Ito's Lemma and SDEs

Tuesday, November 06, 2012
2:08 PM

- Ito's lemma
- SDE, stochastic
- Ito's issometry

Ito integral (stochastic integral)

$$\int_0^t g(s, W_s)\, dW_s = \lim_{\|\Pi\| \downarrow 0} \sum_k \underbrace{g_{k-1}}_{g(t_{k-1},\, W_{t_{k-1}})} \overbrace{\Delta W_k}^{W_{t_k} - W_{t_{k-1}}}$$

$$\int_0^t W_s\, dW_s = \frac{1}{2}\left(W_t^2 - t\right)$$

$$\varepsilon = \sum_k g_{k-1} \Delta W_k - \frac{1}{2}\left(W_t^2 - t\right)$$

$$\mathbb{E}[\varepsilon] = 0, \quad \mathbb{V}[\varepsilon] \xrightarrow[\|\Pi\| \downarrow 0]{} 0$$

Ito's lemma I

$$X_t = f(W_t), \quad f \in C^2$$

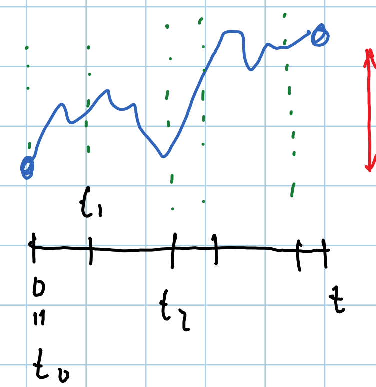
$$dX_t = \underbrace{f'(W_t)\, dW_t}_{} + \underbrace{\frac{1}{2} f''(W_t)\, dt}_{} \qquad \text{(SDE)}$$

$$dX_t = \underbrace{f'(W_t)\, dW_t}_{\text{naive calc}} + \underbrace{\tfrac{1}{2} f''(W_t)\, dt}_{\text{Ito correction}} \qquad \text{(Itô)}$$

$$X_t - X_0 = \int_0^t f'(W_s)\, dW_s + \int_0^t \tfrac{1}{2} f''(W_s)\, ds$$

Sketch of Proof:

Take a partition π



$$X_t - X_0 = f(W_t) - f(W_0)$$

$$= \sum_k \left(f(W_{t_k}) - f(W_{t_{k-1}}) \right)$$

$$W_{t_k} - W_{t_{k-1}} \overset{d}{=} \sqrt{\Delta t_k}\, Z_k \,. \qquad Z_k \sim N(0,1)$$

$\Delta W_{t_k} = (W_{t_k} - W_{t_{k-1}})$

$$f(\underbrace{w}_{W_{t_{k-1}}} + \Delta w) - f(w) = \Delta w\, f'(w) + \tfrac{1}{2} (\Delta w)^2 f''(w) + O(\Delta w)^3$$

$$\Rightarrow X_t = \sum_k \left\{ \left(f'(W_{t_{k-1}})\, \Delta W_{t_k} + \tfrac{1}{2} \underbrace{f''(W_{t_{k-1}})}_{h_{k-1}} \underbrace{(\Delta W_{t_k})^2}_{\Delta W_k} \right) + O(\Delta W_{t_k}^3) \right\}$$

$$\sum_k f'(W_{t_{k-1}})\,\Delta W_{t_k} \xrightarrow[\|\Pi\|\downarrow 0]{} \int_0^t f'(W_s)\,dW_s$$

$$\mathcal{E} = \sum_k h_{k-1}(\Delta W_k)^2 - \underbrace{\int_0^t h_s\,ds}_{\sum_k h_{k-1}\Delta t_k}$$

goal see / show $\mathbb{E}[\mathcal{E}] \xrightarrow[\|\Pi\|\downarrow 0]{} 0$ & $\mathbb{V}[\mathcal{E}] \xrightarrow[\|\Pi\|\downarrow 0]{} 0$

