Ito's Isometry

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$$\mathbb{E}\left[\left(\int_0^t g_s\, dW_s\right)\left(\int_0^t h_s\, dW_s\right)\right]$$

$$= \mathbb{E}\left[\int_0^t g_s h_s\, ds\right]$$

here $g_t = g(t, W_t)$, $h_t = h(t, W_t)$

$$\mathcal{E} = \mathbb{E}\left[\left(\int_0^t g_s\, dW_s\right)\left(\int_0^t h_s\, dW_s\right) - \int_0^t g_s h_s\, ds\right] = 0$$

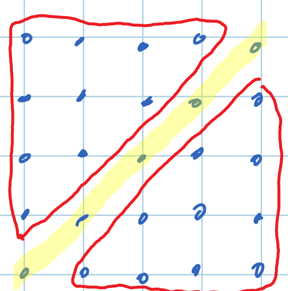
$$\left(\sum_k g_{t_{k-1}} \Delta W_{t_k}\right)\left(\sum_j h_{t_{j-1}} \Delta W_{t_j}\right) - \sum_k g_{t_{k-1}} h_{t_{k-1}} \Delta t_k$$

$$\sum_{k,j} g_{k-1} h_{j-1} \Delta W_k \Delta W_j$$

$$= \sum_{k<j} g_{k-1} h_{j-1} \Delta W_k \Delta W_j \qquad A$$

$$+ \sum_{j<k} g_{k-1} h_{j-1} \Delta W_k \Delta W_j \qquad B$$

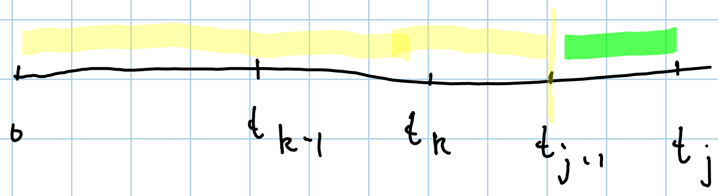
$$+ \sum_{k} g_{k-1} h_{k-1} \Delta W_k^2 \qquad C$$



$$\mathbb{E}[A] = \sum_{k<j} \mathbb{E}[g_{k-1} h_{j-1} \Delta W_k \Delta W_j] = \sum \mathbb{E}[g_{k-1} h_{j-1} \Delta W_k]$$

$$\mathbb{E}[A] = \sum_{k<j} \mathbb{E}[g_{k-1} h_{j-1} \Delta W_k \Delta W_j] = \sum_{k<j} \mathbb{E}[g_{k-1} h_{j-1} \Delta W_k] \cdot \underbrace{\mathbb{E}[\Delta W_j]}_{0}$$

$$= 0$$



similarly $\mathbb{E}[B] = 0$

so

$$\mathcal{E} = \mathbb{E}\left[\sum_k g_{k-1} h_{k-1} (\Delta W_k^2 - \Delta t_k) \right]$$

$$= \sum_k \mathbb{E}[g_{k-1} h_{k-1}] \cdot \underbrace{\mathbb{E}[\Delta W_k^2 - \Delta t_k]}_{0} = 0$$

holds $\forall\ \Pi \Rightarrow \mathbb{E}\left[\int_0^t g_s\, dW_s \int_0^t h_s\, dW_s \right] = \mathbb{E}\left[\int_0^t g_s h_s\, ds \right]$

