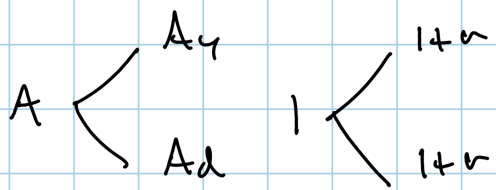
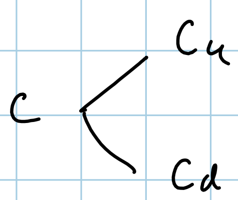
Review

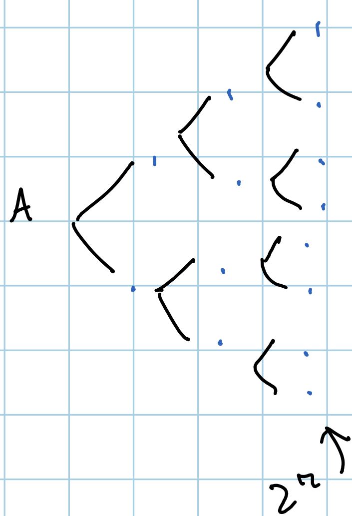
Tuesday, September 18, 2012
2:11 PM

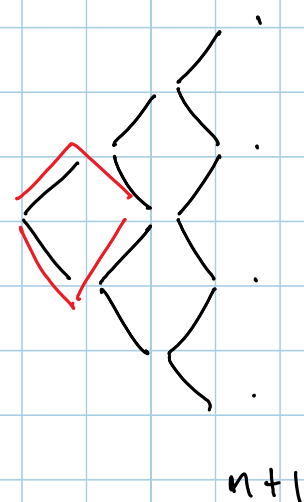


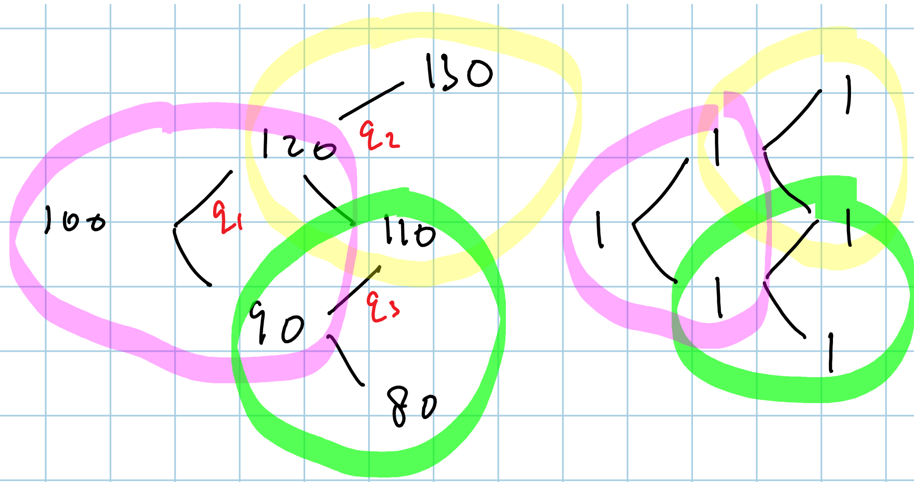
$$\boxed{A_d < A(1+r) < A_u} \iff \nexists \text{ arb.}$$



$$\boxed{\exists\ \mathbb{Q} \text{ s.t. } C = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[C_1]} \iff \nexists \text{ arb}$$







