Binomial Model

Tuesday, September 11, 2012
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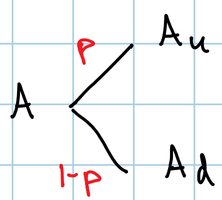
$p \in (0,1)$



$$A = \left(p A_u + (1-p) A_d\right)\frac{1}{1+r}$$

$$A \in (A_u, A_d)$$

$p = \frac{1}{2}$, $\quad A_u = 100$, $A_d = -100$ $\quad A_u = 10^6$, $A_d = -10^6$

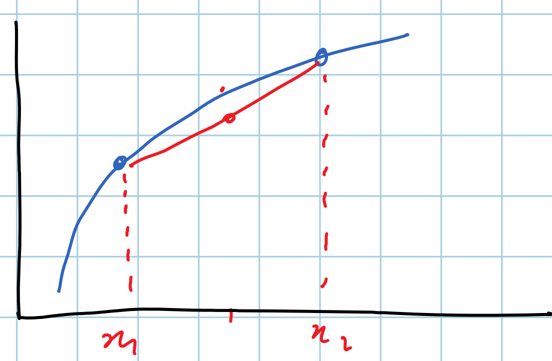
X_1 & X_2 are random outcomes

$X_1 \preceq X_2$ iff $\mathbb{E}[u(X_1)] \le \mathbb{E}[u(X_2)]$

utility fn.

$u(x): \mathbb{R} \mapsto \mathbb{R}$

- increasing
- concave



e.g. $u(x) = -e^{-\gamma x}$ $\quad (\gamma > 0)$

X_1 ~ no play

$X_2 \sim$ play but pay

$X_1 = x$

$X_2 = x - A + A_1$

① $\mathbb{E}[U(X_1)] = -e^{-\gamma x}$

const.

② $\mathbb{E}[U(X_2)] = \mathbb{E}[-e^{-\gamma(x - A + A_1)}]$

$= -e^{-\gamma(x-A)} \mathbb{E}[e^{-\gamma A_1}]$

indifference price A s.t. $\mathbb{E}[U(X_1)] = \mathbb{E}[U(X_2)]$

$\Rightarrow \boxed{A^{buy} = -\frac{1}{\gamma} \ln \mathbb{E}[e^{-\gamma A_1}]}$

$A^{guess} = \mathbb{E}[A_1]$
$\in (A_d, A_u)$

$A^{sell} = \frac{1}{\gamma} \ln \mathbb{E}[e^{\gamma A_1}]$

$e^{-\gamma A_1} \sim 1 - \gamma A_1 + o(\gamma)$

$\mathbb{E}[e^{-\gamma A_1}] \sim 1 - \gamma \mathbb{E}[A_1] + o(\gamma)$

$\ln(\mathbb{E}[e^{-\gamma A_1}]) \sim -\gamma \mathbb{E}[A_1] + o(\gamma)$

$(\ln(1+z) \sim z + o(z))$

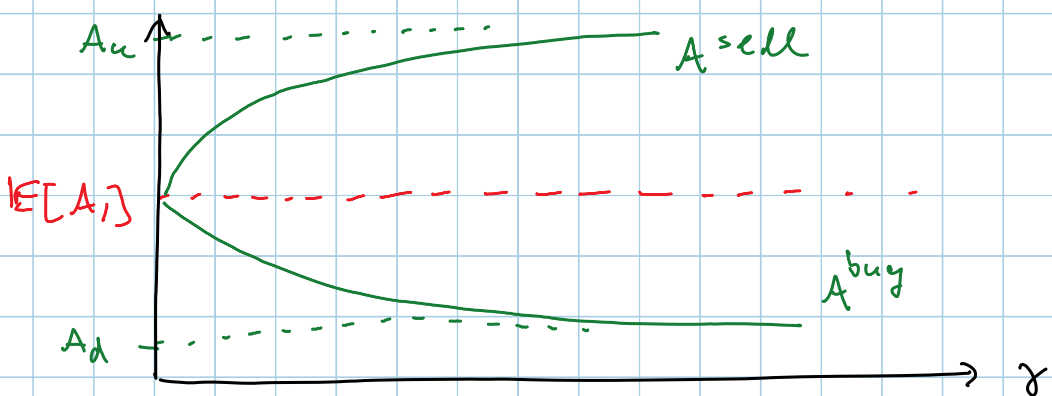
$A^{buy} \sim \mathbb{E}[A_1] + \gamma(\quad)$

