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$$A \begin{cases} A_u \\ A_d \end{cases} \qquad 1 \begin{cases} 1+r \\ 1+r \end{cases}$$

① $\exists$ no arb if and only if $\exists$ $\mathbb{Q}$ s.t.

$$C_0 = \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[C_1]$$

└ risk-neutral prob.

for any traded asset C.

① $\iff$ $A_d < A_0(1+r) < A_u$

$\iff$ $0 < q < 1$

$$\underset{\alpha}{A_0 \begin{cases} A_u \\ A_d \end{cases}} + \underset{\beta}{1 \begin{cases} 1+r \\ 1+r \end{cases}} = \underbrace{A_0\alpha + \beta}_{C_0} \begin{cases} A_u\alpha + (1+r)\beta = C_u \\ A_d\alpha + (1+r)\beta = C_d \end{cases}$$

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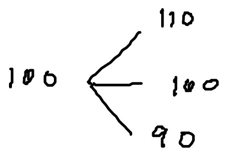
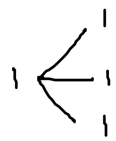
Incomplete Markets

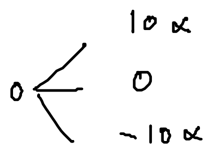
(more outcomes than traded assets)





① direct approach .. (α, β)

$V_0 = 0 \Rightarrow \beta = -100\,\alpha$



for any arb $\mathbb{P}(V_1 \geq 0) = 1$

$\Rightarrow \alpha = 0$

then $\mathbb{P}(V_1 > 0) = 0$

but an arb also has $\mathbb{P}(V_1 > 0) > 0$

$\therefore$ there are no arb in this market!

② $A_0 = \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[A_1]$

$1 - q_u - q_d$

$100 = 1\,(110\, q_u + 100\, q_m + 90\, q_d)$

seek: $q_u + q_m + q_d = 1$, $q_u > 0$, $q_m > 0$, $q_d > 0$

$\Rightarrow 0 = 10\, q_u - 10\, q_d \Rightarrow q_u = q_d = q$

$q_u > 0$ & $q_m > 0 \Rightarrow q > 0$

$q_d > 0 \Rightarrow 1 - q_u - q_d > 0 \Rightarrow 1 - 2q > 0$

$\Rightarrow q < \frac{1}{2}$

if $0 < q < \frac{1}{2}$ $\exists$ no arb

How to value $? \begin{cases} 0 \\ 10 \\ 0 \end{cases}$

$B_0 = \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[B_1]$ but $\mathbb{Q}$ is <u>not unique</u>!

$= 1\,(0 \cdot q + 10 \cdot (1-2q) + 0 \cdot q)$

$= 10(1-2q) = B_0^* \Rightarrow q = (1 - B_0^*/10)\,\frac{1}{2}$

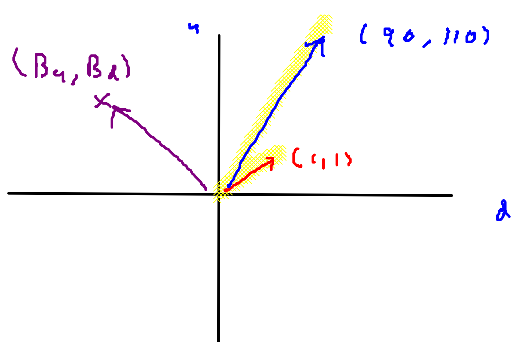
$0 < B_0 < 10$ any such B_0 is a viable no arb. price!

try replication:

