Brownian Motion: W_t
(Wiener process)

* $W_0 = 0$

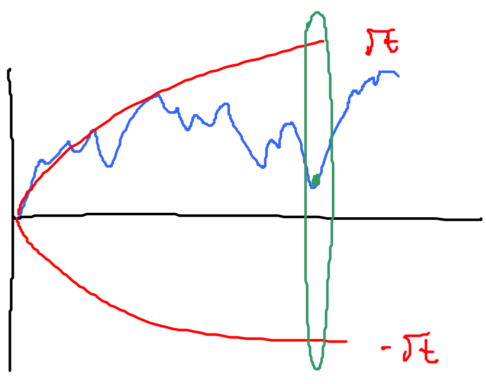
* $W_t \sim \mathcal{N}(0, t)$

* increments are stationary + independent

  - $W_{t+s} - W_t \overset{d}{=} W_s$ stationary

  - $W_{T_2} - W_{T_1}$ is $\perp$ $W_{T_4} - W_{T_3}$ whenever $T_1 < T_2 \leq T_3 < T_4$ independence

* paths are continuous



$\mathbb{V}[W_t] = t$

$TV(W)_t = +\infty$

$[W, W]_t = t$ a.s.

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$\sigma W_t \underset{\mathbb{P}}{\sim} \mathcal{N}(0, \sigma^2 t)$

$$\boxed{S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \underline{\sigma W_t}}}, \quad W_t \text{ is } \mathbb{P}\text{-Brownian motion}$$

from CRR as $n \uparrow +\infty$ ...

$$\rightarrow \quad S_t \overset{d}{=} S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \underline{\sigma\sqrt{t}\, Z}}, \quad Z \underset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$$

¿ What is the joint distribution of S_{T_1} & S_{T_2} ?

$$S_{T_1} = S_0 \; e^{(\mu - \frac{1}{2}\sigma^2)T_1 + \sigma W_{T_1}} \leftarrow \text{normal}$$

$T_2 > T_1$

$$S_{T_2} = S_0 \; e^{(\mu - \frac{1}{2}\sigma^2)T_2 + \sigma W_{T_2}} \leftarrow \text{normal}$$

$$\begin{pmatrix} \ln(S_{T_1}/S_0) \\ \ln(S_{T_2}/S_0) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} ; \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right)$$

$$m_1 = (\mu - \tfrac{1}{2}\sigma^2)T_1, \quad m_2 = (\mu - \tfrac{1}{2}\sigma^2)T_2$$

$$\Sigma_{11} = \mathbb{V}[\sigma W_{T_1}] = \sigma^2 T_1$$

$$\Sigma_{22} = \mathbb{V}[\sigma W_{T_2}] = \sigma^2 T_2$$

$$\Sigma_{12} = \Sigma_{21} = \mathbb{C}[\sigma W_{T_1}, \sigma W_{T_2}]$$

$$= \sigma^2 \left(\mathbb{E}[W_{T_1} W_{T_2}] - \overset{0}{\mathbb{E}[W_{T_1}]} \cdot \overset{0}{\mathbb{E}[W_{T_2}]} \right)$$

$$\hookrightarrow \mathbb{E}[W_{T_1}(W_{T_1} - (W_{T_2} - W_{T_1}))]$$

$$= T_1 - \mathbb{E}[W_{T_1}(W_{T_2} - W_{T_1})]$$

independence

$$\underbrace{\mathbb{E}[W_{T_1}]}_{=0}\ \underbrace{\mathbb{E}[W_{T_2} - W_{T_1}]}_{=0}$$

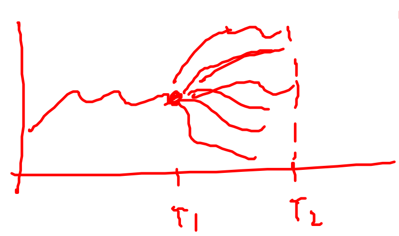
$$\Sigma_{21} = \Sigma_{12} = \sigma^2 T_1$$

$$\begin{pmatrix} \ln(S_{T_1}/S_0) \\ \ln(S_{T_2}/S_0) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} (\mu - \frac{1}{2}\sigma^2) T_1 \\ (\mu - \frac{1}{2}\sigma^2) T_2 \end{pmatrix} ; \begin{pmatrix} \sigma^2 T_1 & \sigma^2 T_1 \\ \sigma^2 T_1 & \sigma^2 T_2 \end{pmatrix} \right)$$

