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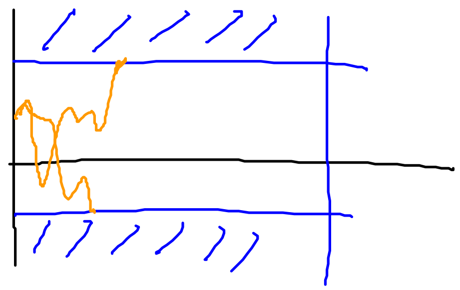
$$\tau = \inf\{t : S_t \notin (L, U)\}$$



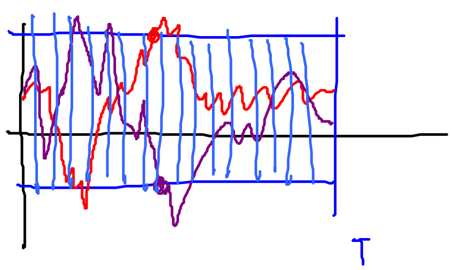
$$Q = \mathbb{1}_{\tau \le T} \, (S_T - K)_+$$

$$\tau_U = \inf\{t : S_t \ge U\}$$

$$\tau_L = \inf\{t : S_t \le L\}$$

$$\tau = \tau_U \vee \tau_L = \max(\tau_U, \tau_L)$$

$$Q = \mathbb{1}_{\tau \le T} \, (S_T - K)_+$$

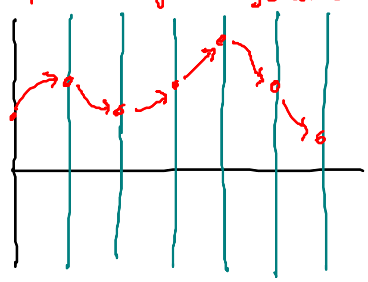


$$S_t \overset{d}{=} S_0 \, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z} \quad , \quad Z \underset{Q}{\sim} N(0,1)$$

$$S_n = S_{n-1} \, e^{(r - \frac{1}{2}\sigma^2)\Delta T + \sigma\sqrt{\Delta t} \, Z_n}$$

$$Z_1, Z_2, \ldots, Z_n \quad \text{iid} \underset{Q}{\sim} N(0,1)$$

sample path of S generated under Q.



$$\{\max(S_n) \ge U \cap \min(S_n) \le L\} = \omega$$

$$\mathbb{1}_{\tau \le T} = \mathbb{1}_{\omega}$$

$$V_0 = e^{-rT}\,\mathbb{E}^{Q}\Big[\underbrace{\mathbb{1}_{\tau\le T}\,(S_T - K)_+}_{X}\Big]$$

$$\approx e^{-rT}\,\frac{1}{M}\sum_{m=1}^{M}\mathbb{1}_{\omega_m}\,(S_T^{(m)} - K)_+$$

$$\boxed{\begin{aligned}&\mathbb{E}[X] \approx \frac{1}{M}\sum_{m=1}^{M} X^{(m)}\\ &\qquad X^{(1)}, X^{(2)}, \dots, X^{(M)} \text{ iid} \sim X.\end{aligned}}$$

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Stochastic Interest Rate Models

$$A_0 \begin{cases} A_u = A_0 e^{\sigma\sqrt{\Delta t}} \\ A_d = A_0 e^{-\sigma\sqrt{\Delta t}} \end{cases}$$

$r_0 e^{\sigma\sqrt{\Delta t}}$, $r_0 + \sigma\sqrt{\Delta t}$

$$r_0 \begin{cases} r_u \\ r_d \end{cases}$$

is not a tree associated with a traded asset!

$r_0 e^{-\sigma\sqrt{\Delta t}}$, $r_0 - \sigma\sqrt{\Delta t}$

short rates of interest

maturity

$$P_0(1) \begin{cases} 100 \\ 100 \end{cases}$$

time

100 into M.M.

get $100(1+r_0)$ @ $t=1$

$100(1+r_0)(1+r_1)$ @ $t=2$

r_u or r_d

$$P_0(1) = \frac{100}{1+r_0} = \frac{\mathbb{E}^{Q}[100]}{1+r_0}$$

can use to "calibrate" to 1 year bond price (I R tree)

$$P_0(2) \begin{cases} P_{1u}(2) \begin{cases} 100 \\ 100 \end{cases} \\ P_{1d}(2) \begin{cases} 100 \\ 100 \end{cases} \end{cases}$$

q, r_0, r_u, r_d

$$P_{1u}(2) = \frac{\mathbb{E}^{Q}[100]}{1+r_u} = \frac{100}{1+r_u}$$

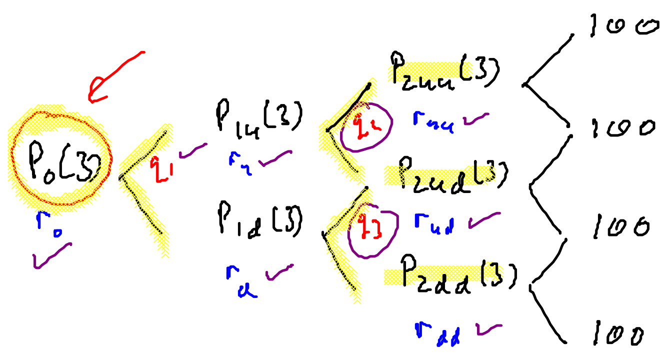
$$P_{1d}(2) = \frac{\mathbb{E}^{Q}[100]}{1+r_d} = \frac{100}{1+r_d}$$