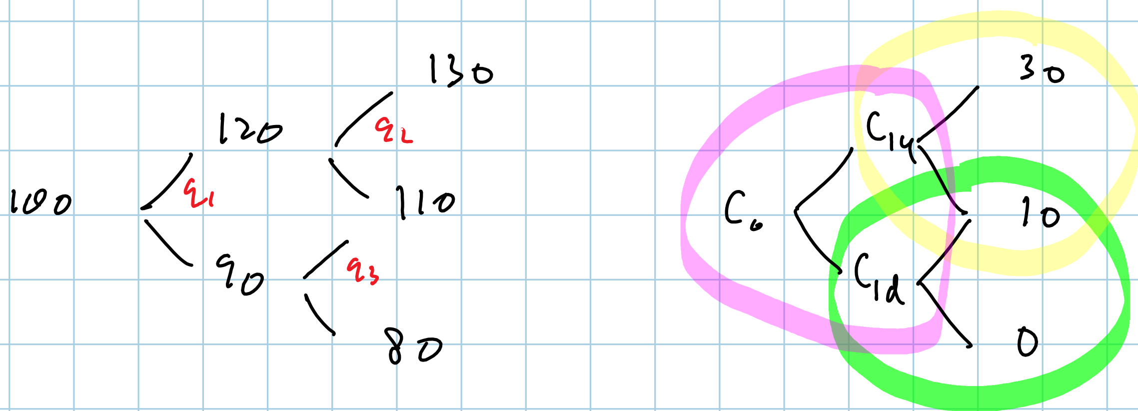
$$120 = \frac{1}{1+0}\left(130\, q_2 + 110\,(1-q_2)\right) \Rightarrow q_2 = \#$$

$$90 = \frac{1}{1+0}\left(110\, q_3 + 80\,(1-q_3)\right) \Rightarrow q_3 = \#$$

$$100 = \frac{1}{1+0}\left(120\, q_1 + 90\,(1-q_1)\right) \Rightarrow q_1 = \#$$

call struck @ 100 maturity $T=2$

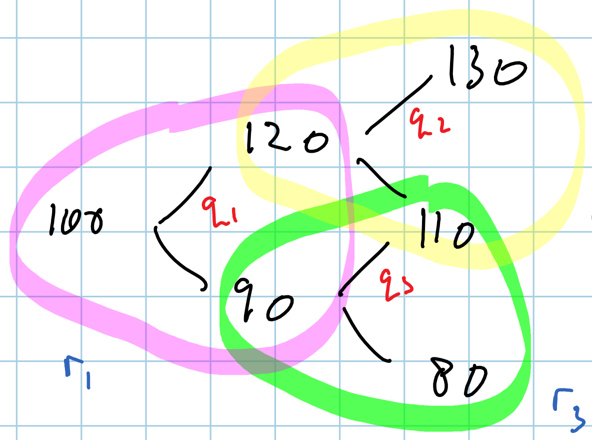
$\max(A_T - K, 0) \triangleq (A_T - K)_+$

