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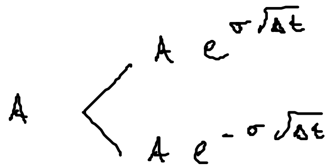
- Trinomial trees
- exotic options
- interest rate models.

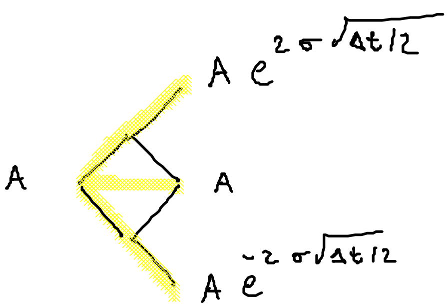


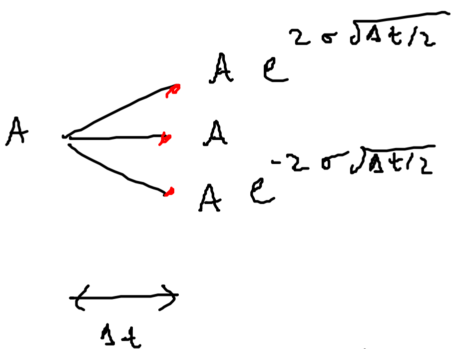
$$A_T \stackrel{d}{=} A_0 \, e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z}$$

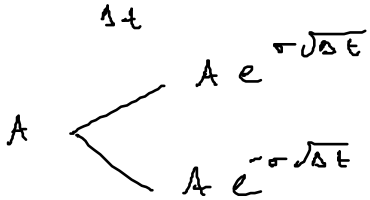
$$Z \underset{\mathbb{P}}{\sim} N(0,1)$$

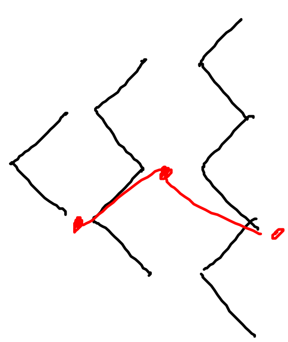
$$A_T \stackrel{d}{=} A_0 \, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z}$$

$$Z \underset{\mathbb{Q}}{\sim} N(0,1)$$









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$$A \begin{cases} A e^{\sigma\sqrt{\lambda \Delta t}} \\ A \\ A e^{-\sigma\sqrt{\lambda\Delta t}} \end{cases} \qquad 1 \begin{cases} e^{r\Delta t} \\ e^{r\Delta t} \\ e^{r\Delta t} \end{cases}$$

$\mathbb{Q}$ is not unique ... but we can pick one ... such that

$$A_T \overset{d}{=} A_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z}$$

where $Z \underset{\mathbb{Q}}{\sim} N(0,1)$

$\Rightarrow$

$$① \quad \mathbb{E}^{\mathbb{Q}}[\ln(A_{t+\Delta t}/A_t)] = (r-\tfrac{1}{2}\sigma^2)\Delta t$$

$$② \quad \mathbb{V}^{\mathbb{Q}}[\ln(A_{t+\Delta t}/A_t)] = \sigma^2 \Delta t$$

force these equalities

$$① \Rightarrow \quad \sigma\sqrt{\lambda\Delta t}\, q_u + 0\, q_m + (-\sigma\sqrt{\lambda \Delta t})\, q_d = (r-\tfrac{1}{2}\sigma^2)\Delta t$$

$$\Rightarrow \quad q_u - q_d = \frac{r-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\frac{\Delta t}{\lambda}} \qquad ③$$

$$② \Rightarrow \quad \mathbb{V}^{\mathbb{Q}}[\ln(A_{t+\Delta t}/A_t)] = \mathbb{E}^{\mathbb{Q}}[(\ln(A_{t+\Delta t}/A_t))^2] - (\mathbb{E}^{\mathbb{Q}}[\ln(A_{t+\Delta t}/A_t)])^2$$