$\blacksquare$

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recall we had an O-U

$$dr_t = \kappa(\theta - r_t)\, dt + \sigma\, dW_t$$

whose solution is

$$r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{-\kappa(t-u)}\, dW_u$$

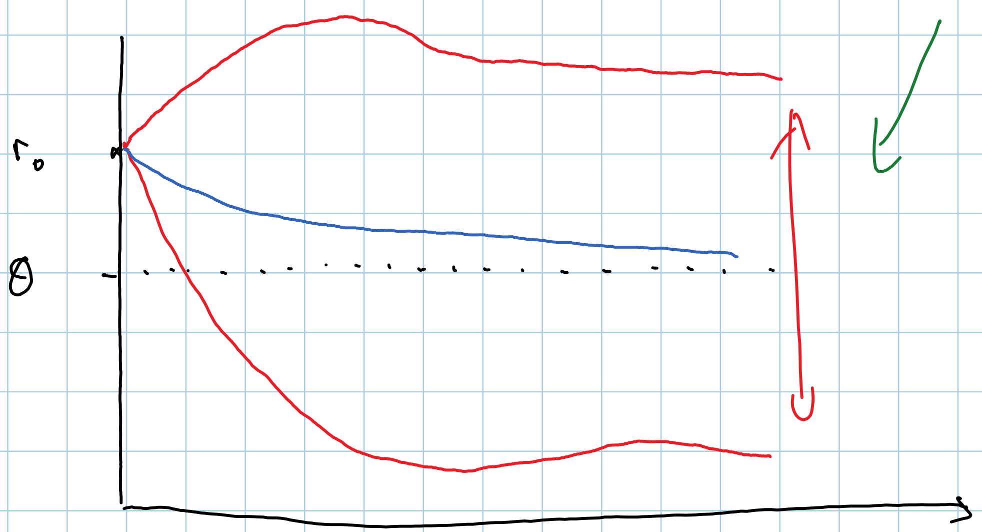
$$\sim \mathcal{N}\left(r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t});\ \Sigma^2 \right)$$

$$\Sigma^2 = \mathbb{E}\left[\left(\sigma \int_0^t e^{-\kappa(t-u)}\, dW_u\right)^2\right]$$

$$= \sigma^2\, \mathbb{E}\left[\int_0^t \left(e^{-\kappa(t-u)}\right)^2 du\right]$$

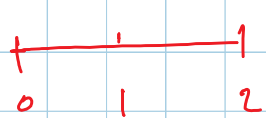
$$= \sigma^2 \int_0^t e^{-2\kappa(t-u)}\, du$$

$$= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right) \xrightarrow[t \to +\infty]{} \frac{\sigma^2}{2\kappa}$$



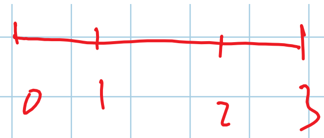
$$I_t = \int_0^t W_s\, dW_s \overset{?}{\sim} \mathcal{N}(\cdot\,;\,\cdot)$$

$$W_0\,(W_t - W_0) = 0$$



$$\underbrace{W_0\,(W_1 - W_0)}_{\to\, 0} + W_1\,(W_2 - W_1)$$

$$0 + W_1(W_2 - W_1) + W_2(W_3 - W_2)$$



$$\mathbb{E}[I_t] = 0$$

$$\mathbb{V}[I_t] = \mathbb{E}[I_t^2] = \mathbb{E}\left[\left(\int_0^t W_s \, dW_s\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t W_s^2 \, ds\right]$$

$$= \int_0^t \mathbb{E}[W_s^2] \, ds = \int_0^t s \, ds = \frac{1}{2} t^2$$

recall $I_t = \int_0^t W_s \, dW_s = \frac{1}{2}(W_t^2 - t)$

$$\mathbb{E}[I_t^2] = \mathbb{E}\left[\frac{1}{4}(W_t^2 - t)^2\right]$$

$$= \frac{1}{4} \mathbb{E}\left[W_t^4 - 2t W_t^2 + t^2\right]$$

$$= \frac{1}{4}\left(3t^2 - 2t \cdot t + t^2\right)$$

$$= \frac{1}{2} t^2$$

$$I_t = \int_0^t e^{W_s} \, dW_s$$

¿ $\mathbb{E}[I_t] = $? 0

¿ $\mathbb{V}[I_t] = $?

$$\mathbb{E}[I_t^2] = \mathbb{E}\left[\int_0^t e^{2W_s} \, ds\right]$$

$$= \int_0^t \mathbb{E}[e^{2W_s}]\, ds$$

$$e^{\frac{1}{2}\cdot 4 \cdot s} = e^{2s}$$

$$= \frac{1}{2}\left(e^{2t} - 1\right)$$

Black Scholes PDE

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$$A_0 \left\langle \begin{array}{l} \overset{p}{\nearrow}\ A_u \\ \searrow\ A_d \end{array} \right. \qquad 1 \left\langle \begin{array}{l} \overset{p}{\nearrow}\ e^{r\Delta t} \\ \searrow\ e^{r\Delta t} \end{array} \right. \qquad C_0 \left\langle \begin{array}{l} \overset{p}{\nearrow}\ C_u = \alpha A_u + \beta e^{r\Delta t} \\ \searrow\ C_d = \alpha A_d + \beta e^{r\Delta t} \end{array} \right.$$