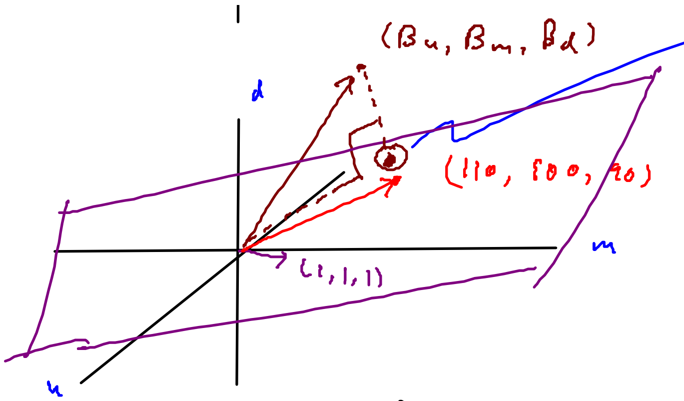
$$100 \begin{cases} 110 \\ 100 \\ 90 \end{cases} \quad 1 \begin{cases} 1 \\ 1 \\ 1 \end{cases} = \begin{cases} 110\alpha + \beta = B_u \leftarrow \\ 100\alpha + \beta = B_m \leftarrow \\ 90\alpha + \beta = B_d \leftarrow \end{cases}$$

$\alpha \qquad \beta$

$$100 \begin{cases} 110 \\ 90 \end{cases} \quad 1 \begin{cases} 1 \\ 1 \end{cases} = \begin{cases} 110\alpha + \beta = B_u \\ 90\alpha + \beta = B_d \end{cases}$$



unique $\alpha \quad \beta$



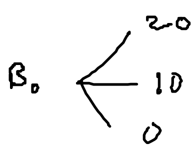
⇒ unique "price"
i.e. price of asset that is "as close to B as possible"

$$\hat{B}_1 = \arg\min_{\hat{B}_1} \mathbb{E}^{\mathbb{P}}\left[(\hat{B}_1 - B_1)^2\right]$$

↳ $\alpha A_1 + \beta$

Value of $B = \alpha A_0 + \beta$ ← is it arb free?

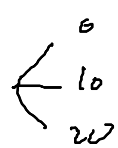
Such value is called the min var hedge price.

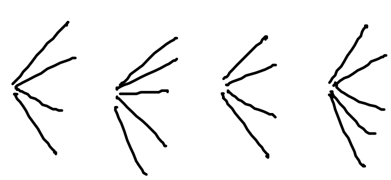


$$B_0 = 1\,(20 \cdot q + (1-2q)\,10 + q \cdot 0)$$

$$= 10$$

what if $B_0 = 11$ ... find the arb.





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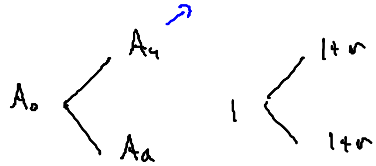
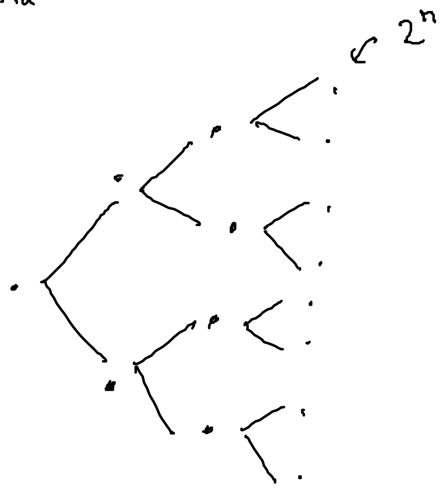
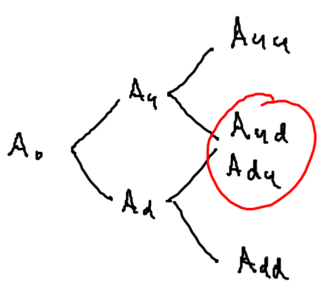
how to write in terms of observables?



