UNIVERSITY OF TORONTO

Faculty of Arts and Science

Final Examination, December 10th, 2008

ACT460H1 F / STA2502H F

DURATION - 3 **hours**

EXAMINER: Prof. S. Jaimungal

COURSE CODE (**circle one**): ACT460H1 F STA2502H F

LAST NAME: ____

FIRST NAME: ____

STUDENT #: ____

Each question is worth 10 points.

Please write clearly!

AIDS: **Calculators Only**

Exam Contains : **29 pages**

1 *[10]*	2 *[10]*	3 *[10]*	4 *[10]*	5 *[10]*	6 *[10]*	7 *[10]*	8 *[10]*	9 *[10]*	Total *[90]*

Formula Page

- Normal cdf: $\Phi(x) := \int_{-\infty}^{x} e^{-\frac{1}{2}x} \frac{dx}{\sqrt{2\pi}}$.

- Moments of Normals: If Z is a normal r.v. with mean 0 and variance 1, then the m.g.f. is $\mathbb{E}[e^{a\,Z}] = e^{\frac{1}{2}a^2}$ and the fourth moment is $\mathbb{E}[Z^4] = 3$.

- 2-D Ito's Lemma: If W_t and Z_t are standard correlated Brownian motions with $d[W,Z] = \rho\, dt$ and X_t and Y_t satisfy the SDEs:

$$dX_t = \mu_t^X\, dt + \sigma_t^X\, dW_t\ , \qquad dY_t = \mu_t^Y\, dt + \sigma_t^Y\, dZ_t$$

and $U_t = f(X_t, Y_t, t)$, where $f(x, y, t)$ is twice differentiable in x and y and once differentiable in t, then

$$\begin{aligned} dU_t \;&=\; \left(\partial_t + \mu_t^X \partial_x + \mu_t^Y \partial_y + \frac{1}{2}(\sigma_t^X)^2\, \partial_{xx} + \frac{1}{2}(\sigma_t^Y)^2\, \partial_{yy} + \rho\, \sigma_t^Y\, \sigma_t^X\, \partial_{xy}\right) f(X_t, Y_t, t)\; dt \\ &\quad + \sigma_t^X \partial_x f(X_t, Y_t, t)\; dW_t + \; \sigma_t^Y \partial_y f(X_t, Y_t, t)\; dZ_t\ . \end{aligned}$$

- Ito's Isometry: If W_t is a standard Brownian motion, then $E\left[\left(\int_0^t g_s\, dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t g_s^2\, ds\right]$ for all $\mathcal{F}_t$-adapted processes g_t.

- Ito's Product and Quotient Rule:

$$\frac{d(X_t\, Y_t)}{X_t\, Y_t} = \frac{dX_t}{X_t} + \frac{dY_t}{Y_t} + \frac{d[X,Y]_t}{X_t, Y_t}\ ; \qquad \frac{d(X_t/Y_t)}{X_t/Y_t} = \frac{dX_t}{X_t} - \frac{dY_t}{Y_t} + \frac{d[Y,Y]_t}{Y_t^2} - \frac{d[X,Y]_t}{X_t\, Y_t}\ .$$

- Girsanov's Theorem: Given two equivalent measures $\mathbb{P}_1$ and $\mathbb{P}_2$, there exists an $\mathcal{F}_t$-adapted vector process $\vec{\lambda}_t$ such that if $\vec{W}_t$ is a vector of $\mathbb{P}_1$-Wiener process, then $\vec{W}_t^*$, satisfying $d\vec{W}_t^* = \vec{\lambda}_t\, dt + d\vec{W}_t$, is a $\mathbb{P}_2$-Wiener process.

1. Briefly explain each of the following concepts:

   (a) *[5]* Arbitrage.

   (b) *[5]* A Brownian motion.

2. *[10]* Please indicate true or false (**no explanations required**).

   $+2$ for correct answer; -0.5 for incorrect answer; 0 for no answer.

   (a) [T] [F]

   In an economy with three tradable assets, it is never possible to replicate contingent claims written on one of the assets.