α $\qquad$ β

$$C_0 = \alpha A_0 + \beta$$

$$= e^{-r\Delta t}\left(q C_u + (1-q) C_d\right)$$

$$q = \frac{e^{r\Delta t} - A_d/A_0}{A_u/A_0 - A_d/A_0}$$

-1 of C

α of A

β of M

$$-C_0 + \alpha A_0 + \beta \left\langle \begin{array}{l} 0 \\ 0 \end{array} \right.$$

$$\| \\ 0$$

Black-Scholes Model

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$$

Geometric Brownian Motion - GBM

$$X_t = \ln S_t$$

$$dX_t = \left(\mu - \tfrac{1}{2}\sigma^2\right)dt + \sigma\, dW_t$$

$$\Rightarrow X_t - X_0 = \left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t$$

$$\Rightarrow S_t = S_0\, e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

$$\overset{d}{=} S_0\, e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\, Z}, \quad Z \underset{\mathbb{P}}{\sim} N(0,1)$$

- can trade in S_t, hold α_t units of S_t.

- interests are constant r

$$\frac{dB_t}{B_t} = r\,dt, \quad B_0 = 1, \qquad B_t = e^{rt}$$

  - can trade in B_t. Hold β_t units of B_t

- want to price a claim C which pays

  $\varphi(S_T)$ @ T.

  call this price $\quad g_t = g(t, S_t)$ $\quad$ Further assume $g \in C^{1,2}$

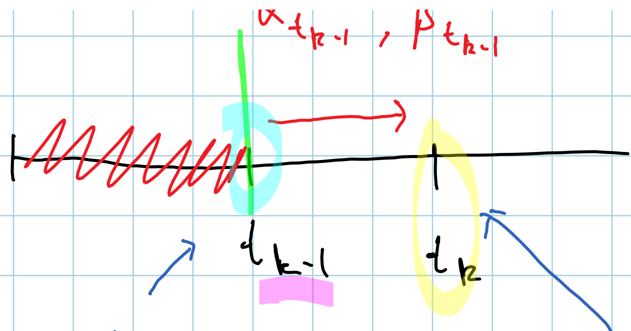
  short this claim ... at all times hold -1 of g_t

- start with nothing $V_0 = 0$

$$V_t = \alpha_t S_t + \beta_t B_t - g_t$$

$$dV_t = d(\alpha_t S_t) + d(\beta_t B_t) - dg_t$$

$\alpha_{t_{k-1}}, \beta_{t_{k-1}}$



$$\alpha_{t_{k-1}} S_{t_{k-1}} + \beta_{t_{k-1}} B_{t_{k-1}} - g_{t_{k-1}}$$

$$\alpha_{t_{k-1}} S_{t_k} + \beta_{t_{k-1}} B_{t_k} - g_{t_k}$$

$$\sum_k \Delta V_k = \sum_k \Big(\alpha_{t_{k-1}} (S_{t_k} - S_{t_{k-1}}) + \beta_{t_{k-1}} (B_{t_k} - B_{t_{k-1}}) - 1 (g_{t_k} - g_{t_{k-1}}) \Big)$$

$$dV_t = \alpha_t \, dS_t + \beta_t \, dB_t - 1 \, dg_t$$

self-financing constraint

$$= \alpha_t (\mu S_t \, dt + \sigma S_t \, dW_t) + \beta_t (r B_t \, dt)$$

$$- \Big[\underbrace{(\partial_t g_t + \underbrace{\mu S_t \partial_s g_t + \tfrac{1}{2} \sigma^2 S_t^2 \partial_{ss} g_t}_{\mathcal{L} g_t})}_{(\partial_t + \mathcal{L}) g_t} dt + \sigma S_t \partial_s g_t \, dW_t \Big]$$