$$P_0(2) = \frac{\mathbb{E}^{Q}[P_1(2)]}{1+r_0} = \frac{1}{1+r_0}\left(q\,\frac{100}{1+r_u} + (1-q)\,\frac{100}{1+r_d} \right)$$

require $q \in (0,1)$

$r_0 + \sigma\sqrt{\Delta t}$ $\qquad$ $r_0 - \sigma\sqrt{\Delta t}$

can be calibrated from historical data,



$$r_n = r_{n-1} + \sigma\sqrt{\Delta t}\, x_n$$

$$\left(r_{n-1}\, e^{\sigma\sqrt{\Delta t}\, x_n}\right)$$

$$A \begin{cases} A e^{\sigma\sqrt{\Delta t}} & p \\ A e^{-\sigma\sqrt{\Delta t}} \end{cases}$$

$$p = \frac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \dots$$

↳ real-world prob

$$\boxed{A_0 = \frac{1}{1+r}\,\mathbb{E}^{\mathbb{Q}}[A_1]}$$

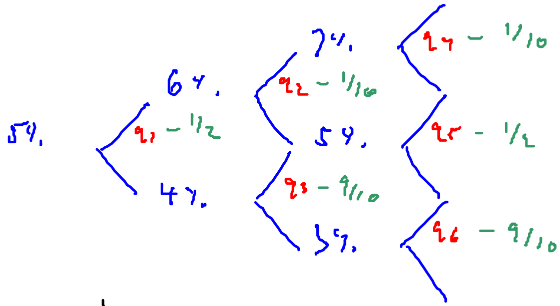
$$q = \frac{1}{2}\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \dots$$

↳ risk-neutral prob (not actual movement of asset prices)

$$\boxed{\begin{aligned} \mathbb{E}^{\mathbb{Q}}[A_T] &= e^{rT} A_0 \\ \mathbb{E}^{\mathbb{P}}[A_T] &= e^{\mu T} A_0 \end{aligned}}$$

$$\mathbb{V}^{\mathbb{P}}[\ln(A_T/A_0)] = \mathbb{V}^{\mathbb{Q}}[\ln(A_T/A_0)] = \sigma^2 T$$

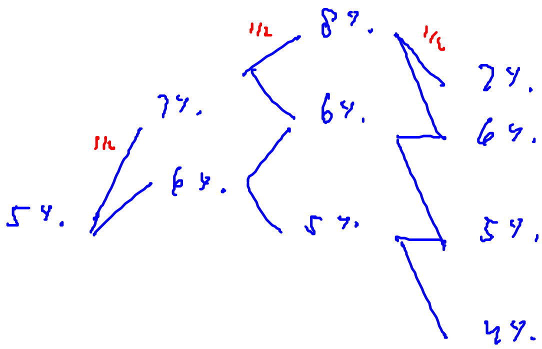
instead of $r_n = r_{n-1} \pm \sigma\sqrt{\Delta t}$ and finding $\mathbb{Q}$ $\{q_1, q_2, q_3, \dots\}$



Fix q's in particular $q = \frac{1}{2}$

$$r_n = r_{n-1} \pm \sigma\sqrt{\Delta t} + \underbrace{\theta_{n-1}\,\Delta t}_{\text{drift}}$$

2.



$$r_n = r_{n-1} + \theta_{n-1}\,\Delta t \pm \sigma\sqrt{\Delta t}$$

r_0

$\underbrace{r_0 + \theta_0\,\Delta t + \sigma\sqrt{\Delta t}}_{r_u}$

$\underbrace{r_0 + \theta_0\,\Delta t - \sigma\sqrt{\Delta t}}_{r_d}$

$\underline{r_0 + (\theta_0 + \theta_1)\,\Delta t + 2\sigma\sqrt{\Delta t}}$

$\underline{r_0 + (\theta_0 + \theta_1)\,\Delta t}$

$\underline{r_0 + (\theta_0 + \theta_1)\,\Delta t - 2\sigma\sqrt{\Delta t}}$

$$P_0(1) = \frac{100}{1+r_0} \Rightarrow r_0 = \#$$

$$P_0(2) = \frac{1}{1+r_0}\left(\frac{100}{1+r_u}\,\frac{1}{2} + \frac{100}{1+r_d}\,\frac{1}{2}\right) \Rightarrow \theta_0 = \#$$

$$P_0(3) = f(\theta_1) \Rightarrow \theta_1 = \#$$

$\vdots$

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call option on 3-mth bd, $K = 98.99$, $T = 1$-mth

$$\begin{cases} P_{1u}(3) \\ \\ P_{1d}(3) \end{cases}$$

$$C_0 \begin{cases} (P_{1u}(3) - K)_+ \\ \\ (P_{1d}(3) - K)_+ \end{cases}$$

$$C_0 = e^{-r_0 \Delta t} \mathbb{E}^{Q}[C_1]$$

call $K = 98.99$, $T = 2$
bond $T = 3$