   (b) [T] [F]

   You have sold a put option on XYZ shares and you are simultaneously delta-hedging the position. Suppose that important (unexpected) news arrives declaring poor sales of XYZ products resulting in a drop in share value. You must sell shares of XYZ to maintain your hedge.

   (c) [T] [F]

   If interest rates are modeled as $dr_t = \theta\, dt + \sigma\, dW_t$ where W_t is a Brownian motion, then interest rates mean-revert.

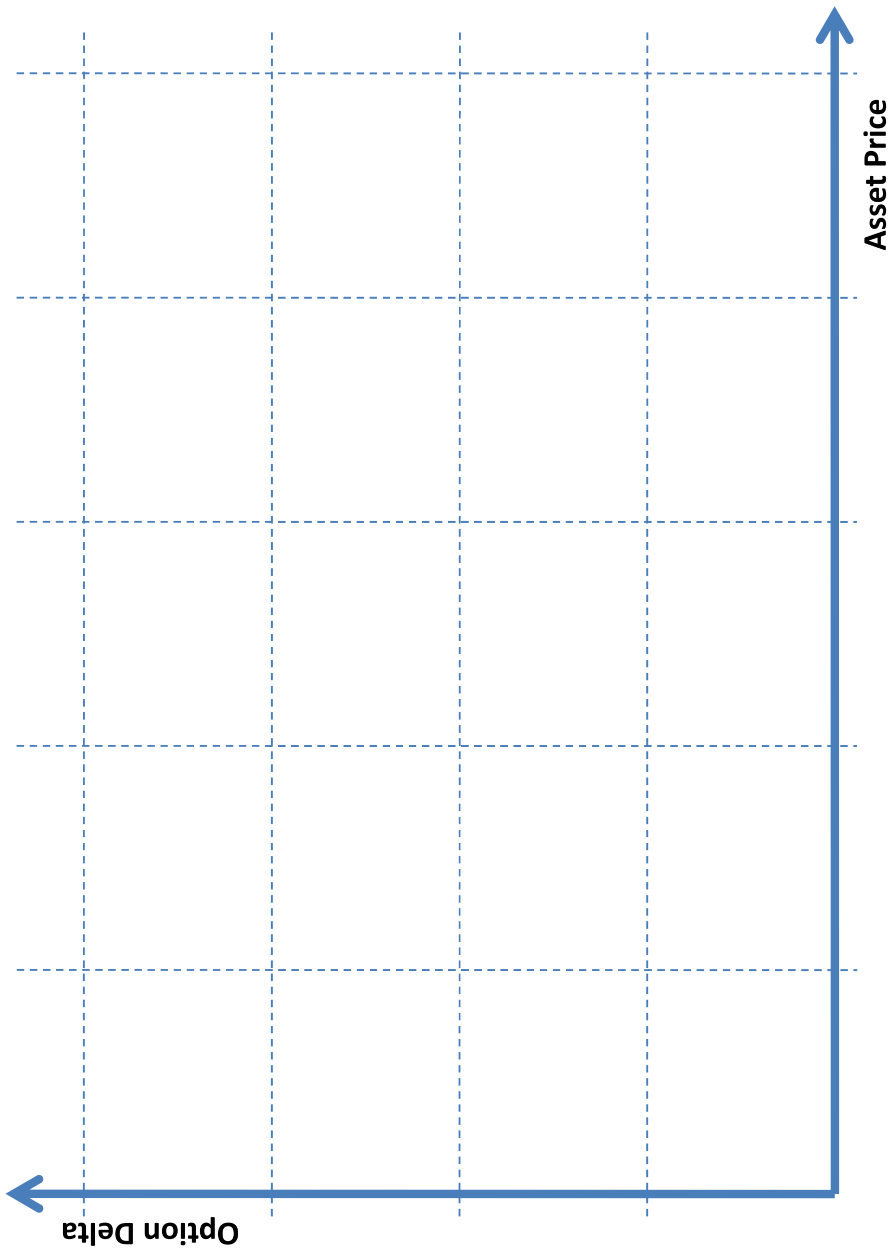
   (d) [T] [F]

   If the real-world evolution of share prices evolves with a vol of 20% and you delta-gamma hedge a put option with a vol of 25% on a daily basis, then the net PnL will be symmetric.