$$\rho = \frac{\sigma^2 T_1}{\sqrt{\sigma^2 T_1 \cdot \sigma^2 T_2}} = \sqrt{\frac{T_1}{T_2}}$$

$$\mathbb{E}[W_{T_1} W_{T_2}] = \mathbb{E}[\,\mathbb{E}[W_{T_1} W_{T_2} \mid W_{T_1}]\,]$$

$$= \mathbb{E}[W_{T_1} \underbrace{\mathbb{E}[W_{T_2} \mid W_{T_1}]}\,] = T_1$$



W_{T_1}

$$\mathbb{V}[W_t W_{t+s}] = \mathbb{E}[(W_t W_{t+s})^2] - (\underbrace{\mathbb{E}[W_t W_{t+s}]}_{t})^2$$

$$\mathbb{E}[(W_t(W_t + (W_{t+s} - W_t)))^2]$$

$$= \mathbb{E}[W_t^2(W_t^2 + 2W_t(W_{t+s} - W_t) + (W_{t+s} - W_t)^2)]$$

$$= 3t^2 + 2\underbrace{\mathbb{E}[W_t^3(W_{t+s} - W_t)]}_{=\underbrace{\mathbb{E}[W_t^3]}_{=0}\underbrace{\mathbb{E}[W_{t+s} - W_t]}_{=0}} + \underbrace{\mathbb{E}[W_t^2(W_{t+s} - W_t)^2]}_{=\underbrace{\mathbb{E}[W_t^2]}_{=t}\underbrace{\mathbb{E}[(W_{t+s} - W_t)^2]}_{=s}}$$

$$= 3t^2 + ts$$

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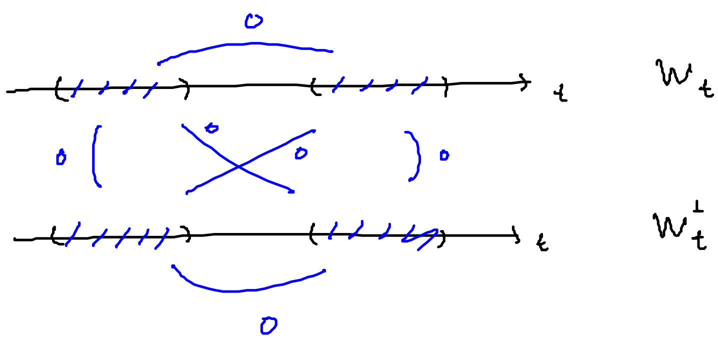
Correlated Brownian Motions

W_t and $W_t^\perp$ are independent B. motions.

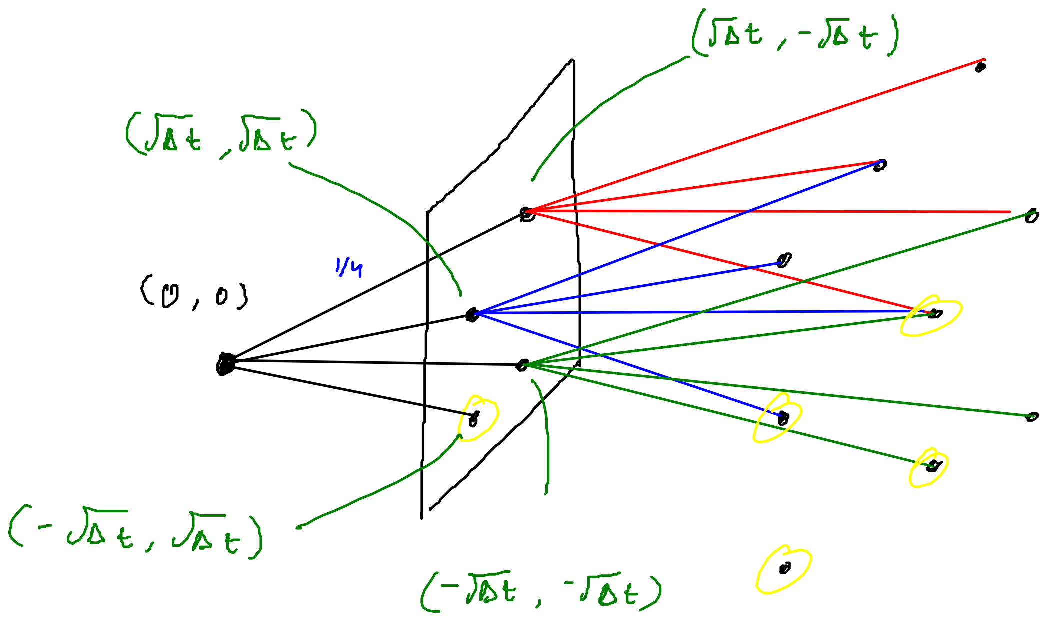
- $W_0 = 0$
  $W_0^\perp = 0$

- $$\begin{pmatrix} W_t \\ W_t^\perp \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \right)$$