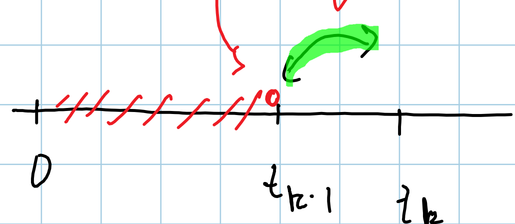
if true then can claim

$$\sum_k h_{k-1}(\Delta W_k)^2 \longrightarrow \int_0^t h_s\,ds$$

$$\mathbb{E}[\mathcal{E}] = \sum_k \left\{ \mathbb{E}[h_{k-1}(\Delta W_k)^2] - \mathbb{E}[h_{k-1}\,\Delta t_k] \right\} = 0$$

$\mathbb{E}[h_{k-1}]\cdot\mathbb{E}[(\Delta W_k)^2]$

$\hookrightarrow \Delta t_k$

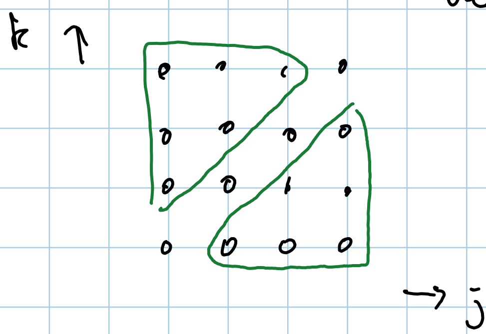


$$\mathbb{V}[\mathcal{E}] = \mathbb{V}\left(\sum_k h_{k-1}\left[(\Delta W_k)^2 - \Delta t_k\right]\right)$$

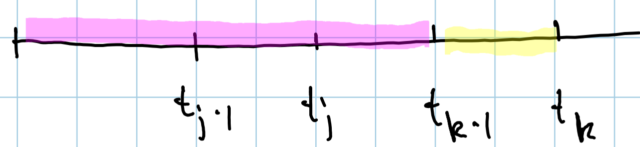
$$= \mathbb{E}\left[\left(\sum_k h_{k-1}\left[(\Delta W_k)^2 - \Delta t_k\right]\right)^2\right] \quad \text{b/c } \mathbb{E}[\mathcal{E}] = 0$$

$$= \mathbb{E}\left[\sum_{j,k} h_{k-1}\, h_{j-1}\, [(\Delta W_k)^2 - \Delta t_k]\cdot[(\Delta W_j)^2 - \Delta t_j] \right]$$

but $$\sum_{j,k} \alpha_j \alpha_k = \underbrace{2\sum_{j<k} \alpha_j \alpha_k}_{\to L} + \underbrace{\sum_k \alpha_k^2}_{\hookrightarrow D}$$



$$\mathbb{E}[L] = \sum_{j<k} \mathbb{E}\left[h_{k-1}\, h_{j-1}\, [(\Delta W_j)^2 - \Delta t_j]\cdot((\Delta W_k)^2 - \Delta t_k) \right]$$



$$\to \mathbb{E}[\quad]\;\underbrace{\mathbb{E}[\quad]}_{\hookrightarrow 0}$$

$$\mathbb{E}[D] = \sum_k \mathbb{E}\left[h_{k-1}^2 \cdot (\Delta W_k^2 - \Delta t_k)^2 \right]$$

$$= \sum_k \mathbb{E}[h_{k-1}^2]\cdot \underbrace{\mathbb{E}[(\Delta W_k^2 - \Delta t_k)^2]}$$

$$\mathbb{E}\left[\left((\sqrt{\Delta t_k}\, Z_k)^2 - \Delta t_k \right)^2 \right]$$

$$= \underbrace{\mathbb{E}[(Z_k^2 - 1)^2]}\, \Delta t_k^2$$

$= \beta < +\infty$

$$= \beta \sum_k \mathbb{E}[h_{k-1}^2] \Delta t_k^2$$

$\leq \Delta t_k \|\pi\|$

$$\leq \beta \|\pi\| \Big(\underbrace{\mathbb{E} \sum_k h_{k-1}^2 \Delta t_k}_{\xrightarrow[\|\pi\|\downarrow 0]{} \int_0^t h_s^2 \, ds < +\infty} \Big) \xrightarrow[\|\pi\|\downarrow 0]{} 0$$

$$\Rightarrow \mathbb{V}[\mathcal{E}] \xrightarrow[\|\pi\|\downarrow 0]{} 0$$

$\therefore$ by LLN $\boxed{\sum_k h_{t_{k-1}} (\Delta W_{t_k})^2 \xrightarrow[\|\pi\|\downarrow 0]{} \int_0^t h_s \, ds}$ a.s.