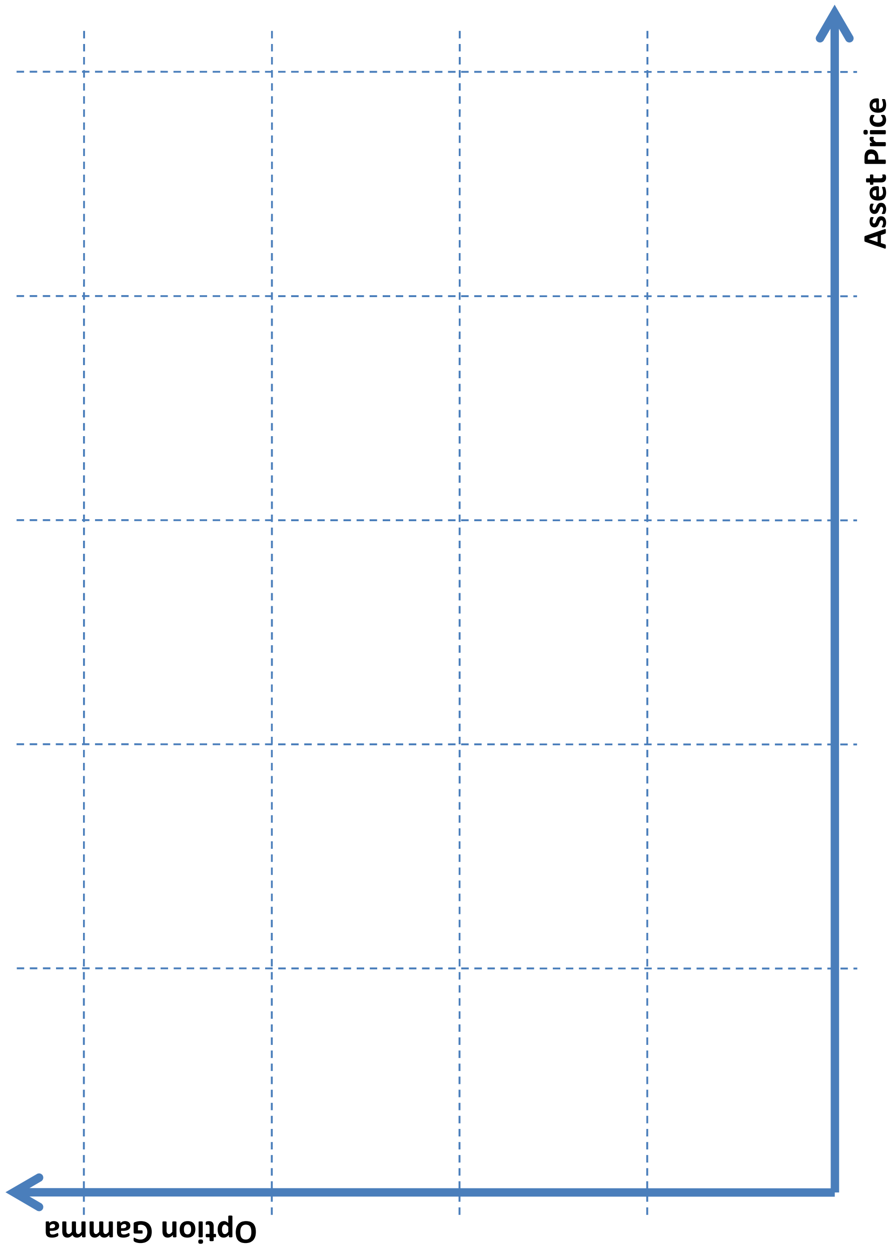
   (e) [T] [F]

   Suppose that a put option struck at 1 is selling for 0.10; while a put option struck at 2 is selling for 0.2. Both puts have the same maturity. This economy admits an arbitrage.

3. (a) *[5]* Consider the following portfolio: long put struck at 80 and a long call struck at 100. Sketch the **delta** of the portfolio (i) at maturity (ii) 1-month from maturity (iii) 1-year from maturity all on the same graph. Label any important points clearly.