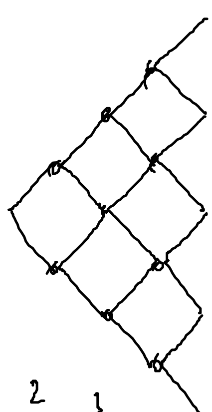


$O(2^n)$

recombining trees:



equal

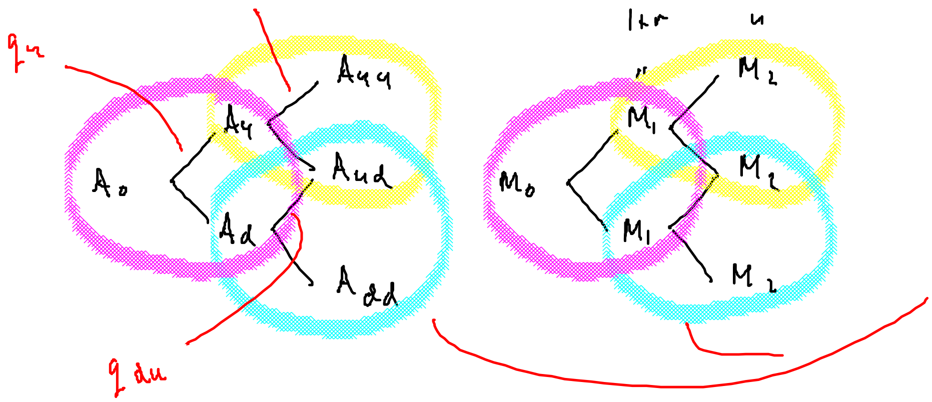


$O(n^2)$

2 3 4

quu

$(1+r)^2$



$$A_d = \frac{1}{1+r}\left(q_{du} A_{ud} + (1-q_{du}) A_{dd}\right)$$

$$(\alpha_0, \beta_0) \quad \begin{cases} (\alpha_{1u}, \beta_{1u}) \\ (\alpha_{1d}, \beta_{1d}) \end{cases}$$

$$\alpha_{1u} A_u + \beta_{1u} M_u = \alpha_0 A_u + \beta_0 M_u$$

$$\alpha_{1d} A_d + \beta_{1d} M_d = \alpha_0 A_d + \beta_0 M_d$$

$$\alpha_1 A_1 + \beta_1 M_1 = \alpha_0 A_1 + \beta_0 M_1$$

Self-Financing conditions

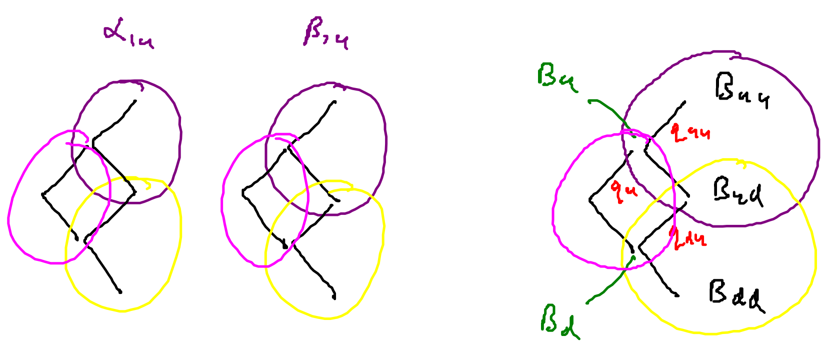
$$\Rightarrow \quad (\alpha_1 - \alpha_0) A_1 + (\beta_1 - \beta_0) M_1 = 0$$

$\exists$ a $\mathbb{Q}$ s.t. $\quad A_t = \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[A_{t+1}]$ iff

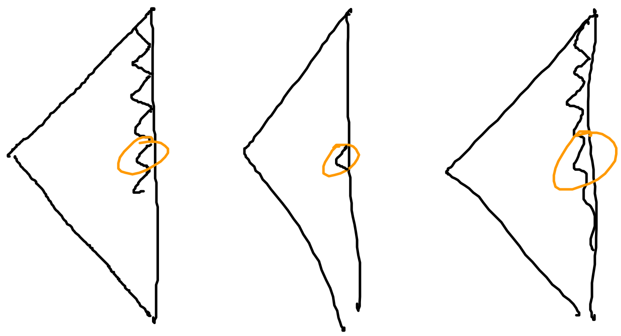
there is no arb.

so find q_u, q_{ud}, q_{du} and check that

$q_u \in (0,1)$, $\quad q_{ud} \in (0,1)$, $\quad q_{du} \in (0,1)$.