(can be seen as a generalization of quadratic variation)

$$[W, W]_t = \lim_{\|\pi\|\downarrow 0} \sum_k (\Delta W_k)^2 = t \quad \text{a.s.}$$

$$X_t - X_0 = \int_0^t f'(W_s) \, dW_s + \frac{1}{2} \int_0^t f''(W_s) \, ds$$

$$+ \lim_{\|\pi\|\downarrow 0} \sum_k O(\Delta W_k^3)$$

$\Delta W_k = (\Delta t_k)^{1/2} Z_k$

$$\lim_{\|\pi\|\to 0} \sum_k O(\Delta W_k^3)$$

$$\Delta W_k = (\Delta t_k)^{1/2} Z_k$$

$$\hookrightarrow \le \sum_k \Delta t_k^{3/2} |Z_k|^3 \cdot |R_{k-1}|$$

$$\Delta t_k^{3/2} \hookrightarrow \Delta t_k \cdot \|\pi\|^{1/2}$$

$$\le \|\pi\|^{1/2} \underbrace{\sum_k \Delta t_k |Z_k^3| |R_{k-1}|}_{\hookrightarrow\ \mathbb{E}[|Z^3|] \int_0^t |R_s| \, ds} \xrightarrow[\|\pi\|\to 0]{} 0$$

$$\Rightarrow \boxed{\begin{aligned} X_t - X_0 &= \int_0^t f'(W_s)\, dW_s + \int_0^t \tfrac{1}{2} f''(W_s)\, ds \\ \text{or} \quad dX_t &= f'(W_t)\, dW_t + \tfrac{1}{2} f''(W_t)\, dt \end{aligned}}$$

<u>Ito lemma II</u>:

$X_t = f(t, W_t)$ and $f \in C^{1,2}$ then

$$X_t - X_0 = \int_0^t \partial_w f(s, W_s)\, dW_s$$

$$+ \int_0^t \left(\partial_t f(s, W_s) + \tfrac{1}{2} \partial_{ww} f(s, W_s) \right) ds$$

e.g., $f(t, \omega) = t\, \omega^2$,

$$X_t = t\,W_t^2$$

$$t\,W_t^2 - 0 = \int_0^t \underbrace{2sW_s}_{\partial_w f(s,W_s)}\, dW_s$$

$$+ \int_0^t \Big(\underbrace{W_s^2}_{\partial_t f(s,W_s)} + \tfrac{1}{2} \overbrace{2}^{\partial_{ww} f(s,W_s)} s \Big)\, ds$$

$$\Big(dX_t = 2tW_t\, dW_t + (W_t^2 + \tfrac{1}{2} 2t)\, dt \Big)$$

$$W_t \longmapsto f(t, W_t) = X_t$$

$$dX_t = \mu(t, W_t)\, dt + \sigma(t, W_t)\, dW_t$$

$$X_t \longmapsto g(t, X_t) = Y_t$$

$$dY_t = \alpha(t, X_t)\, dt + \beta(t, X_t)\, dW_t$$

Ito's Lemma III:

Suppose that $dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$

& $Y_t = g(t, X_t)$, $g \in C^{1,2}$ then

$$dY_t = \partial_x g(t, X_t)\, dX_t + \left[\partial_t g(t, X_t) + \frac{1}{2}\sigma^2(t, X_t)\, \partial_{xx} g(t, X_t)\right] dt$$

or

$$dY_t = \sigma(t, X_t)\, \partial_x g(t, X_t)\, dW_t + \Big[\partial_t g(t, X_t) + \mu(t, X_t)\, \partial_x g(t, X_t) + \frac{1}{2}\sigma^2(t, X_t)\, \partial_{xx} g(t, X_t)\Big] dt$$