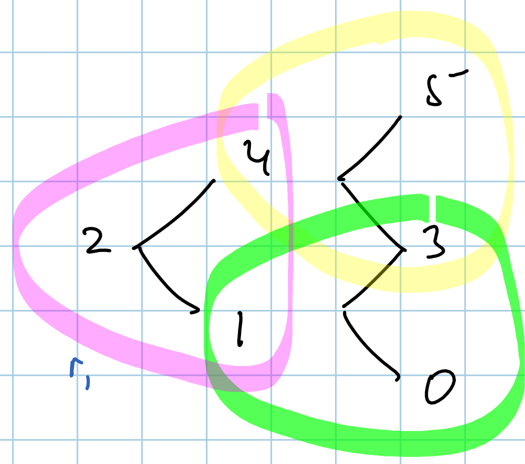


$$C_{1d} = \frac{1}{1+0}\left(10\, q_3 + 0\,(1-q_3)\right) = \#$$

$$C_{1u} = \frac{1}{1+0}\left(30\, q_2 + 10\,(1-q_2)\right) = \#$$

$$C_0 = \frac{1}{140}\left(C_{1u}\, q_1 + C_{1d}\,(1-q_1)\right) = \#$$



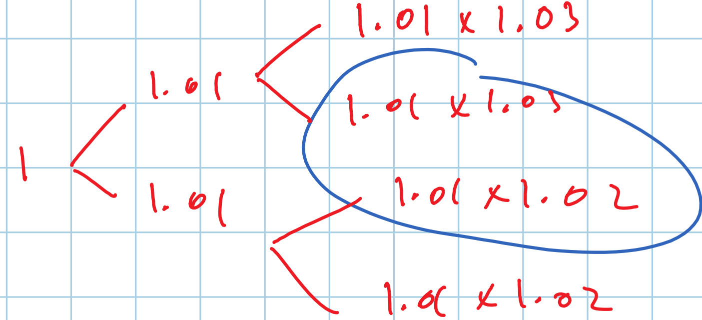


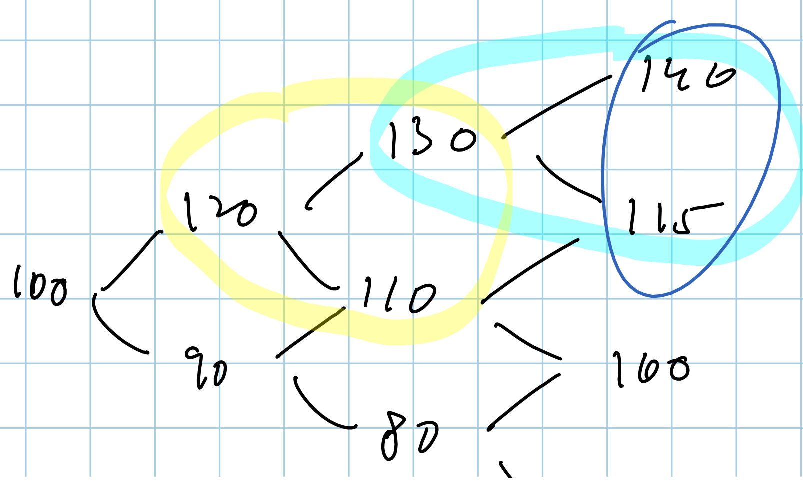
$$120 = \frac{1}{1+r_2}\left[130\, q_2 + 110\,(1-q_2)\right]$$

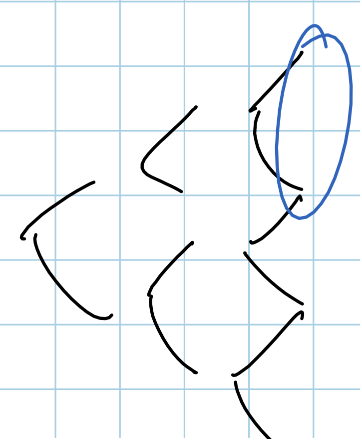
$$4 = \frac{1}{1+r_2}\left[5\, q_2 + 3\,(1-q_2)\right]$$