Itô's lemma on $g(t, S_t)$

Ito's lemma on $g(t, S_t)$

$$\Rightarrow dV_t = \left\{ \alpha_t \mu S_t + r \beta_t B_t - (\partial_t + \mathcal{L}) g_t \right\} dt$$

$$+ \underbrace{(\alpha_t - \partial_s g_t)}_{=0} \sigma S_t \, dW_t$$

choose $\boxed{\alpha_t = \partial_s g_t}$ to remove all local risk.

now $dV_t = \{ \cdot \} dt$

- to avoid arbitrage $\{ \cdot \} = 0 \Rightarrow dV_t = 0$

  but $V_0 = 0 \Rightarrow V_t = 0 = \alpha_t S_t + \beta_t B_t - g_t$

  $$\Rightarrow \boxed{\beta_t = B_t^{-1} (g_t - \alpha_t S_t)}$$

- $\{ \cdot \} = 0$

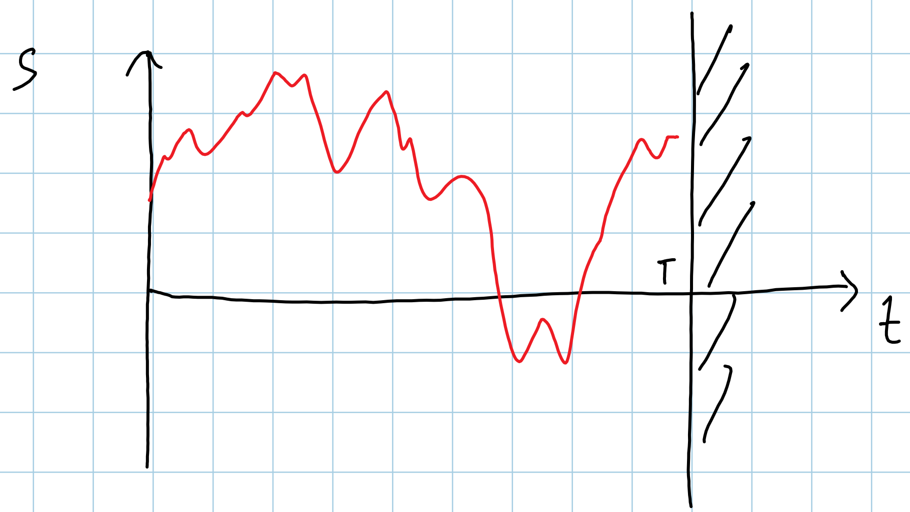
$$\Rightarrow 0 = \alpha_t \mu S_t + r \beta_t B_t - \left(\partial_t g_t + \mu S_t \partial_s g_t + \tfrac{1}{2} \sigma^2 S_t^2 \partial_{ss} g_t \right)$$

$$= \mu S_t (\overset{\partial_s g_t}{\alpha_t} - \partial_s g_t) + r (g_t - \underset{\hookrightarrow \partial_s g_t}{\alpha_t} S_t)$$

$$- \partial_t g_t - \tfrac{1}{2} \sigma^2 S_t^2 \partial_{ss} g_t$$

$$\Rightarrow \quad \partial_t g_t + r S_t \partial_s g_t + \tfrac{1}{2}\sigma^2 S_t^2 \partial_{ss} g_t = r g_t \quad \leftarrow$$

must hold $\forall$ paths S_t!



$$\therefore \begin{cases} \partial_t g(t,s) + \boxed{r} S \partial_s g(t,S) + \tfrac{1}{2}\sigma^2 S^2 \partial_{ss} g(t,S) = r g(t,S) \\ g(T,S) = \varphi(S) \end{cases}$$