(note: $\mu = 0$, $\sigma = 1$ you get the Itô's lemma II)

e.g. suppose that $dX_t = X_t\, dW_t$

$Y_t = \ln(X_t)$. $\quad g = \ln x$

$\partial_t g = 0$, $\quad \partial_x g = \frac{1}{x}$, $\quad \partial_{xx} g = -\frac{1}{x^2}$

$$\Rightarrow \quad dY_t = \underbrace{X_t}_{\sigma(t,X_t)} \cdot \underbrace{\frac{1}{X_t}}_{\partial_x g(t,X_t)}\, dW_t + \Big(\underbrace{0}_{\partial_t g(t,X_t)} + \underbrace{0}_{\mu(t,X_t)} \cdot \frac{1}{x} + \frac{1}{2} \cdot \underbrace{X_t^2}_{\sigma^2(t,X_t)} \cdot \overbrace{\left(-\frac{1}{X_t^2}\right)}^{\partial_{xx} g(t,X_t)}\Big)\, dt$$

$$= dW_t - \frac{1}{2}\, dt$$

$$= dW_t - \frac{1}{2}\, dt$$

$$\left(\text{naive: } dY_t = \frac{dX_t}{X_t} = \frac{X_t\, dW_t}{X_t} = dW_t\right)$$

---

e.g. derive an integration by parts formula for

$$I_t = \int_0^t (e^{W_s} \cdot s)\, dW_s$$

---

suppose $\quad \dfrac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t \quad$ (Geometric Brownian motion)

$$Y_t = \ln X_t, \qquad g = \ln x$$

$$\partial_t g = 0, \quad \partial_x g = \frac{1}{x}, \quad \partial_{xx} g = -\frac{1}{x^2}$$

$$dY_t = \underbrace{\frac{1}{X_t}}_{\partial_x g(t, X_t)} \cdot \underbrace{\sigma X_t}_{\sigma(t, X_t)} \cdot dW_t$$

$$+ \Big[\underbrace{0}_{\partial_t g(t, X_t)} + \underbrace{\mu X_t}_{\mu(t, X_t)} \cdot \underbrace{\frac{1}{X_t}}_{\partial_x g(t, X_t)} + \frac{1}{2} \cdot \underbrace{\sigma^2 X_t^2}_{\sigma^2(t, X_t)} \cdot \underbrace{\left(-\frac{1}{X_t^2}\right)}_{\partial_{xx} g(t, X_t)} \Big]\, dt$$

$$= \sigma\, dW_t + \left(\mu - \frac{1}{2}\sigma^2\right) dt$$

---

volatility

$$\underbrace{\frac{dX_t}{X_t}}_{\text{asset's return}} = \mu\, dt + \sigma\, dW_t$$

μ: drift/ mean return

Black-Scholes Model

(Geometric Brownian Motion)

Want to solve the SDE for X_t.

[analog of solving for X_t in

$$\frac{dX_t}{X_t} = \mu\, dt$$

$$\frac{dX_t}{X_t} = \underbrace{d(\ln X_t) = \mu\, dt}$$

$$\ln X_t - \ln X_0 = \mu t$$

$$X_t = X_0\, e^{\mu t}.$$]

we just showed that

$$d(\ln X_t) = (\mu - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, dW_t$$

$$\ln X_t - \ln X_0 = (\mu - \tfrac{1}{2}\sigma^2)\, t + \sigma W_t$$

$\Rightarrow$

$$\boxed{X_t = X_0\, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}} \quad \text{(GBM)}$$

$$\boxed{X_t = X_0\, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}} \qquad (GBM)$$

$$\mathbb{E}[X_t] = X_0\, e^{(\mu - \frac{1}{2}\sigma^2)t}\, \mathbb{E}[e^{\sigma W_t}] \qquad W_t \sim N(0, t)$$

$$= X_0\, e^{(\mu - \frac{1}{2}\sigma^2)t}\, e^{\frac{1}{2}\sigma^2 t}$$

$$= X_0\, e^{\mu t}$$

NOTE: expected continuously compounded rate of return is μ – irrespective of σ.

$$\left(\begin{aligned} dX_t &= \mu X_t\, dt + \sigma X_t\, dW_t \\ d\,\mathbb{E}[X_t] &= \mu\, \mathbb{E}[X_t]\, dt + \sigma\, \overset{\to 0}{\mathbb{E}[X_t\, dW_t]} \end{aligned} \right)$$