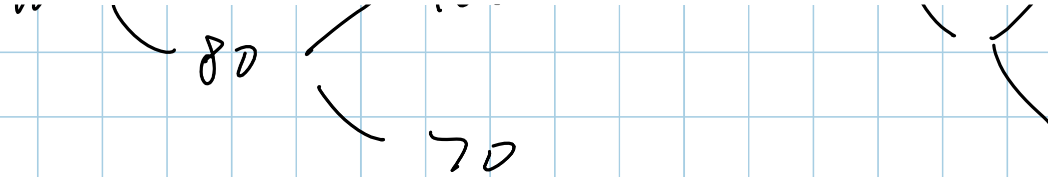
$\Rightarrow$ $q_2 = \#$

$r_2 = \#$








80
70

CRR Model

Tuesday, September 18, 2012
2:50 PM

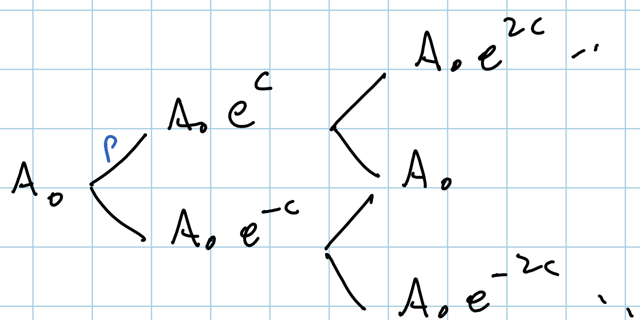
Δt time step

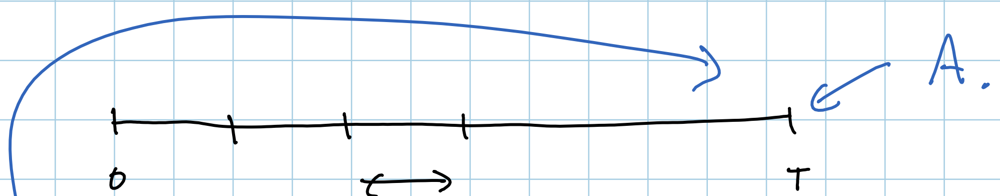
r = const. continuous compounding

$(1+r) \to e^{r\Delta t}$

$$A_n = A_{n-1} e^{c x_n}, \qquad x_1, x_2, \ldots \text{ iid Bernoulli r.v.}$$

$$\mathbb{P}(x_k = +1) = p$$
$$\mathbb{P}(x_k = -1) = 1-p$$





$\Delta t = \frac{T}{N}$

N — number of steps

$$A_T = A_0 \, e^{c x_1} e^{c x_2} \cdots e^{c x_{N-1}}$$

$$= A_0 \exp\Big\{ \underbrace{c \sum_{n=1}^{N} x_n}_{X^N} \Big\}$$

$$\mathbb{E}[X^N] = c \sum_{n=1}^{N} \mathbb{E}[x_n]$$

$$= c\, N\, \mathbb{E}[x_1] = c N \big(+1\, p + (-1)(1-p)\big)$$

$$= c N (2p-1) = \mu \Delta t N$$

$$= c N (2p-1) = \mu \Delta t \, N$$

(to match data)

$$\mathbb{V}[X^N] = c^2 N \mathbb{V}[x_1]$$

$$= c^2 N \left(\mathbb{E}[x_1^2] - (\mathbb{E}[x_1])^2 \right)$$

$$= c^2 N \left(1 - (2p-1)^2 \right) = \sigma^2 \Delta t \, N$$

(to match data)

<u>data</u>

S_1

S_2

S_3

$\vdots$

S_M

$\ln(S_2/S_1) = r_1$

$\ln(S_3/S_2) = r_2$

annualized return

$$\mathbb{E}[r] = \mu \Delta t \sim \frac{1}{M} \sum_{m=1}^{M} r_m$$

$$\mathbb{V}[r] = \sigma^2 \Delta t \sim \frac{1}{M} \sum_{m=1}^{M} (r_m - \mu \Delta t)^2$$

annualized variance

$$\Rightarrow \boxed{\begin{aligned} p &= \frac{1}{2}\left(1 + \frac{\mu}{c} \Delta t\right) \\ c &= \sigma \sqrt{\Delta t} + \ldots \end{aligned}}$$

CRR

Cox, Rubenstein, Ross

$$p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma} \sqrt{\Delta t}\right)$$

$A e^{\sigma\sqrt{\Delta t}}$ $\quad$ $e^{r \Delta t}$

A $\quad$ p $\quad$ 1

$$A \begin{cases} A e^{\sigma\sqrt{\Delta t}} \\ A e^{-\sigma\sqrt{\Delta t}} \end{cases} \quad p \qquad\qquad 1 \begin{cases} e^{r\Delta t} \\ e^{r\Delta t} \end{cases}$$

$$A = e^{-r\Delta t}\, \mathbb{E}^{\mathbb{Q}}[A_{\Delta t}]$$

$$= e^{-r\Delta t}\left(q\, A e^{\sigma\sqrt{\Delta t}} + A e^{-\sigma\sqrt{\Delta t}}(1-q)\right)$$

$$\Rightarrow \quad \boxed{q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}}$$

$$e^{z} \sim 1 + z + \tfrac{1}{2}z^2 + \cdots$$

$$q \sim \frac{1}{2}\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \cdots$$

$$\mu \to \mu - \tfrac{1}{2}\sigma^2$$

$$p \sim \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right) + \cdots$$

$$\mathbb{E}^{\mathbb{P}}\left[\ln\left(\frac{A_T}{A_0}\right)\right] = \mu T \qquad \mathbb{E}^{\mathbb{Q}}\left[\ln\left(\frac{A_T}{A_0}\right)\right] = \left(r - \tfrac{1}{2}\sigma^2\right) T$$

$$\mathbb{V}^{\mathbb{P}}\left[\ln\left(\frac{A_T}{A_0}\right)\right] = \sigma^2 T \qquad \mathbb{V}^{\mathbb{Q}}\left[\ln\left(\frac{A_T}{A_0}\right)\right] = \sigma^2 T$$

"$\mathbb{P}$ & $\mathbb{Q}$ variances are identical"!