- W_t & $W_t^\perp$ have ind. & stationary increments



- paths are continuous



Z, $Z^\perp$ ind std. normal r.v. how to define X & Y s.t.

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$$

$X = Z$

$Y = a\, Z + b\, Z^\perp$

$\mathbb{E}[\,Y\,] = 0$, $\quad \mathbb{V}[\,Y\,] = a^2 + b^2 = 1$

$$\mathbb{C}[\,X, Y\,] = \mathbb{C}[\,Z,\ aZ + bZ^\perp\,]$$
$$= a\,\mathbb{C}[Z, Z] + b\,\mathbb{C}[Z, Z^\perp]$$
$$= a$$

$a = \rho$
& $b = \sqrt{1-\rho^2}$ achieves the goal.

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create correlated B.motions X_t & Y_t out of uncorrelated B. motions W_t & $W_t^{\perp}$

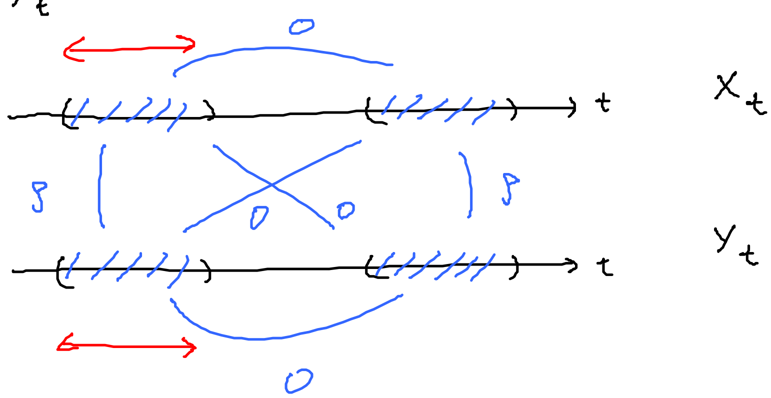
$$X_t = W_t$$

$$Y_t = \rho\, W_t + \sqrt{1-\rho^2}\; W_t^{\perp}$$

ρ - correlation

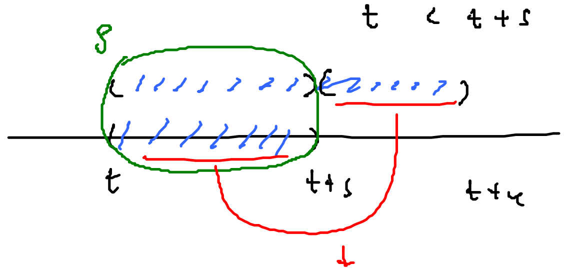
- $\begin{pmatrix} X_t \\ Y_t \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} t & \rho t \\ \rho t & t \end{pmatrix} \right)$

- X_t has stationary & independent increments
- Y_t " " " "



$$c = \mathbb{C}[\,(X_{t+s} - X_t)\,;\,(Y_{t+u} - Y_t)\,] = \rho\, s$$

$t < t+s < t+u$



$$c = \mathbb{C}[\,(X_{t+s} - X_t)\,;\,(Y_{t+u} - Y_{t+s}) + (Y_{t+s} - Y_t)\,]$$

$$= \mathbb{C}[\,(X_{t+s} - X_t)\,;\,(Y_{t+u} - Y_{t+s})\,] \to 0 \quad \text{ind. increments.}$$

$$+\ \mathbb{C}[\,(X_{t+s} - X_t)\,;\,(Y_{t+s} - Y_t)\,]$$

stationary $\hookrightarrow$ $\mathbb{C}[\,X_s\,;\,Y_s\,] = \mathbb{C}[\,W_s\,;\,\rho W_s + \sqrt{1-\rho^2}\, W_s^{\perp}\,]$

$$= \rho\ \mathbb{C}[W_s ; W_s] + \sqrt{1-\rho^2}\ \mathbb{C}[W_s, W_s^{\perp}]$$