Vasicek Model of Interest Rates

Tuesday, November 06, 2012
4:35 PM

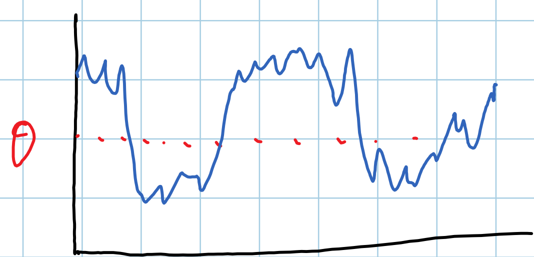
Vasicek Model

in discrete time: $r_n - r_{n-1} = \kappa(\theta - r_{n-1})\Delta t + \sigma\sqrt{\Delta t}\, x_n$

$x_1, x_2, \ldots$ iid Bernoulli $P(x_k = \pm 1) = \frac{1}{2}$.

continuous time version:

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t$$



$$dx_t = \kappa(\theta - x_t)\,dt$$

$$d(e^{\kappa t} x_t) = \kappa e^{\kappa t} x_t\, dt + e^{\kappa t} dx_t$$

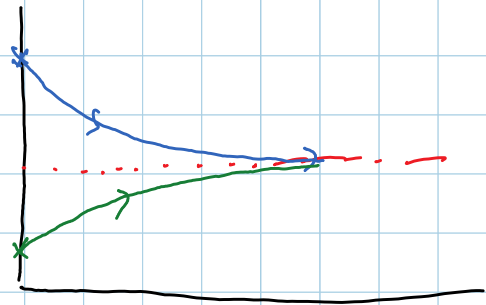
$$= e^{\kappa t}[\kappa x_t\, dt + \kappa(\theta - x_t)\, dt]$$

$$= e^{\kappa t} \kappa\theta\, dt$$

$$\Rightarrow e^{\kappa t} x_t - x_0 = \int_0^t e^{\kappa s} \kappa\theta\, ds$$

$$= \frac{e^{\kappa t} - 1}{\kappa} \cdot \kappa\theta$$

$$\Rightarrow x_t = x_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})$$



$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dW_t$$

$$Y_t = e^{\kappa t} r_t\,, \qquad g(t,r) = e^{\kappa t} r$$

$$dY_t = \underbrace{e^{\kappa t}}_{\partial_r g(t,r_t)} \cdot \underbrace{\sigma}_{\sigma(t,r_t)}\, dW_t$$

$$+ \Big[\underbrace{\kappa e^{\kappa t} r_t}_{\partial_t g(t,r_t)} + \underbrace{\kappa(\theta - r_t)}_{\mu(t,r_t)} \cdot \underbrace{e^{\kappa t}}_{\partial_r g(t,r_t)} + \frac{1}{2} \underbrace{\sigma^2}_{\sigma^2(t,r_t)} \cdot \underbrace{0}_{\partial_{rr} g(t,r_t)}\Big]\,dt$$

$$= e^{\kappa t}\left[\sigma\,dW_t + \kappa\theta\,dt\right]$$

$$\Rightarrow \underset{\displaystyle e^{\kappa t} r_t}{\underset{\downarrow}{Y_t}} - \underset{\displaystyle r_0}{\underset{\downarrow}{Y_0}} = \int_0^t \sigma e^{\kappa s}\,dW_s + \kappa\theta \underbrace{\int_0^t e^{\kappa s}\,ds}_{\hookrightarrow \frac{e^{\kappa t}-1}{\kappa}}$$

$$\Rightarrow \boxed{r_t = r_0 e^{-\kappa t} + \theta(1-e^{-\kappa t}) + \int_0^t \sigma e^{-\kappa(t-s)} dW_s}$$

$$\neq h(t, W_t)$$

Ornstein-Uhlenbeck process

(OU)

$$\mathbb{E}\left[\int_0^t g_s \, dW_s\right] = 0$$

$$\mathbb{E}\left[\sum_k g_{t_{k-1}} \Delta W_{t_k}\right] = \sum_k \mathbb{E}[g_{t_{k-1}} \Delta W_{t_k}] = 0$$

0

true for "any" g.

Suppose g is deterministic... what can you say about $\int_0^t g_s \, dW_s$?

We know that