"$\mathbb{P}$ & $\mathbb{Q}$ means are not"!

limiting distribution:

$$\ln\left(\frac{A_T}{A_0}\right) = X_N = c \sum_{n=1}^{N} x_n = \sigma\sqrt{\Delta t} \sum_{n=1}^{N} x_n \xrightarrow[N\uparrow+\infty]{d,\,\mathbb{P}} \mathcal{N}(\mu T\,;\ \sigma^2 T)$$

by CLT

$$\ln\left(\frac{A_T}{A_0}\right) = X_N \xrightarrow[N\uparrow+\infty]{d,\,\mathbb{Q}} \mathcal{N}\left((r-\tfrac{1}{2}\sigma^2)T\,;\ \sigma^2 T\right)$$

by CLT

$$A_T \overset{d}{=} A_0\, e^{\mu T + \sigma\sqrt{T}\, Z_{\mathbb{P}}}, \qquad Z_{\mathbb{P}} \underset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$$

also $$A_T \overset{d}{=} A_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z_{\mathbb{Q}}}, \qquad Z_{\mathbb{Q}} \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

Asset prices (in the limit as $N\uparrow+\infty$) are lognormal r.v. (at a fixed pt. in time)

$$\mathbb{E}^{\mathbb{P}}[A_T] = A_0\, e^{\mu T}\, \mathbb{E}^{\mathbb{P}}\left[e^{\sigma\sqrt{T}\, Z_{\mathbb{P}}}\right]$$

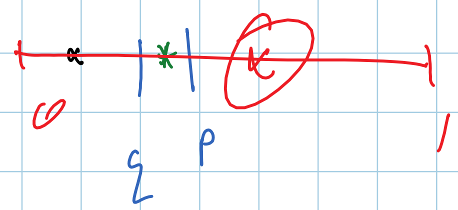
$$= A_0\, e^{\mu T}\, e^{\frac{1}{2}\sigma^2 T} = A_0\, e^{(\mu+\frac{1}{2}\sigma^2)T}$$

(recall $\mathbb{E}[e^{uZ}] = e^{\frac{1}{2}u^2}$, $Z \sim \mathcal{N}(0,1)$)

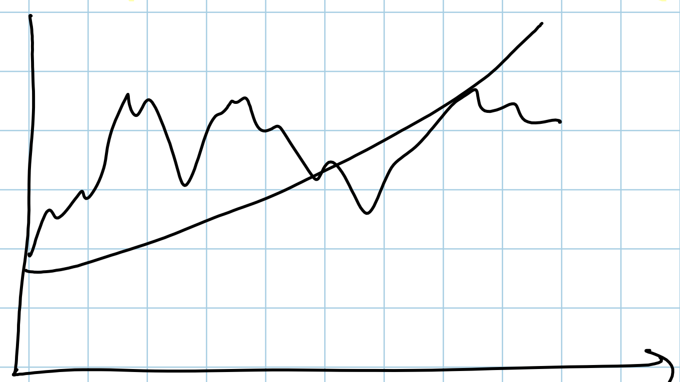
$$\mathbb{E}^{\mathbb{Q}}[A_T] = A_0\, e^{(r-\frac{1}{2}\sigma^2)T}\, \mathbb{E}^{\mathbb{Q}}\left[e^{\sigma\sqrt{T}\, Z_{\mathbb{Q}}}\right]$$

$$= A_0\, e^{rT}$$

IF we started with calibrating to daily returns (continuously compounded) and <u>not</u> daily log-returns then $\mu \to \mu - \frac{1}{2}\sigma^2$.



A_t , $e^{(r-\frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\, Z}$



American Options

Tuesday, September 18, 2012
4:26 PM

$$A_T \xrightarrow[N \uparrow +\infty]{d} A_0\, e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z}, \quad Z \underset{\mathbb{P}}{\sim} N(0,1)$$

$$\rightarrow \quad A_T \xrightarrow[N \uparrow +\infty]{d} A_0\, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z}, \quad Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

European Options pay $\varphi(A_T)$ @ T

$$V_t = e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}[\, \underbrace{\varphi(A_T)}_{\hookrightarrow V_T}\,]$$

American Option allows you to decide when (τ) to exercise (meaning receive $\varphi(A_\tau)$)

bounded above by T.