$$B_u = \frac{1}{1+r}\left(q_{uu} B_{uu} + (1-q_{uu}) B_{ud}\right)$$



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$A_{t_0}, A_{t_1}, A_{t_2}, \dots, A_{t_n}$ ⟵ observed

$$r_{t_n} = \frac{A_{t_n} - A_{t_{n-1}}}{A_{t_{n-1}}}$$

$$\rightarrow \quad r_{t_n} = \ln \underbrace{(A_{t_n} / A_{t_{n-1}})}_{\uparrow} \quad \longleftarrow$$

$$\ln(\alpha - 1 + 1) = \ln(1 - \underline{(1-\alpha)}) \sim \alpha - 1 + \dots$$

$$= \frac{A_{t_n}}{A_{t_{n-1}}} - 1$$

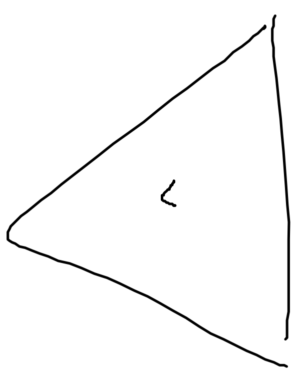
$$\ln(1+x) \sim x$$

$$= \frac{A_{t_n} - A_{t_{n-1}}}{A_{t_{n-1}}}$$

convexity correction

$$\mathbb{E}^{\mathbb{P}}[r] = \left(\mu^* - \tfrac{1}{2}(\sigma^*)^2\right)\Delta t = \frac{1}{N}\sum_{n=1}^{N} r_{t_n}$$

$$\mathbb{V}^{\mathbb{P}}[r] = (\sigma^*)^2 \Delta t = \frac{1}{N}\sum_{n=1}^{N} (r_{t_n} - \mathbb{E}^{\mathbb{P}}[r])^2$$



$$A \begin{cases} \overset{p}{\nearrow} A e^{c} \\ \searrow A e^{-c} \end{cases}$$

$\hookrightarrow A_n = A_{n-1} e^{c x_n}$

$x_1, x_2, \dots$ iid Bernoulli r.v.

$\mathbb{P}(x_1 = +1) = p$

$\mathbb{P}(x_1 = -1) = 1 - p$

$$r = \ln(A_n / A_{n-1}) = c x_n$$

$$\mathbb{E}^{\mathbb{P}}[r] = (\mu^* - \tfrac{1}{2}(\sigma^*)^2)\Delta t = c\,\mathbb{E}^{\mathbb{P}}[x_n] = c\,(1\cdot p + (-1)(1-p))$$

$$= c\,(2p-1)$$

$$\Rightarrow \quad p = \frac{1}{2}\left[1 + \frac{\mu^* - \frac{1}{2}(\sigma^*)^2}{c} \Delta t \right]$$

$$\mathbb{V}^{\mathbb{P}}[r] = (\sigma^*)^2 \Delta t = c^2 \, \mathbb{V}^{\mathbb{P}}[x_1] = c^2 \left(\mathbb{E}^{\mathbb{P}}[x_n^2] - (\mathbb{E}^{\mathbb{P}}[x_1])^2 \right)$$

$$= c^2 \left(1 - (2p-1)^2\right) = c^2 \left(1 - \frac{(\mu^* - \frac{1}{2}(\sigma^*)^2)^2 \Delta t^2}{c^2} \right)$$

$$\Rightarrow \quad c^2 = (\sigma^*)^2 \Delta t + (\mu^* - \tfrac{1}{2}(\sigma^*)^2)^2 \Delta t^2$$

$$\Rightarrow \quad c = \sigma^* \sqrt{\Delta t} \sqrt{1 + \left(\frac{\mu^* - \frac{1}{2}(\sigma^*)^2}{\sigma^*} \right)^2 \Delta t}$$

