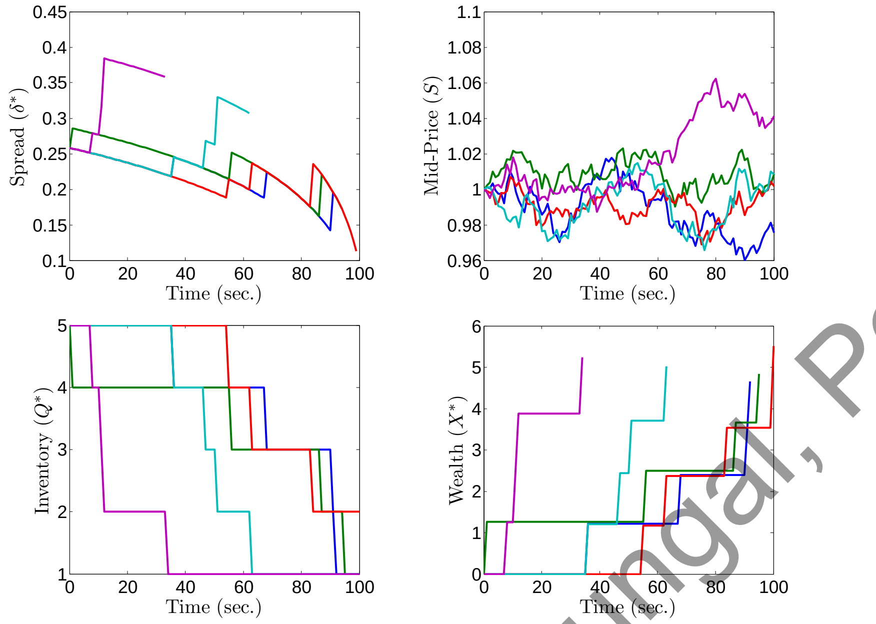

Figure 11.4 Some sample paths for the agent following the optimal strategy.

denominator. Hence, it is possible to write

$$\frac{\omega(t,q)}{\omega(t,q-1)} = \frac{\frac{\tilde{\lambda}^{q-1}}{(q-1)!}e^{-\kappa\alpha(q-(q-1))^2}(T-t)^{q-1} + \frac{\tilde{\lambda}^{q}}{q!}e^{-\kappa\alpha(q-q)^2}(T-t)^{q}}{\frac{\tilde{\lambda}^{q-1}}{(q-1)!}e^{-\kappa\alpha(q-1-(q-1))^2}(T-t)^{q-1}} + o\left((T-t)^{-1}\right)$$

$$= e^{-\kappa\alpha} + \frac{\tilde{\lambda}}{q}(T-t) + o\left((T-t)^{-1}\right). \tag{11.10}$$

Therefore, far from the terminal time, the agent posts spread that grows like

$$\delta^*(t,q) = \frac{1}{\kappa}\left[1 + \log\left(e^{-\kappa\alpha} + \frac{\tilde{\lambda}}{q}(T-t)\right)\right] + o\left((T-t)^{-1}\right).$$

11.3 Liquidation With Limit and Market Orders

In the previous section, the agent considered posting only limit orders and is found to post more aggressively as maturity approaches. However, it is also interesting to consider the situation in which we allow the agent to also post market orders in addition to limit orders. In this case, when the agent is far behind schedule, i.e., when maturity is approaching but the agent still has many shares to acquire, then

11.4 Liquidation With Limit and Market Orders Targeting TWAP

11.5 Liquidation With Limit and Market Orders Targeting VWAP

11.6 Further Readings / Extensions

11.7 Exercises

Exercise I. Show that (11.9) is indeed the solution to (11.8) by completing the following steps

(a) Compute $\omega(t,q)$ for $q = 1,2,3$ by explicit integration of (11.8).

(b) Notice that the solutions are all polynomials in $(T-t)$ and increase in order as q increases. Hence guess that

$$\omega(t,q) = \sum_{n=0}^{q-1} a_n^{(q)} (T-t)^n$$

and show that the coefficients $a_n^{(q)}$ satisfy the recursion

$$a_n^{(q)} = \frac{\tilde{\lambda}}{n}\, a_{n-1}^{(q-1)} \tag{11.11}$$

for $n = 1,\dots,q-1$ and $q = 1,2,\dots$, and $a_0^{(q)} = e^{-\kappa\alpha\, q^2}$.

(c) Prove via induction that the above form of the solution is indeed correct.

(d) Solve the recursion and show that

$$a_n^{(q)} = \frac{\tilde{\lambda}^n}{n!} e^{-\kappa\alpha\,(q-n)^2} \tag{11.12}$$

for $n = 0,\dots,q-1$ and $q = 1,2,\dots$.

Exercise II. In the optimization problem (11.3), the terminal penalty is assumed to be $-\alpha q^2$. Suppose instead that terminal penalty is a generic bounded and increasing function of the terminal inventory $\ell(q)$, so that the agent's optimization problem is

$$H(x,S) = \sup_{\delta\in\mathcal{A}} \mathbb{E}_{0,x,S,\mathfrak{N}}\left[X_\tau + Q_\tau\, S_\tau - \ell(Q_\tau)\right] . \tag{11.13}$$

(a) Derive the corresponding DPE for the associated value function, and solve for the value function and the optimal trading strategy using the same methods as outlined in Section 11.2.

(b) Many markets provide rebates to liquidity providers. This means that each time that an agent posts a limit order and it is filled before being canceled, the agent receives a rebate β. Account for such rebates in the formulation of the agent's optimization problem and determine the modified optimal posting strategy.

Exercise III. Suppose that the agent wishes to penalize inventories different from zero not just at the terminal time, but also throughout the entire duration of trading. In this case, the agent adds a running penalty term to the optimization problem and wishes to optimize

$$H(x,S) = \sup_{\delta \in \mathcal{A}} \mathbb{E}_{0,x,S,\mathfrak{N}} \left[X_\tau + Q_\tau \,(S_\tau - \alpha\, Q_\tau) - \phi\, \sigma^2 \int_0^\tau Q_s^2 \, ds \right] \tag{11.14}$$

instead of (11.3), with $\phi \geq 0$. When $\phi = 0$, the agent solves the old optimization problem, but with $\phi > 0$, the agent modifies their behaviour to reflect their risk attitudes towards holding inventory.

(a) Show that the corresponding value function can be written as $H(t,x,S,q) = x + q\,S + h(t,q)$ where $h(t,q)$ satisfies the coupled non-linear system of ODEs

$$\begin{cases} \partial_t h + \tilde{\lambda} \exp\{-\kappa\,[h(t,q) - h(t,q-1)]\} & = & \phi\, q^2, \\ h(t,0) & = & 0, \\ h(T,q) & = & -\alpha\, q^2. \end{cases} \tag{11.15}$$

and that the optimal spread δ^* is still provided in feedback form as

$$\delta^*(t,q) = \frac{1}{\kappa} + [h(t,q) - h(t,q-1)] \; . \tag{11.16}$$

(b) By writing $h(t,q) = \frac{1}{\kappa}\omega(t,q)$ solve for $\omega(t,q)$ and the optimal control δ.

(c) Demonstrate that if $\phi > 0$, the limit $\lim_{T\to+\infty} \delta^*(t,q)$ is finite for each q and independent of current time t.