Black-Scholes PDE

if you can solve this PDE then, the value of the claim is

$$g_t = g(t, S_t)$$

Discrete Hedging

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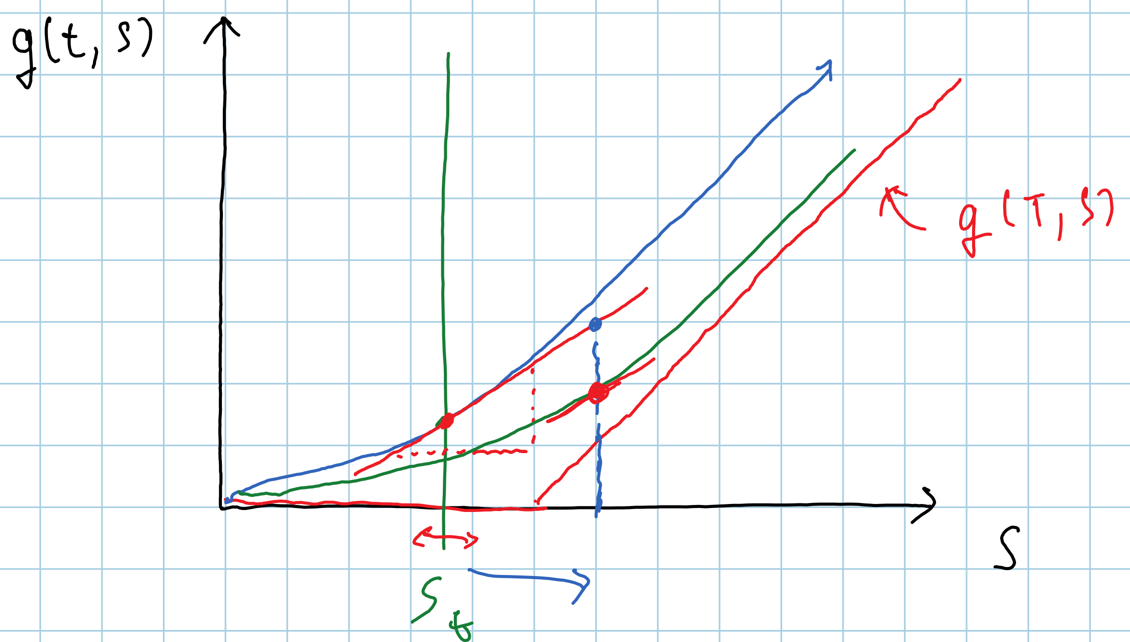
How to hedge an option you sold?

Found that $\alpha_t = \partial_S g(t, S_t) \rightarrow$ the option's "Delta" Δ

locally removes risk.



$\underline{t=0}$: sold an option we get $g_0 = g(0, S_0)$

buy Δ_0 units of $S \rightarrow$ costs $\Delta_0 S_0$

bank account has $M_0 = (g_0 - \Delta_0 S_0)$
(earns r)

$\underline{t = \Delta t}$ $\Big\{$ asset position now worth $\Delta_0 S_1$

$r \Delta t$

$t = \Delta t$ $\begin{cases} \text{asset position now worth} & \Delta_0 S_1 \\ \text{Money} & M_0 e^{r\Delta t} \end{cases}$

rebalance
- want Δ_1 units of S costs $\Delta_1 S_1$
- Money $M_1 = M_0 e^{r\Delta t} + (\Delta_0 - \Delta_1) S_1$

$t = 2\Delta t$ $\begin{cases} \Delta_1 S_2 & \text{asset value} \\ M_1 e^{r\Delta t} & \text{money} \end{cases}$

rebalance $\begin{cases} \Delta_2 \text{ units of } S & \Delta_2 S_2 \\ M_1 e^{r\Delta t} + (\Delta_1 - \Delta_2) S_2 & \text{money} \end{cases}$

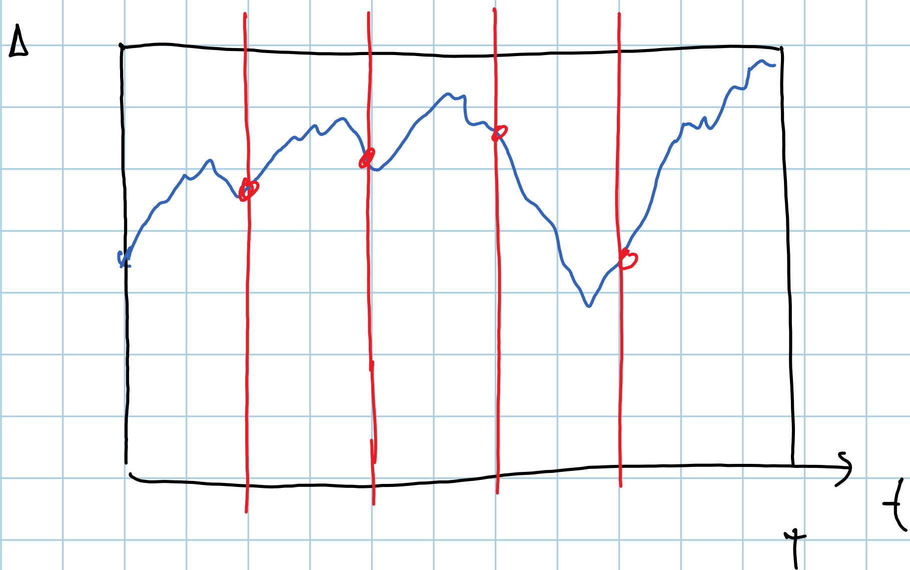
$t = k\Delta t$ $\begin{cases} \Delta_{k-1} S_k \\ M_{k-1} e^{r\Delta t} \end{cases}$