(b) *[5]* Sketch the **gamma** of a **digital call** option (i) at maturity (ii) 1-month from maturity (iii) 1-year from maturity all on the same graph. Label any important points clearly. *[Recall that a digital call option pays 1 at maturity if the asset price exceeds the strike K otherwise it pays nothing.]*



4. Consider a simple two-step binomial model of interest rates in which $r_0 = 5\%$, and $r_n = r_{n-1} \pm 1\%$. (Treat these rates as per period discount rates – e.g. discounting over the first period is $1/1.05$).

   (a) *[5]* Determine the risk-neutral branching probabilities consistent with a market price of 100 for a coupon bearing bond which pays 5 at $t = 1$ and 105 at $t = 2$.

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(b) *[5]* Suppose that the risk-neutral branching probabilities are $q = 1/2$.

Consider a European call option on a 3-period bond with notional 100. The option matures at $t = 2$ and the strike of the option is 95. Determine the value of the option.

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5. Consider a digital call option in the Black-Scholes model with <u>zero interest rates</u>.

(a) *[5]* Show that the price of the digital call option is

$$V(S,t) = \Phi(d_-)\ , \quad d_- = \frac{\ln(S/K)}{\sigma(T-t)^{1/2}} - \tfrac{1}{2}\sigma(T-t)^{1/2}\ .$$

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(b) *[5]* Confirm that the price satisfies the Black-Scholes partial differential equation.

[Hint: use the fact that $\Phi''(x) = -x\Phi'(x)$*]*

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6. You are given that W_t and B_t are correlated Brownian motions with correlation ρ.

   (a) *[5]* Obtain an integration by parts formula for $\int_0^t e^{W_s}\, dB_s$.

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(b) *[5]* Determine the mean and variance of $X_t = \int_0^t W_s \, dB_s - \int_0^t B_s \, dW_s$.

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7. *[10]* Suppose that two stocks U_t and V_t satisfy the following SDEs:

$$\frac{dU_t}{U_t} = \alpha\, dt + \sigma\, dX_t\ , \qquad \frac{dV_t}{V_t} = \beta\, dt + \eta\, dY_t\ ,$$

where X_t and Y_t are $\mathbb{P}$-Wiener processes with correlation $d[X,Y]_t = \rho\, dt$ and $\alpha, \beta, \sigma, \eta$ are all constants. The risk-free rate is zero.

Determine the price at time $t = 0$ of an option which pays

$$\varphi = U_T \times \mathbb{I}_{V_S > \gamma}$$

at the maturity date T and $T > S > 0$. Here, γ is a constant.

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8. *[10]* Prove that

$$\int_0^t W_s dZ_s + \int_0^t Z_s dW_s = W_t \, Z_t - \rho \, t \quad a.s.$$

Do not use Ito's lemma, but rather use the fundamental definition of the stochastic integrals.

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9. Consider the Vasicek model for the short rate of interest:

$$dr_t = \kappa(\theta - r_t)\, dt + \sigma\, dW_t$$

where W_t is a $\mathbb{Q}$-Wiener process. The solution to this SDE is

$$r_s = \theta + (r_t - \theta)\, e^{-\kappa\,(s-t)} + \sigma \int_t^s e^{-\kappa(s-u)} dW_u \qquad \text{for} \quad t \le s.$$

(a) *[5]* Show that the distribution of $I_t^T = \int_t^T r_s\, ds$ is normal with mean m and variance v with

$$\begin{aligned} m &= \theta((T-t) - B(T-t;\kappa)) + B(T-t;\kappa)\, r_t\,, \\ v &= \frac{\sigma^2}{\kappa^2}\left((T-t) + B(T-t;2\kappa) - 2B(T-t;\kappa)\right) \end{aligned}$$

where, $B(\tau;\kappa) = \frac{1}{\kappa}(1 - e^{-\kappa\tau})$.

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(b) *[5]* Show that price of a T-maturity bond satisfies the SDE

$$\frac{dP_t(T)}{P_t(T)} = r_t\, dt - \sigma\, B(T-t;\kappa)\, dW_t\ .$$

Blank intentionally. Continue work here...