$= \beta S \qquad + \qquad 0$

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$$\mathbb{V}[X_t Y_t] = \mathbb{E}[X_t^2 Y_t^2] - (\mathbb{E}[X_t Y_t])^2$$

$$\mathbb{E}[X_t Y_t] = \mathbb{E}[W_t (\rho W_t + \sqrt{1-\rho^2} W_t^\perp)]$$

$$= \rho\, \mathbb{E}[W_t^2] + \sqrt{1-\rho^2}\, \underbrace{\mathbb{E}[W_t W_t^\perp]}_{= \underbrace{\mathbb{E}[W_t]}_{=0} \underbrace{\mathbb{E}[W_t^\perp]}_{=0}}$$

$$= \rho t$$

$$\mathbb{E}[X_t^2 Y_t^2] = \mathbb{E}[W_t^2 (\rho W_t + \sqrt{1-\rho^2} W_t^\perp)^2]$$

$$= \rho^2\, \mathbb{E}[W_t^4] + 2\rho\sqrt{1-\rho^2}\, \mathbb{E}[W_t^3 W_t^\perp]$$

$$+ (1-\rho^2)\, \mathbb{E}[W_t^2 \cdot (W_t^\perp)^2]$$

$$= 3\rho^2 t^2 + 0 + (1-\rho^2) t^2$$

$$= (1 + 2\rho^2) t^2$$

$$\Rightarrow \mathbb{V}[X_t Y_t] = (1+\rho^2) t^2$$

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X_t & Y_t are correl. B.mtn correl $= \rho$.

$\mathbb{V}[\ X_t\ Y_{t+s}\ ] = ?$

$\mathbb{E}[\ X_t\ Y_{t+s}] = \mathbb{E}[\ X_t\ (Y_t + (Y_{t+s} - Y_t))]$

$= \mathbb{E}[\ X_t\ Y_t\ ] + \mathbb{E}[\ X_t\ (Y_{t+s} - Y_t)]$

$\underset{\text{ind.}}{=}$ $\mathbb{E}[\ X_t]\,\mathbb{E}[\ Y_{t+s} - Y_t] = 0$

$= \rho\, t$

$\mathbb{E}[\ X_t^2\ Y_{t+s}^2] = \mathbb{E}[\ X_t^2\ (\ Y_t + (Y_{t+s} - Y_t))^2]$

$= \mathbb{E}[\ X_t^2\ (\ Y_t^2 + 2\ Y_t\ (Y_{t+s} - Y_t) + (Y_{t+s} - Y_t)^2)]$

$= \mathbb{E}[\ X_t^2\ Y_t^2\ ] + 2\ \mathbb{E}[\ X_t^2\ Y_t\ (Y_{t+s} - Y_t)]$

$\mathbb{E}[\ X_t^2\ Y_t^2\ ] = (1+2\rho^2)\, t^2$ from previous calc.

$\underset{\text{ind.}}{=}$ $\mathbb{E}[\ X_t^2\ Y_t]\,\mathbb{E}[\ Y_{t+s} - Y_t] = 0$

$+ \mathbb{E}[\ X_t^2\ (Y_{t+s} - Y_t)^2]$

$\underset{\text{ind.}}{=}$ $\mathbb{E}[\ X_t^2]\,\mathbb{E}[(Y_{t+s} - Y_t)^2]$

$\underset{\text{st.}}{=}$ $\mathbb{E}[\ X_t^2]\ \mathbb{E}[\ Y_s^2]$

$= (1+2\rho^2)\, t^2 + t\ s$

$\Rightarrow \mathbb{V}[\ X_t\ Y_{t+s}] = (1+\rho^2)\ t^2 + t\ s$

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$$[X, Y]_t = \lim_{\|\pi\| \downarrow 0} \sum_k (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) = \rho t \quad \text{a.s.}$$