$$\sim \sigma^* \sqrt{\Delta t} + o(\sqrt{\Delta t})$$

$$\lim_{\Delta t \downarrow 0} \frac{o(\sqrt{\Delta t})}{\sqrt{\Delta t}} = 0$$

$\Rightarrow$

$$\boxed{\begin{aligned} p &= \frac{1}{2}\left(1 + \frac{\mu^* - \frac{1}{2}(\sigma^*)^2}{\sigma^*} \sqrt{\Delta t} \right) + o(\sqrt{\Delta t}) \\ c &= \sigma^* \sqrt{\Delta t} + o(\sqrt{\Delta t}) \end{aligned}}$$

$$A \to e^{\sigma^* \sqrt{\Delta t}} A$$

$$A \to e^{-\sigma^* \sqrt{\Delta t}} A$$

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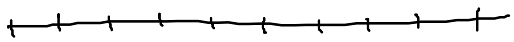

$0 \qquad \Delta t = \frac{T}{N} \qquad T < +\infty \qquad N \uparrow +\infty \iff \Delta t \downarrow 0$

$$A \begin{cases} A\, e^{\sigma^* \sqrt{\Delta t}} \\ A\, e^{-\sigma^* \sqrt{\Delta t}} \end{cases}$$

$$A_T = A_0 \exp\Big\{ \underbrace{\sigma^* \sqrt{\Delta t}\,(x_1 + x_2 + \cdots + x_N)}_{X_T} \Big\}$$

$$X_T \xrightarrow[N \to +\infty]{d} \mathcal{N}(\cdot\,;\,\cdot) \qquad \text{by CLT}$$

$$\mathbb{E}^{\mathbb{P}}[X_T] = \sigma^* \sqrt{\Delta t} \,.\, N \,.\, \mathbb{E}^{\mathbb{P}}[x_1]$$

$$= \sigma^* \sqrt{\Delta t} \,.\, N \,.\, (2p-1)$$

$$= \underbrace{\sigma^* \sqrt{\Delta t}}_{\sqrt{T/N}} \,.\, N \,.\, \left(\frac{\mu^* - \frac{1}{2}(\sigma^*)^2}{\sigma^*} \underbrace{\sqrt{\Delta t}}_{\sqrt{T/N}} + o(\sqrt{\Delta t}) \right)$$

$$\xrightarrow[\Delta t \downarrow 0]{} \left(\mu^* - \tfrac{1}{2}(\sigma^*)^2\right) T$$

$$\mathbb{V}^{\mathbb{P}}[X_T] = (\sigma^*)^2 \Delta t \,.\, N \,.\, \mathbb{V}^{\mathbb{P}}[x_1]$$

$$= (\sigma^*)^2 \underbrace{\Delta t}_{T/N} \,.\, N \,.\, (1 - \underbrace{(2p-1)^2}_{\left(\frac{\mu^* - \frac{1}{2}(\sigma^*)^2}{\sigma^*}\right)^2 \Delta t + o(\Delta t)})$$

$$\xrightarrow[\Delta t \downarrow 0]{} (\sigma^*)^2\, T$$

$$\boxed{X_T \xrightarrow[\Delta t \downarrow 0]{d} {}^{\mathbb{P}}\ \mathcal{N}\left((\mu^* - \tfrac{1}{2}(\sigma^*)^2) T \, ; \, (\sigma^*)^2 T \right)}$$

$A_T \overset{d}{=} A_0 \, e^{X_T}$ is a log-normal dist.

$$\overset{d}{=} A_0 \, e^{(\mu^* - \frac{1}{2}(\sigma^*)^2) T + \sigma^* \sqrt{T} Z}$$

$$Z \underset{\mathbb{P}}{\sim} \mathcal{N}(0 ; 1)$$

$$\mathbb{E}^{\mathbb{P}}[A_T] = A_0 \, \mathbb{E}^{\mathbb{P}}[e^{X_T}]$$

$$= A_0 \, e^{(\mu^* - \frac{1}{2}(\sigma^*)^2) T} \, \mathbb{E}^{\mathbb{P}}[e^{\sigma^* \sqrt{T} Z}]$$

