UNIVERSITY OF TORONTO

Faculty of Arts and Science

Final Examination December 14, 2009

ACT460H1 F / STA2502H F

DURATION - 3 hours

EXAMINER: Prof. S. Jaimungal

LAST NAME: ____

FIRST NAME: ____

STUDENT #: ____

Each question is worth 10 points.

Please write clearly!

AIDS: **Calculators Only**

PLEASE HAND IN AND WRITE YOUR ANSWERS IN THIS BOOKLET.

Exam Contains : **30 pages**

1 *[10]*	2 *[10]*	3 *[10]*	4 *[10]*	5 *[10]*	6 *[10]*	7 *[10]*	8 *[10]*	Total *[80]*

Formula Page

- Normal cdf: $\Phi(x) := \int_{-\infty}^{x} e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}}$.

- Moments of Normals: If Z is a normal r.v. with mean 0 and variance 1, then the m.g.f. is $\mathbb{E}[e^{a\,Z}] = e^{\frac{1}{2}a^2}$ and the fourth moment is $\mathbb{E}[Z^4] = 3$.

- 2-D Ito's Lemma: If W_t and Z_t are standard correlated Brownian motions with $d[W,Z] = \rho\, dt$ and X_t and Y_t satisfy the SDEs:

$$dX_t = \mu_t^X\, dt + \sigma_t^X\, dW_t\ , \qquad dY_t = \mu_t^Y\, dt + \sigma_t^Y\, dZ_t$$

and $U_t = f(X_t, Y_t, t)$, where $f(x,y,t)$ is twice differentiable in x and y and once differentiable in t, then

$$\begin{aligned} dU_t &= \partial_x\ f(X_t,Y_t,t)dX_t + \partial_y\ f(X_t,Y_t,t)dY_t \\ &\quad + \left(\frac{1}{2}(\sigma_t^X)^2\,\partial_{xx} + \frac{1}{2}(\sigma_t^Y)^2\,\partial_{yy} + \rho\,\sigma_t^Y\,\sigma_t^X\,\partial_{xy}\right) f(X_t,Y_t,t)\ dt \\ &= \left(\partial_t + \mu_t^X\partial_x + \mu_t^Y\partial_y + \frac{1}{2}(\sigma_t^X)^2\,\partial_{xx} + \frac{1}{2}(\sigma_t^Y)^2\,\partial_{yy} + \rho\,\sigma_t^Y\,\sigma_t^X\,\partial_{xy}\right) f(X_t,Y_t,t)\ dt \\ &\quad + \sigma_t^X\partial_x f(X_t,Y_t,t)\ dW_t + \ \sigma_t^Y\partial_y f(X_t,Y_t,t)\ dZ_t\ . \end{aligned}$$

- Ito's Isometry: If W_t is a standard Brownian motion, then $E\left[\left(\int_0^t g_s\, dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t g_s^2\, ds\right]$ for all $\mathcal{F}_t$-adapted processes g_t.

- Feynman-Kac: Suppose a function $f(y,t)$ satisfies the PDE:

$$\begin{cases} \left(\partial_t + a(y,t)\partial_y + \frac{1}{2}b^2(y,t)\partial_{yy}\right) f(y,t) &= c(y,t)\ f(y,t) \\ f(y,T) = \varphi(y) \end{cases}$$

Then $f(y,t)$ admits the unique solution

$$f(y,t) = \mathbb{E}^{\mathbb{M}}[e^{-\int_t^T c(Y_s,s)ds}\ \varphi(Y_T)\mid Y_t = y]$$

where,

$$dY_t = a(Y_t,t)dt + b(Y_t,t)dW_t$$

and W_t is a $\mathbb{M}$-standard Brownian motion.

1. Briefly explain each of the following concepts:

    (a) *[5]* Arbitrage

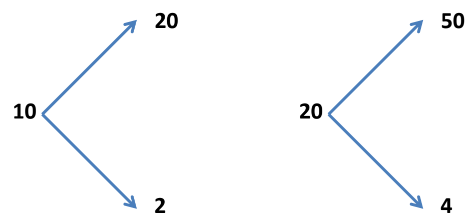
    (b) *[5]* A Brownian motion.

2. *[10]* Please indicate true or false (**no explanations required**).

   +2 for correct answer; -0.5 for incorrect answer; 0 for no answer.

   (a) [T] [F]

   An economy has the two traded assets shown below. This economy admits an arbitrage.