$$\left(\ln(1+z) \sim z + o(z) \right)$$

$$A^{buy} \sim \mathbb{E}[A_1] + \gamma \left(\quad \right)$$

$$A^{buy} \xrightarrow[\gamma \downarrow 0]{} \mathbb{E}[A_1]$$

$$A^{sell} \xrightarrow[\gamma \downarrow 0]{} \mathbb{E}[A_1]$$



Multi Asset

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$$A \begin{cases} A_u \\ A_d \end{cases} \qquad B \begin{cases} B_u = B(1+r) \\ B_d = B(1+r) \end{cases}$$

α $\qquad\qquad$ β

$t=0 \quad t=1 \qquad\qquad t=0 \quad t=1$

$$(A, B) \begin{cases} (A_u, B_u) \\ (A_d, B_d) \end{cases}$$

Portfolio Value: $V_0 = \alpha A + \beta B$

$$V_1 = \alpha A_1 + \beta B_1$$

An arbitrage strategy (portfolio) (α, β) is one s.t.

i) $V_0 = 0$

ii) $\exists$ a t s.t. a) $\mathbb{P}(V_1 \geq 0) = 1$ (never lose)

b) $\mathbb{P}(V_1 > 0) > 0$ (sometimes win)

$$0 \begin{cases} 1 \\ 0 \end{cases}$$ is an arb.

$$1$$

$0 \left\langle \begin{matrix} 1 \\ -1 \end{matrix} \right.$ is not arb

$0 \left\langle \begin{matrix} 1 \\ 1 \end{matrix} \right.$ is an arb.

$1 \left\langle \begin{matrix} 2 \\ 1 \end{matrix} \right.$ $\quad 1 \left\langle \begin{matrix} 1 \\ 1 \end{matrix} \right.$

$+1 \qquad -1$

$\rightarrow \quad 0 \left\langle \begin{matrix} 1 \\ 0 \end{matrix} \right.$

$1 \left\langle \begin{matrix} 2 \\ 1 \end{matrix} \right.$ $\quad 1 \left\langle \begin{matrix} 1+r \\ 1+r \end{matrix} \right.$

$\alpha \qquad -\alpha$

$\rightarrow \quad 0 \left\langle \begin{matrix} (1-r)\alpha \\ -\alpha r \end{matrix} \right.$

$\begin{pmatrix} 1-r > 0 \\ -r < 0 \end{pmatrix}$ or $\begin{pmatrix} 1-r < 0 \\ -r > 0 \end{pmatrix}$

$\Downarrow \qquad\qquad \Downarrow$

$\underline{\underline{(r>0,\ r<1)}} \qquad (r<0,\ r>1)$

$A \in (A_u, A_d)$

$\underline{A_u > A_d}$, $B > 0$

$A \left\langle \begin{matrix} A_u \\ A_d \end{matrix} \right.$ $\quad B \left\langle \begin{matrix} B(1+r) \\ B(1+r) \end{matrix} \right.$

$$A \begin{cases} \\ A_d \end{cases} \qquad B \begin{cases} \\ B(1+r) \end{cases}$$

$$\alpha \qquad\qquad \beta$$

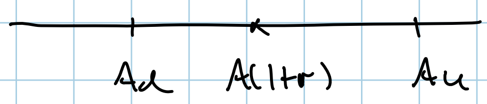
$$V_0 = 0 \Rightarrow A\alpha + B\beta = 0 \Rightarrow \beta = -\alpha \frac{A}{B}$$

$$0 \begin{cases} A_u \alpha + B(1+r)\beta = (A_u - (1+r)A)\alpha \\ A_d \alpha + B(1+r)\beta = (A_d - (1+r)A)\alpha \end{cases}$$

to avoid arb:

$$\left.\begin{aligned} A_u - (1+r)A &> 0 \\ A_d - (1+r)A &< 0 \end{aligned}\right\} \Rightarrow \boxed{A_d < A(1+r) < A_u}$$

no arb $\Leftrightarrow$



$$75 \begin{cases} 50 \\ 75 \end{cases} \qquad 1 \begin{cases} 1 \\ 1 \end{cases}$$

$$A \qquad\qquad B$$

$A_u > A_d$, $A(1+r) \in (A_d, A_u)$

~~$(C_d > C_u)$~~

$$A \begin{cases} A_u \\ A_d \end{cases} \qquad B \begin{cases} B_u = B(1+r) \\ B_d = B(1+r) \end{cases} \qquad C \begin{cases} C_u \\ C_d \end{cases}$$

$$\alpha \qquad\qquad \beta$$

~~$C(1+r) \in (C_d, C_u)$~~

choose α & β s.t.