American Put $\varphi(A_\tau) = (K - A_\tau)_+$

(LaTeX \varphi ϕ, Φ, φ)

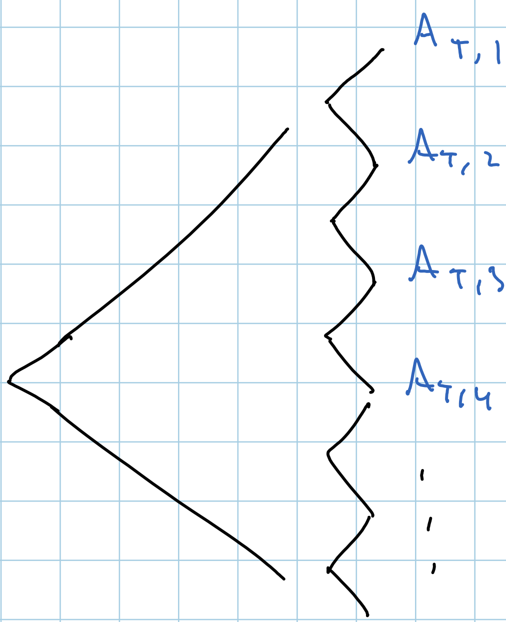
$$\begin{tikzcd}[row sep=small] & A e^{\sigma\sqrt{\Delta t}} \\ A \arrow[ur, "q"] \arrow[dr] & \\ & A e^{-\sigma\sqrt{\Delta t}} \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & e^{r\Delta t} \\ 1 \arrow[ur, "q"] \arrow[dr] & \\ & e^{r\Delta t} \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & P_u \\ P \arrow[ur, "q"] \arrow[dr] & \\ & P_d \end{tikzcd}$$

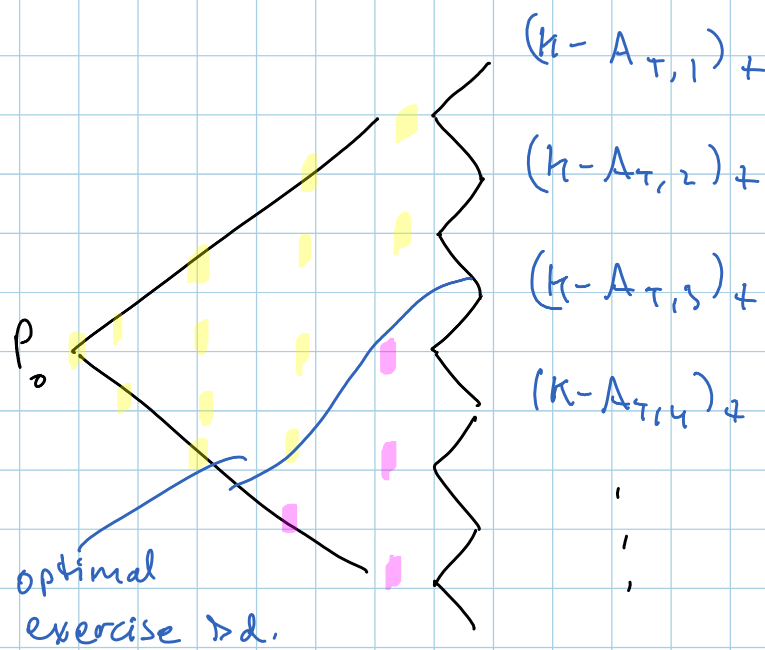
$$P^{hold} = e^{-r\Delta t}\left(q P_u + (1-q) P_d\right)$$

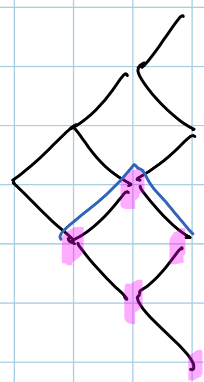
$-$ Intrinsic

$$P^{I} = (K - A)_+$$

$$P = \max(P^{H}, P^{I})$$







Quiz

Tuesday, September 18, 2012
5:03 PM