   (b) [T] [F]

   You have sold a put option on XYZ shares and you are simultaneously delta-hedging the position. Suppose that important (unexpected) news arrives declaring poor sales of XYZ products resulting in a drop in share value. You must sell shares of XYZ to maintain your hedge.

   (c) [T] [F]

   If $X_t = \mu t + W_t$ where $\mu > 0$ and W_t is a standard Brownian motion, then the variance of X_t is equal to t.

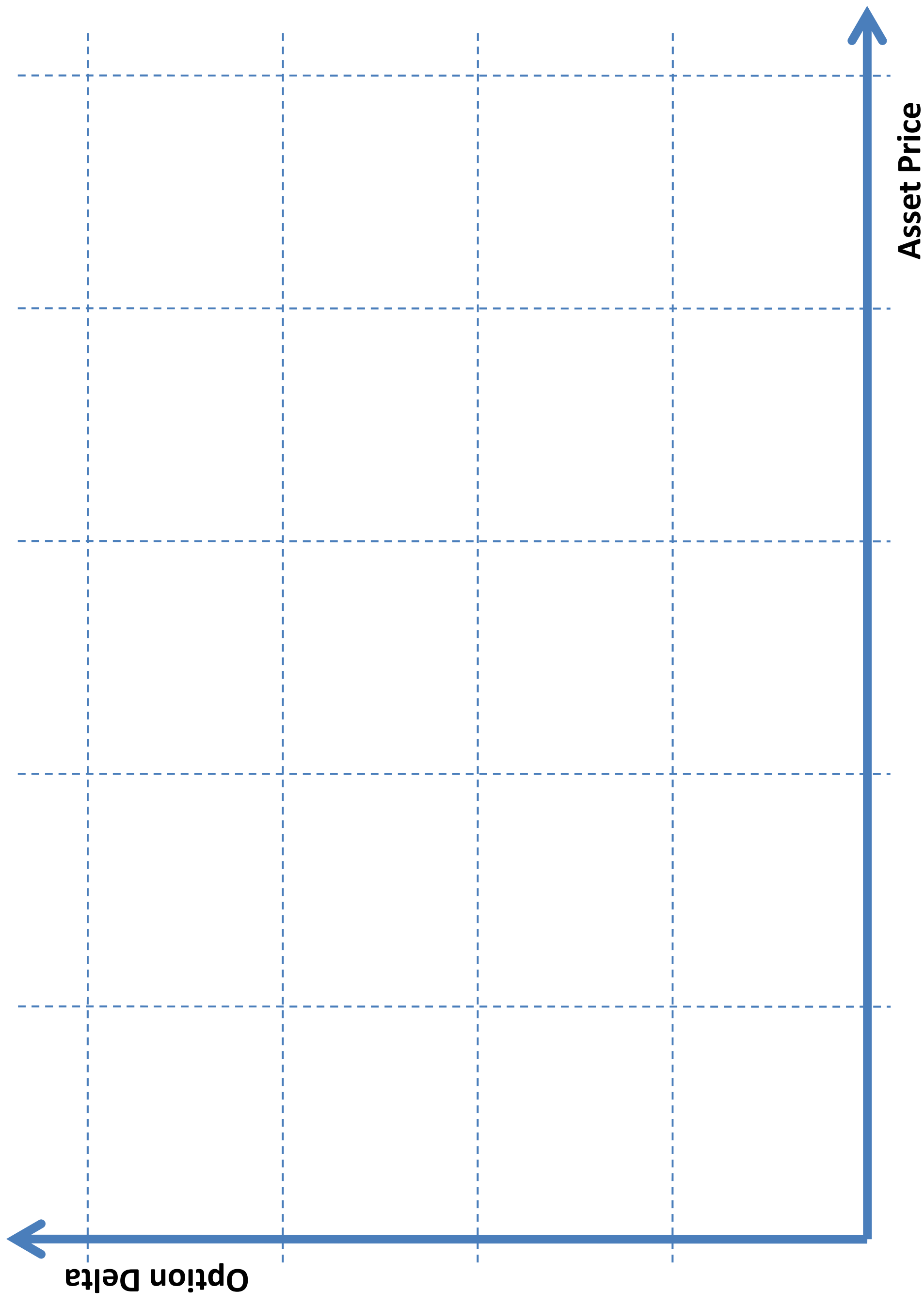
   (d) [T] [F]

   Delta hedging using a move-based approach always outperforms hedging using a time-based approach.