reminder mgf of normal $\mathbb{E}[e^{uZ}] = e^{\frac{1}{2}u^2}$ $Z \sim \mathcal{N}(0,1)$

$$\Rightarrow \quad \boxed{\mathbb{E}^{\mathbb{P}}[A_T] = A_0 \, e^{\mu^* T}}$$

μ^* is the "drift" of the asset

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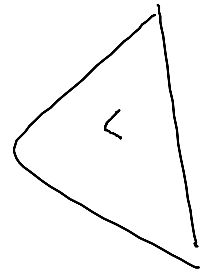
$$A \begin{cases} \overset{q}{\nearrow}\; A e^{\sigma^* \sqrt{\Delta t}} \\ \searrow\; A e^{-\sigma^* \sqrt{\Delta t}} \end{cases} \qquad 1 \begin{cases} e^{r \Delta t} \\ e^{r \Delta t} \end{cases}$$



$$\cancel{A} = e^{-r\Delta t} \left(q \cancel{A} e^{\sigma^* \sqrt{\Delta t}} + (1-q) \cancel{A} e^{-\sigma^* \sqrt{\Delta t}} \right)$$

$$\Rightarrow \quad \boxed{q = \frac{e^{r\Delta t} - e^{-\sigma^* \sqrt{\Delta t}}}{e^{\sigma^* \sqrt{\Delta t}} - e^{-\sigma^* \sqrt{\Delta t}}}} \qquad e^x \sim 1 + x + \tfrac{1}{2}x^2 + \cdots$$

$$= \frac{(1 + r\Delta t) - (1 - \sigma^* \sqrt{\Delta t} + \frac{1}{2}(\sigma^*)^2 \Delta t + \cdots)}{(1 + \sigma^* \sqrt{\Delta t} + \frac{1}{2}(\sigma^*)^2 \Delta t + \cdots) - (1 - \sigma^* \sqrt{\Delta t} + \frac{1}{2}(\sigma^*)^2 \Delta t)}$$

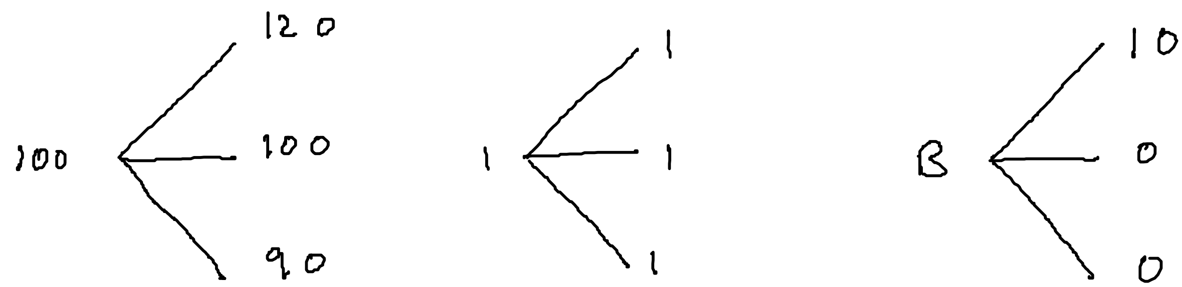
$$= \frac{\sigma^* \sqrt{\Delta t} + (r - \frac{1}{2}(\sigma^*)^2)\Delta t}{2\sigma^* \sqrt{\Delta t}} + \cdots$$

$$q = \frac{1}{2}\left(1 + \frac{r - \frac{1}{2}(\sigma^*)^2}{\sigma^*} \sqrt{\Delta t}\right) + \cdots$$

$$p = \frac{1}{2}\left(1 + \frac{\mu^* - \frac{1}{2}(\sigma^*)^2}{\sigma^*} \sqrt{\Delta t}\right) + \cdots$$

$$A_T \overset{d}{=} A_0 e^{X_T}$$

$$X_T \underset{Q}{\sim} N\left((r - \tfrac{1}{2}(\sigma^*)^2) T \, ; \; (\sigma^*)^2 T \right)$$



Find no and range for B.