1.

choose α & β s.t.

$$\alpha A + \beta B = C \quad \begin{cases} \alpha A_u + \beta B_u = C_u & \times B_d \\ \alpha A_d + \beta B_d = C_d & \times B_u \end{cases}$$

$$\Rightarrow \alpha (A_u B_d - A_d B_u) = C_u B_d - C_d B_u$$

$$\Rightarrow \begin{cases} \alpha = \dfrac{C_u B_d - C_d B_u}{A_u B_d - A_d B_u} \\ \beta = \dfrac{C_u A_d - C_d A_u}{B_u A_d - B_d A_u} \end{cases}$$

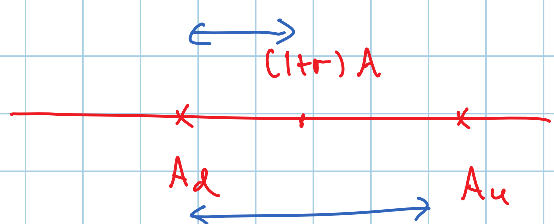
* Since (α, β) replicates outcomes of C then value of (α, β) now is value of C otherwise $\exists$ an arbitrage.

$$C = \alpha A + \beta B = \frac{1}{1+r}\left(q\, C_u + (1-q)\, C_d\right)$$

$$q = \frac{(1+r) - A_d/A}{A_u/A - A_d/A} = \frac{A(1+r) - A_d}{A_u - A_d}$$

• If $q \in (0,1)$ then write: $C = \dfrac{1}{1+r} \mathbb{E}^{Q}[C_1]$

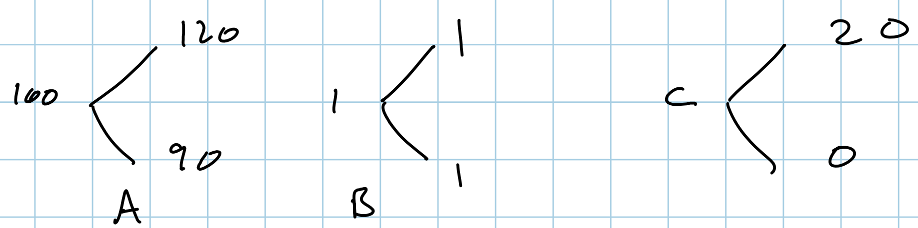
$\hookrightarrow q\, C_u + (1-q)\, C_d$



$$A_d \longleftrightarrow A_u$$

$$\boxed{C = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[C_1]}$$ otherwise $\exists$ an arb.

$\mathbb{Q}$ is called the risk-neutral measure.

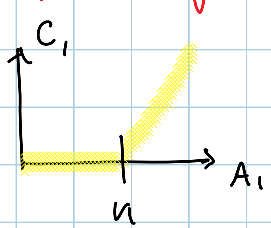


Suppose $C = 6$ construct an arb.

Call option on A struck @ 100

(right to purchase A at fixed price of 100)

$$\max(A_1 - K, 0)$$



$$120\alpha + \beta = 20$$

$$90\alpha + \beta = 0$$

$$\Rightarrow \quad 30\alpha = 20 \Rightarrow \alpha = 2/3 \Rightarrow \beta = -60$$

no arb value of $C = \frac{2}{3} \times 100 - 60 = 6\frac{2}{3}$

$$100 = \frac{1}{1+0}\left(q\,120 + (1-q)\,90\right)$$

$\Rightarrow \quad 10 = 3q + 9 \quad \Rightarrow \quad q = \frac{1}{3}$

$\Rightarrow \quad C = \frac{1}{1+0}\left(q\,20 + (1-q)\,0\right) = \frac{1}{3}20 = 6\frac{2}{3}$