$$(\sigma^2 \lambda\Delta t\, q_u + 0\, q_m + \sigma^2\lambda\Delta t\, q_d) - (r-\tfrac{1}{2}\sigma^2)^2\Delta t^2 = \sigma^2\Delta t$$

$$\Rightarrow \quad q_u + q_d = \frac{1}{\lambda} + \left(\frac{r-\frac{1}{2}\sigma^2}{\sigma}\right)^2\left(\frac{\Delta t}{\lambda}\right) \qquad ④$$

$$\frac{③ + ④}{2} \Rightarrow$$

$$q_u = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{r-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\frac{\Delta t}{\lambda}} + \underbrace{\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma}\right)^2\left(\frac{\Delta t}{\lambda}\right)}\right)$$

$o(\sqrt{\Delta t})$ can be thrown away for

$$\frac{(9) - (1)}{2} \Rightarrow$$

$$q_d = \frac{1}{2}\left(\frac{1}{\lambda} - \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\frac{\Delta t}{\lambda}} + \overbrace{\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma}\right)^2 \left(\frac{\Delta t}{\lambda}\right)}^{\Delta t \ll 1} \right)$$

and $\quad q_m = 1 - (q_u + q_d)$

$$= 1 - \frac{1}{\lambda} + \overbrace{\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma}\right)^2 \left(\frac{\Delta t}{\lambda}\right)}$$

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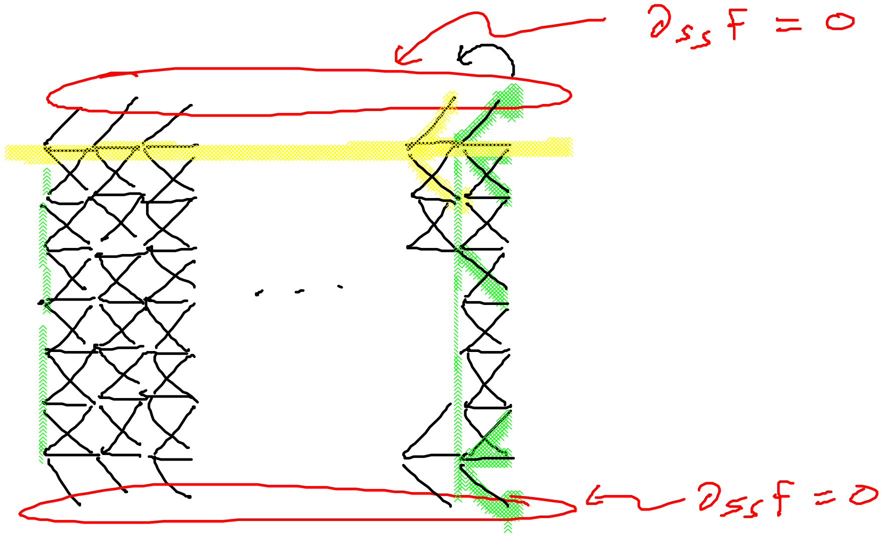
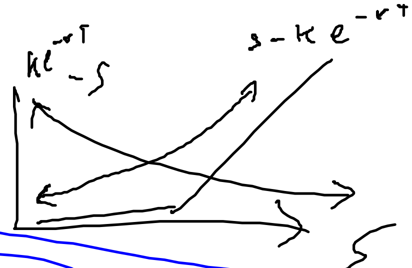


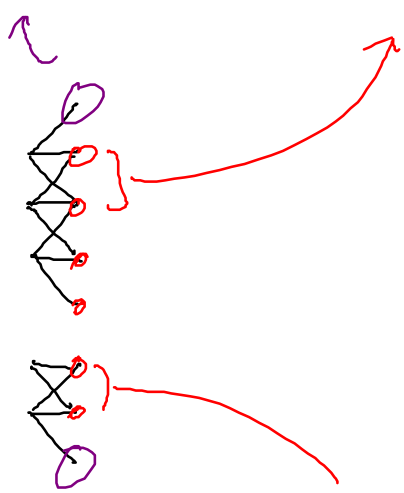


$$\partial_{ss} f \sim \frac{f(S+\Delta S) - 2f(S) + f(S-\Delta S)}{(\Delta S)^2}$$