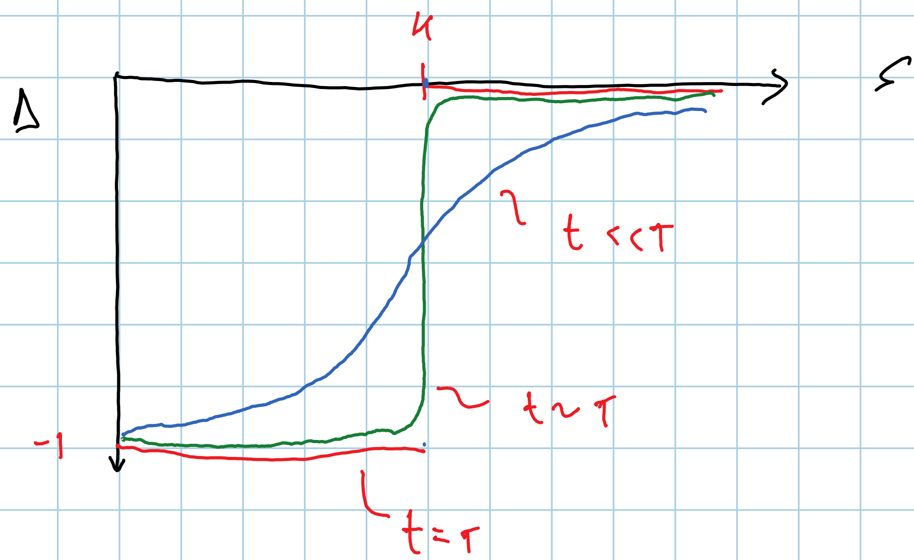
$\rightarrow \begin{cases} \Delta_k \text{ units of } S, & \Delta_k S_k \\ M_k = M_{k-1} e^{r\Delta t} + (\Delta_{k-1} - \Delta_k) S_k \end{cases}$

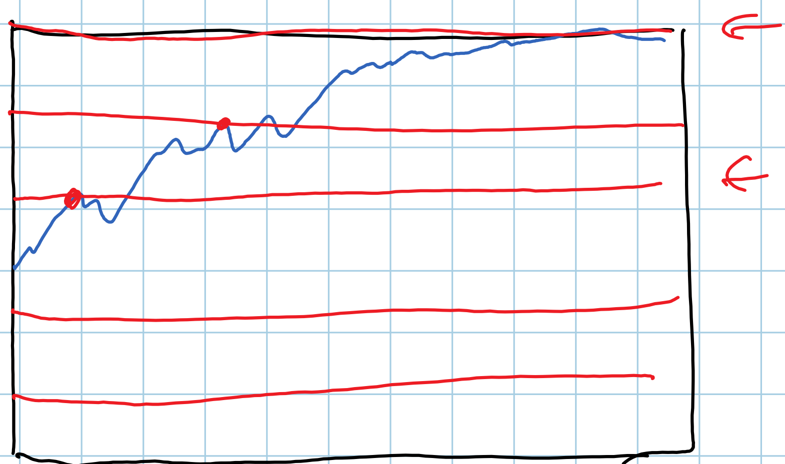
repeat until the end

$\Pi_N = \Delta_N S_N + M_N e^{r\Delta t}$ $\quad V(S_N)$

$$\text{PnL} = \Delta_{n-1} S_n + M_{n-1} e^{r\Delta t} - \Phi(S_n)$$







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