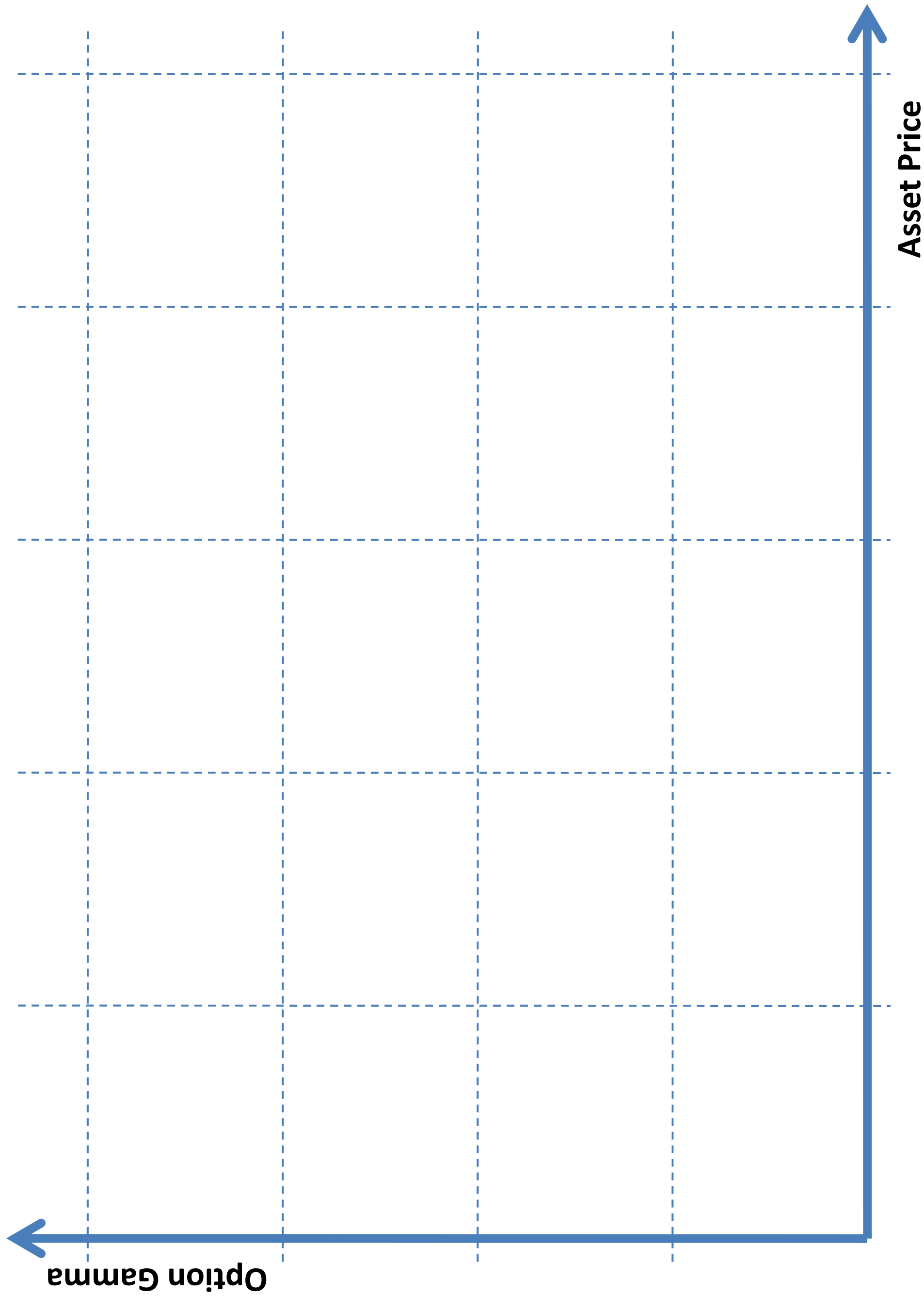
   (e) [T] [F]

   Suppose that a call option struck at 10 is selling for 1; while a call option struck at 20 is selling for 2. Both call options have the same maturity. This economy admits an arbitrage.

3. (a) *[5]* Consider the following portfolio: 4 long puts struck at 1 and one long call struck at 1. Sketch the **delta** of the portfolio (i) at maturity (ii) 1-year from maturity on the same graph below. Label any important points clearly.