$$\to f(S) + \Delta S\, \partial_S f(S) + \tfrac{1}{2}(\Delta S)^2\, \partial_{SS} f(S) + o(\Delta S^2)$$

$$\to f(S) - \Delta S\, \partial_S f(S) + \tfrac{1}{2}(\Delta S)^2\, \partial_{SS} f(S) + o(\Delta S^2)$$

$$\partial_{SS} f\Big|_{\text{up edge}} = 0 \Rightarrow \underset{\text{just outside}}{f(S+\Delta S)} = \underbrace{2f(S) - f(S-\Delta S)}_{\text{inside}}$$



$$f(s-\Delta s) \approx 2f(s) - f(s+\Delta s)$$

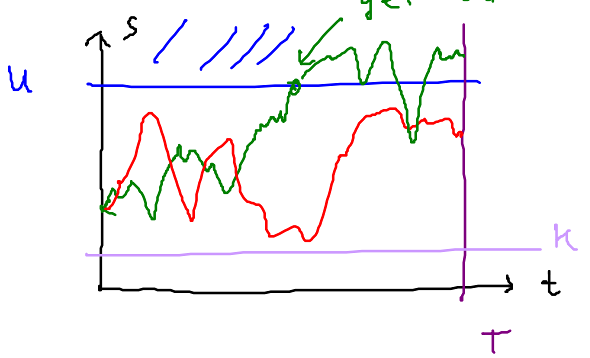
Barrier options: knock-in and knock-out

Knock-in Euro call:
(up-and-in)

payoff @ T

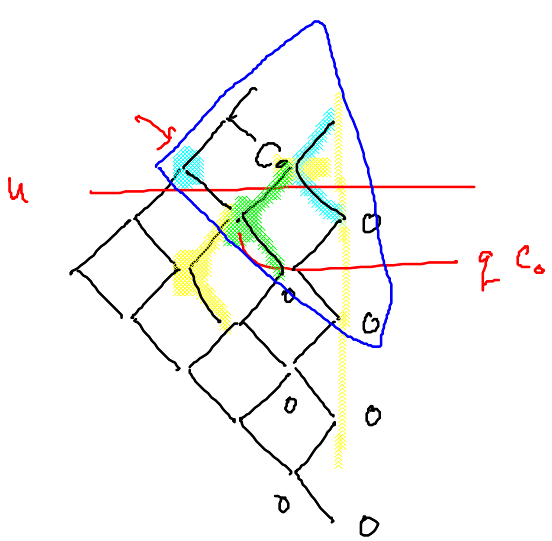
$$\Phi = \mathbb{1}_{\tau \in [0,T]} (S_T - K)_+$$

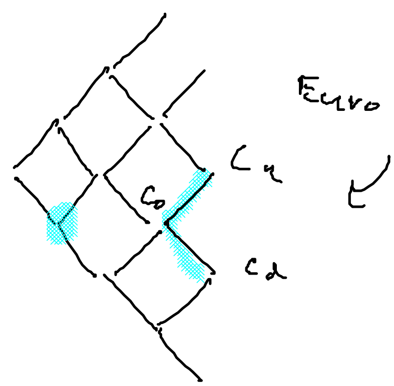
$$\tau = \inf\{ t : S_t \geq U \}$$

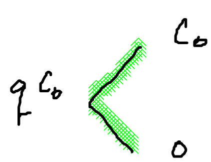


double knock-in Euro call:

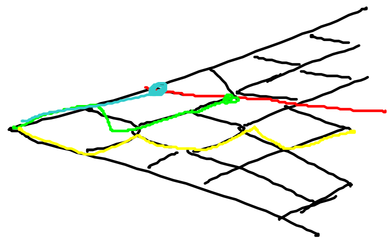
$$\tau = \inf\{ t : S_t \notin (L, U) \}$$