Covariation $\uparrow$

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$$\int_0^t W_s\, dW_s = \lim_{\|\Pi\| \downarrow 0} \sum_k \overset{\text{l.h.p.}}{W_{t_{k-1}}} (W_{t_k} - W_{t_{k-1}})$$

$$\overset{?}{=} \frac{1}{2}(W_t^2 - t) \quad \text{a.s.} \qquad \sum_k (X_{t_k} - X_{t_{k-1}}) \to X_t$$

$$R^\Pi = \sum_k W_{t_{k-1}}(W_{t_k} - W_{t_{k-1}}) - \frac{1}{2}(W_t^2 - t)$$

$$= \sum_k \left(W_{t_{k-1}}(W_{t_k} - W_{t_{k-1}}) - \frac{1}{2}\left((W_{t_k}^2 - t_k) - (W_{t_{k-1}}^2 - t_{k-1})\right)\right)$$

$$= -\frac{1}{2} \sum_k \left[\Delta W_{t_k}^2 - (t_k - t_{k-1}) \right]$$

$$\Delta W_{t_k}^2 = (W_{t_k} - W_{t_{k-1}})^2$$

$$= W_{t_k}^2 - 2 W_{t_k} W_{t_{k-1}} + W_{t_{k-1}}^2$$

$$\mathbb{E}[R^\Pi] = 0$$

$$\mathbb{V}[R^\Pi] = \frac{1}{4} \sum_k \mathbb{V}[\Delta W_{t_k}^2]$$

$$\mathbb{V}[\Delta W_{t_k}^2] = \mathbb{E}[\Delta W_{t_k}^4] - (\mathbb{E}[\Delta W_{t_k}^2])^2$$

$$= 3\Delta t_k^2 - \Delta t_k^2 = 2\Delta t_k^2$$

$$= \frac{1}{2} \sum_k \Delta t_k^2$$

$$\le \frac{1}{2} \underbrace{\left(\sum_k \Delta t_k\right)}_{= t} \|\Pi\| \xrightarrow[\|\Pi\| \downarrow 0]{} 0$$

$$\boxed{\int_0^t W_s\, dW_s \triangleq \lim_{\|\Pi\| \downarrow 0} \sum_k W_{t_{k-1}} \Delta W_{t_k} = \frac{1}{2}(W_t^2 - t) \quad a.s.}$$

$$\text{"} \quad W_t\, dW_t = \frac{1}{2} d(W_t^2) - \frac{1}{2} dt \quad \text{"}$$

$$\Rightarrow \text{"} \quad d(W_t^2) = \underline{2 W_t\, dW_t} + \underbrace{dt}_{\text{Ito correction}} \quad \text{"}$$

$$W_t \longmapsto g(W_t) = X_t$$

$g \in C^2$ (twice differentiable)

Ito's Lemma:

$$\text{"} \quad d\, g(W_t) = \underbrace{g'(W_t)\, dW_t}_{\text{"std. calc"}} + \underbrace{\frac{1}{2} g''(W_t)\, dt}_{\text{Ito correction}} \quad \text{"}$$

$$g(W_t) - g(W_0) = \underbrace{\int_0^t g'(W_s)\, dW_s}_{\text{Stoch Integral}} + \underbrace{\frac{1}{2}\int_0^t g''(W_s)\, ds}_{\text{Riemann-Integral}}$$

$$\int_0^t h(W_s)\, dW_s \triangleq \lim_{\|\Pi\| \downarrow 0} \sum_k h(W_{t_{k-1}}) (W_{t_k} - W_{t_{k-1}})$$

$$\int_0^t h(W_s)\, ds \triangleq \lim_{\|\Pi\| \downarrow 0} \sum_k h(W_{t_k^*})\, \Delta t_k$$

$\hookrightarrow t_{k-1}$

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$$\int_0^t W_s^2 \, dW_s = ?$$

consider a fn $g(x) = x^3$

Ito's lemma $\Rightarrow$

$$dg(W_t) = g'(W_t)\,dW_t + \frac{1}{2} g''(W_t)\,dt$$

$$= 3 W_t^2 \, dW_t + \frac{1}{2}\, 3.\, 2.\, W_t \, dt$$

$\Rightarrow$

$$W_t^2 \, dW_t = \frac{1}{3}\, dg(W_t) - W_t \, dt$$

$$\int_0^t W_s^2 \, dW_s = \frac{1}{3} \int_0^t dg(W_s) - \int_0^t W_s \, ds$$

$$= \frac{1}{3}\left(g(W_t) - g(W_0) \right) - \int_0^t W_s \, ds$$

$$\boxed{\int_0^t W_s^2 \, dW_s = \frac{1}{3} W_t^3 - \int_0^t W_s \, ds}$$