(b) *[5]* Sketch the **gamma** of an **asset-or-nothing call option** struck at 1 (i) at maturity (ii) 1-year from maturity on the same graph below. Label any important points clearly. *[Recall that an asset-or-nothing call has a payoff of S_T if $S_T > K$, otherwise it pays 0.]*



4. Consider a simple two-step binomial model of interest rates in which $r_0 = R$, and $r_n = r_{n-1} \pm 1\%$. (Treat these rates as per period discount rates – e.g. discounting over the first period is $1/(1+R)$).

   (a) *[5]* Determine R and the risk-neutral branching probabilities over the first period consistent with the following market prices:

   - A 1 period zero coupon bond costs \$95.
   - A 2 period zero coupon bond costs \$90.

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(b) *[5]* Suppose that $R = 5\%$ and the risk-neutral branching probabilities are $q = 1/2$.

Consider a European call option on a 3-period zero coupon bond with notional 100. The option matures at $t = 2$ and the strike of the option is 95. Determine the value of the option.

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5. Consider an asset-or-nothing call option in the Black-Scholes model with zero interest rates. Recall that an asset-or-nothing option pays $\varphi = S_T \mathbb{I}(S_T > K)$ at maturity T.

   (a) *[5]* **Show** that the price of the option is

$$V(S,t) = S\Phi(d_+)\ , \qquad d_+ = \frac{\ln(S/K)}{\sigma(T-t)^{1/2}} + \tfrac{1}{2}\sigma(T-t)^{1/2}\ .$$

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(b) *[5]* Confirm that the price satisfies the Black-Scholes partial differential equation.

[Hint: use the fact that $\Phi''(x) = -x\Phi'(x)$*]*

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6. You are given that W_t and B_t are correlated Brownian motions with correlation ρ.

   (a) *[5]* Obtain an integration by parts formula for $\int_0^t W_s^2 \, dB_s$.

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(b) *[5]* Determine the mean and variance of $X_t = \int_0^t (W_s + B_s)\, dB_s$.

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7. Suppose that two stocks U_t and V_t satisfy the following SDEs:

$$\frac{dU_t}{U_t} = \alpha\, dt + \sigma\, dX_t\ , \qquad \frac{dV_t}{V_t} = \beta\, dt + \eta\, dY_t\ ,$$

where X_t and Y_t are $\mathbb{P}$-Wiener processes with correlation $d[X,Y]_t = \rho\, dt$ and $\alpha, \beta, \sigma, \eta$ are all constants. The risk-free rate is zero.

(a) *[5]* Determine the SDE which $G_t := U_t/V_t$ satisfies and the distribution of G_t for a fixed t conditional on G_0.

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(b) *[5]* Determine the price at time $t = 0$ of an option which pays

$$\varphi = \frac{V_S}{U_T}\ \mathbb{I}(V_S > \gamma)$$

at the maturity date T and $T > S > 0$. Here, γ is a positive constant.

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8. Suppose that two stocks X_t and Y_t satisfy the following SDEs:

$$\frac{dX_t}{X_t} = \alpha\, dt + \sigma\, dW_t^X \ , \qquad \frac{dY_t}{Y_t} = \beta\, dt + \eta\, dW_t^Y \ ,$$

where W_t^X and W_t^Y are $\mathbb{P}$-Wiener processes with correlation $d[W^X, W^Y]_t = \rho\, dt$ and $\alpha, \beta, \sigma, \eta$ are all constants. The risk-free interest rate r is constant. Furthermore, an option written on the two stocks has a payoff at maturity of $\varphi(X_T, Y_T)$.

(a) *[5]* Through a **dynamic hedging argument** (analogous to what we covered in class), **prove** that the price of the option $g(x, y, t)$ satisfies the following PDE:

$$\begin{cases} \left(\partial_t + rx\partial_x + ry\partial_y + \frac{1}{2}\sigma^2 x^2\, \partial_{xx} + \frac{1}{2}\eta^2 y^2\, \partial_{yy} + \rho\,\sigma\,\eta\, xy\partial_{xy}\right) g &= rg \ , \\ g(x, y, T) &= \varphi(x, y) \ . \end{cases}$$

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(b) *[5]* Suppose that $\varphi(x, y) = x\mathbb{I}(y > \gamma)$. Assume that $g(x, y, t)$ can be written as $g(x, y, t) = x\, f(y, t)$ for some function $f(y, t)$. Using the PDE from part (a), derive a PDE for $f(y, t)$ and solve for it using the Feynman-Kac result.

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