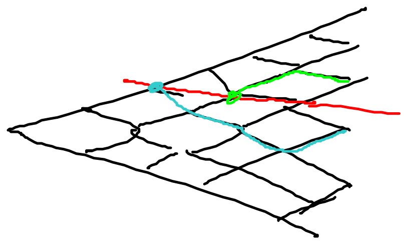




↳ value of a barrier opt conditional on $\tau > t$



Barrier



Euro

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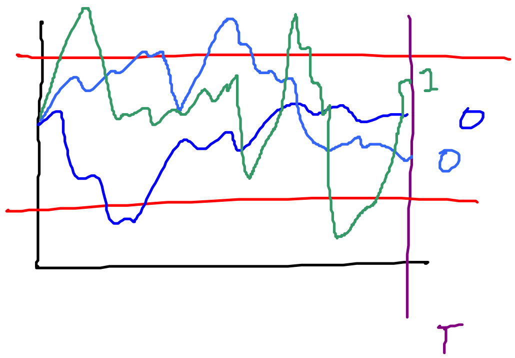
A exotic barrier options..

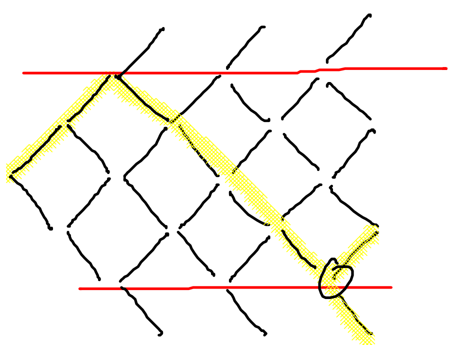
get 1 if: ① S first rises above 2
@ T

② S then drops below 1



$r = 0$

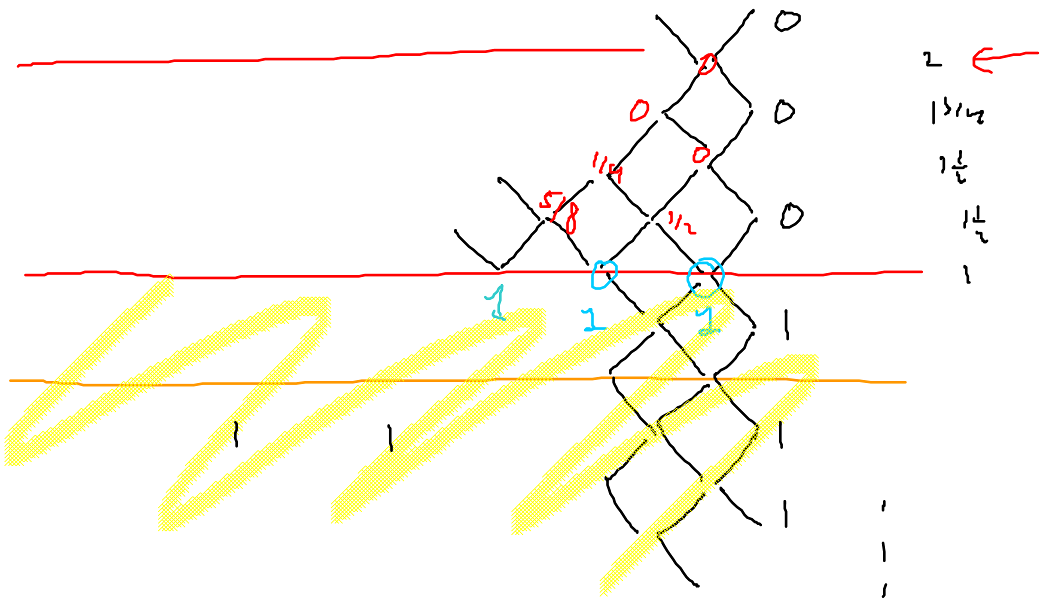
$q = 1/2$



$\mathbb{E}^{Q}[e^{-rT} Q]$

$= \left[\left(\frac{1}{2}\right)^6 \times 1 + \left(1-\left(\frac{1}{2}\right)^6\right) \cdot 0\right]$

$= \frac{1}{2^6}$



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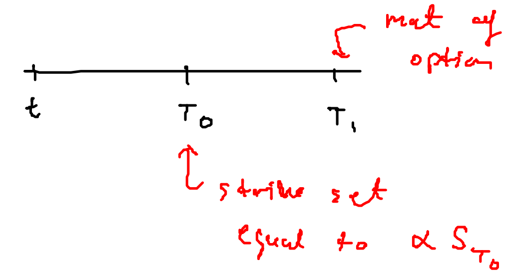
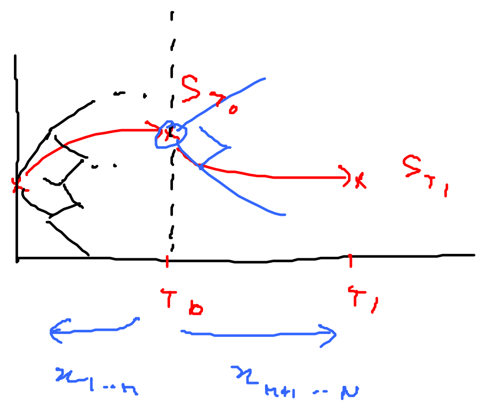
Forward-starting options...



$$Q^{FSC} = (S_{T_1} - \alpha S_{T_0})_+$$

$$V_t = e^{-r(T_1 - t)} \mathbb{E}^{Q}[(S_{T_1} - \alpha S_{T_0})_+]$$



$$S_{T_0} \overset{d}{=} S_0 e^{(r - \frac{1}{2}\sigma^2)T_0 + \sigma\sqrt{T_0} Z}$$

$$S_{T_1} \overset{d}{=} S_0 e^{(r - \frac{1}{2}\sigma^2)T_1 + \sigma\sqrt{T_1} Z}$$

$$Z \underset{Q}{\sim} N(0,1)$$

$$S_{T_0} \overset{d}{=} S_0 e^{(r - \frac{1}{2}\sigma^2)T_0 + \sigma\sqrt{T_0} Z_1}$$

$$S_{T_1} \overset{d}{=} S_{T_0} e^{(r - \frac{1}{2}\sigma^2)(T_1 - T_0) + \sigma\sqrt{T_1 - T_0} Z_2}$$

$$Z_1, Z_2 \text{ iid. } \underset{Q}{\sim} N(0,1)$$

$$S_T = S_0 \exp\left\{ \sigma\sqrt{\Delta t} \sum_{m=1}^{N} x_m \right\}$$

$$\mathbb{E}^{Q}[(S_{T_1} - \alpha S_{T_0})_+]$$

$$= \mathbb{E}^{Q}[\mathbb{E}^{Q}[(S_{T_1} - \alpha S_{T_0})_+ \mid S_{T_0}]]$$

$$\underbrace{e^{r(T_1-T_0)}\left(S_{T_0}\, \Phi(d_+) - \alpha\, S_{T_0}\, e^{-r(T_1-T_0)}\, \Phi(d_-) \right)}_{\text{"h"}}$$

$$d_\pm = \frac{\ln(\alpha S_{T_0} / S_{T_0}) + (r \pm \frac{1}{2}\sigma^2)(T_1 - T_0)}{\sigma\sqrt{T_1 - T_0}}$$

$$= \text{const.}$$

$$= e^{r(T_1-T_0)}\, c\, S_{T_0}$$

$$c = \Phi(d_+) - \alpha\, e^{-r(T_1-T_0)}\, \Phi(d_-)$$

$$= e^{r(T_1-T_0)}\, c\, \mathbb{E}^{\mathbb{Q}}[S_{T_0}]$$

$$= e^{r(T_1-T_0)}\, c\, e^{rT_0}\, S_0$$

$$\Rightarrow \boxed{V_0 = c\, S_0}$$